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Zero-term Rank Preservers Of Fuzzy Matrices

濟州大學校 大學院

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Zero-term Rank Preservers Of Fuzzy Matrices

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CONTENTS

< **Abstract** >

ZERO-TERM RANK PRESERVERS OF FUZZY MATRICES

Linear preserver problem is an important topic on linear algebra and matrix theory. We studied the research papers on the fuzzy rank preserver and term rank preserver. They gave us the motivation to the research on the zero-term rank preserver. Zero-term rank of a matrix is the minimum number of lines (rows or columns) needed to cover all the zero entries of the given matrix. In this thesis, we study on the fuzzy semiring, zero-term rank of fuzzy matrices and linear operator on the vector space of fuzzy matrices. We also characterize the linear operators that preserve zero-term rank of the $m \times n$ matrices over a fuzzy semiring.

1. Introduction

A semiring is essentially a ring in which only zero is required to have an additive inverse (a formal definition is given in chapter 2). Thus all rings with multiplicative identity are semirings. So are such combinatorially interesting systems as the Boolean algebra of subsets of a finite set (with addition being union and multiplication being intersection) and the nonnegative integers (with the usual arithmetic). Fuzzy matrices provide another example of matrices over a semiring. In this case, the semiring of scalars consists of the real numbers $0 \leq x \leq 1$ with $x + y = max(x, y)$ and $xy = min(x, y)$. The concepts of matrix theory are defined over a semiring as over a field. There is much literature on the study of those linear operators on matrices that leave certain properties or subsets invariant. Boolean matrices also have been the subject of research by many authors. Beasley and Pullman characterized those linear operators that preserve Boolean rank in [1] and term rank of matrices over semirings in [4]. L.B.Beasley, S.Z.Song, and S.G.Lee obtained characterization of linear operators that preserve zero-term rank of Boolean matrices in [3]. In this thesis, we consider the zero-term rank of fuzzy matrices. We obtain characterizations of those linear operators that preserve zero-term rank of $m \times n$ matrices over a fuzzy semiring $\mathbb{F}$. In chapter 2, we introduce most of the definitions, notations, and preliminary results. In chapter 3, we characterize a fuzzy rank preserver, and in chapter 4, we give some characterizations of linear operators that preserve zero-term rank of $m \times n$ matrices over a fuzzy semiring $\mathbb{F}$.

2. Preliminaries

We start this chapter by introducing some basic definitions. A formal definition of a semiring is as follows;

Definition 2.1. A *semiring* consists of a set $\mathbb{S}$, and two binary operations on $\mathbb{S}$, addition and multiplication, such that ;

(1) $\mathbb{S}$ is an Abelian monoid under addition (identity denoted by 0);

(2) $\mathbb{S}$ is a monoid under multiplication (identity denoted by 1);

(3) multiplication distributes over addition ;

(4) $s0 = 0s = 0$ for all $s \in \mathbb{S}$; and

(5) $0 \neq 1$.

Usually $\mathbb{S}$ denotes both the semiring and the set. The set of all $m \times n$ matrices with entries in a semiring $\mathbb{S}$ is denoted by $M_{m,n}(\mathbb{S})$. The zero matrix and the $n \times n$ identity matrix I_n are defined as if $\mathbb{S}$ were a field. Addition, multiplication by scalar, and the product of matrices are also defined as if $\mathbb{S}$ were a field.

Definition 2.2. The *rank* (or *semiring rank*) of a nonzero matrix A in $M_{m,n}(\mathbb{S})$ is the least integer k such that $A = BC$ for some B in $M_{m,k}(\mathbb{S})$ and some C in $M_{k,n}(\mathbb{S})$.
The rank of a zero matrix is 0. We denote the rank of A by $r(A)$ or $r_{\mathbb{S}}(A)$.

Definition 2.3. A square matrix A is said to be *invertible* if there exist a square matrix X such that $AX = XA = I_n$.

Lemma 2.1. The rank of a nonzero matrix A is the minimum number of rank-1 matrices which sum to A.

Proof. Suppose B and C are $m \times k$ and $k \times n$ matrices over $\mathbb{S}$. Let b_j and c^j denote the jth column of B and the jth row of C. Then $A = BC = \sum_{j=1}^{k} b_j c^j$. The lemma follows from this expansion. □

Since $M_{n,n}(\mathbb{S})$ is a semiring, we can consider the invertible members of its multiplicative monoid. The permutation matrices (obtained by permuting the columns of I_n) are all invertible. If 1 is the only invertible member of the multiplicative monoid of $\mathbb{S}$, then the permutation matrices are the only invertible members of $M_{n,n}(\mathbb{S})$.

Lemma 2.2. The rank of a matrix is unchanged by transposition, pre- and post-multiplication by an invertible matrix.

Proof Let P, Q be fixed invertible matrices. Let T be any of the mappings $A \mapsto A^t, A \mapsto PA, A \mapsto AQ$. Then $r(T(A)) \leq r(A)$ by Lemma 2.1. But T is bijective, so $r(A) = r(T^{-1}(T(A))) \leq r(T(A))$.

Suppose $\mathbb{S}$ is a semiring. A function T mapping $M_{m,n}(\mathbb{S})$ into itself is called an *operator* on $M_{m,n}(\mathbb{S})$.

Definition 2.4. Let T be an operator on $M_{m,n}(\mathbb{S})$.

(i) T is *linear* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $\alpha, \beta \in \mathbb{S}$ and all $A, B \in M_{m,n}(\mathbb{S})$;

(ii) T *preserves rank* k if $r(T(A)) = r(A)$ whenever $r(A) = k$;

(iii) T is a (U, V) *-operator* if there exist invertible matrices U, V in $M_{m,m}(\mathbb{S})$ and $M_{n,n}(\mathbb{S})$ respectively such that for all $A \in M_{m,n}(\mathbb{S})$

(1) $T(A) = UAV$ or

(2) $m = n$, $T(A) = UA^tV$.

Lemma 2.3. (U, V) - operators are linear, are bijective, and preserve all ranks.

Proof. Linearity follows from the linearity of matrix multiplication. The rest follows from Lemma 2.2. □

Definition 2.5. Let $\mathbb{S}$ be any set of two or more elements. If $\mathbb{S}$ is totally ordered by $<$ (that is, $x < y$ or $y < x$ for all distinct x, y in $\mathbb{S}$), then define $x + y = max(x, y)$ and $xy = min(x, y)$ for all $x, y \in \mathbb{S}$.
If $\mathbb{S}$ has a universal lower bound and a universal upper bound, then $\mathbb{S}$ becomes a semiring. This semiring is said to be a *chain semiring*.

Let H be any nonempty family of sets ordered by inclusion; $0 = \cap_{x \in H} x$ and $1 = \cup_{x \in H} x$. Then $\mathbb{S} = H \cup \{0, 1\}$ is a chain semiring. Let α, ω be real numbers with $\alpha < \omega$. Define $\mathbb{S} = \{\beta : \beta \in [\alpha, \omega]\}$. Then $\mathbb{S}$ is a chain semiring with $\alpha = 0$ and $\omega = 1$. It is isomorphic to the chain semiring in the previous example with $H = \{[\alpha, \beta] : \alpha \leq \beta \leq \omega\}$.
Let $\mathbb{R}_+$ be the nonnegative real numbers. Then $(\mathbb{R}_+, +, \times)$ is a semiring under real addition $+$, and multiplication $\times$. The real numbers 0, 1 are the additive and multiplicative identities for this semiring. But $0, 1$ is not a

subsemiring, because for example $1 + 1 \neq 1$ (real addition). So $(\mathbb{R}_+, +, \times)$ is a semiring but not a chain semiring.

3. Fuzzy Rank Preserver

Definition 3.1. Let $\mathbb{F} = \{\beta \mid 0 \leq \beta \leq 1 \ , \ \beta \in \mathbb{R}\}$ denote a subset of reals. Define $x + y$ as $max\{x, y\}$ and xy as $min\{x, y\}$ for all $x,\ y$ in $\mathbb{F}$. Then $(\mathbb{F},\ +,\ \cdot)$ is called a *fuzzy semiring.* In the followings, $\mathbb{F}$ denotes both the fuzzy semiring and the set. Let $M_{m,n}(\mathbb{F})$ denote the set of all $m \times n$ matrices with entries in a fuzzy semiring $\mathbb{F}$. We call a matrix in $M_{m,n}(\mathbb{F})$ as a *fuzzy matrix.*

According to the definition of rank, a matrix A has rank 1 if and only if $A = xw^t$ and x, w are nonzero vector in $M_{m,1}(\mathbb{F})$ and $M_{n,1}(\mathbb{F})$, respectively. Let j_k denote the column vector of length k all of whose entries are 1 and $J_{m,n}$ the matrix all of whose entries are 1. Then $J_{m,n} = j_m j_n^t$ is a rank-1 matrix. If we denote e_p^k by the pth column of I_k, then $E_{i,j}^{m,n} = e_i^m (e_j^n)^t$ is a rank-1 matrix. From now on, all matrices denote fuzzy matrices.

Definition 3.2. The *norm* of an arbitrary $m \times n$ matrix X is defined by $||X|| = j^t X j$ the sum of all entries in X. That is, $||X||$ is the maximum entry in X.

Note that the mapping $X \mapsto ||X||$ preserves sums and scalar multiples of matrices X, and $||BC|| \leq ||B||||C||$ for all matrices B, C. The symbol $\leq$ is read entrywise, that is, $X \leq Y$ if and only if $x_{ij} \leq y_{ij}$ for all (i, j).

Definition 3.3. If A is any rank-1 $m \times n$ matrix, let $a = Aj$, and $u = A^t j$. Then $au^t = A$, and Furthermore for all x, w, if $A = xw^t$, then $a \leq x$ and $u \leq w$. For that reason $A = au^t$ is called the *minimum factorization* of A.

Lemma 3.1. If $r(A) = 1$ and λ is the minimum nonzero entry of A, then A has a line consisting only of λ's and $0'$s.

Proof. Let au^t be the minimum factorization of A. Then $\lambda = a_p u_q$ for some (p, q). Either $a_p = \lambda$ or $u_q = \lambda$. If $a_p = \lambda$, then $a_p u_j = \lambda$ unless $u_j = 0$, so the pth row of A contains only 0's and λ's. □

Lemma 3.2. If $r(A) = 1$ and $a_{pq} = 0$, then $a_{pi} = 0$ for all i or $a_{jp} = 0$ for all j.

Proof. Similar to Lemma 3.1's proof. □

Definition 3.4. We define the *floor* of X , $\mu(X)$, as the minimum entry in X.

Lemma 3.3. If $r(A) = 1$, then every entry in some line of A is $\mu(A)$.

Proof. This is Lemma 3.2 if $\mu(A) = 0$, and it is Lemma 3.1 if $\mu(A) > 0$.□

Lemma 3.4. Suppose $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $r(A) = 2$ if and only if $ad \neq bc$.

Proof. Let $\alpha = \mu(A)$. If $r(A) = 1$, then A has a line of α's(Lemma 3.3), so $ad = \alpha$ and $bc = \alpha$. Conversely, if $ad = bc$, then $ad = \alpha$ and $bc = \alpha$, Say $a = \alpha$; then $b = \alpha$ or $c = \alpha$.

So $A = \begin{pmatrix} \alpha \\ c + d \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix}$ or $\begin{pmatrix} b \\ d \end{pmatrix} \begin{pmatrix} \alpha & b + d \end{pmatrix}$.

Hence $r(A) \leq 1$. Similarly, $r(A) \leq 1$ if $d = \alpha$. □

Lemma 3.5. If H is a submatrix of A, then $r(H) \leq r(A)$.

Proof. A factorization $A = BC$ where B is $m \times k$ and C is $k \times n$ induces a factorization $H = KL$ where K has k columns and L has k rows. □

Lemma 3.6. If Y is the sum of two rank-1 matrices and Y has a submatrix of rank 2, then $r(Y) = 2$.

Proof. By Lemma 2.1, $1 \leq r(Y) \leq 2$. Apply Lemma 3.5. □

Lemma 3.7. If $r(A) = 1$, then $r(A + \alpha J) = 1$ for all scalars α.

Proof. $a_i u_j + \alpha = (a_i + \alpha)(u_j + \alpha)$ for all i, j. □

We just made use of the fact that addition distributes over multiplication in a chain semiring. If X is any $k \times l$ matrix we let $\begin{pmatrix} X & \alpha \\ \alpha & \alpha \end{pmatrix}$ denote the $m \times n$ matrix $\begin{pmatrix} X & \alpha J \\ \alpha J & \alpha J \end{pmatrix}$.

Lemma 3.8. If $\mu(X) \geq \alpha$, then the rank of $\begin{pmatrix} X & \alpha \\ \alpha & \alpha \end{pmatrix}$ is $r(X)$.

Proof. We may assume $X \neq 0$. Let $M = \begin{pmatrix} X & \alpha \\ \alpha & \alpha \end{pmatrix}$. Suppose first that $r(X) = 1$, so $X = uv^t$. Then for some (p, q), $\mu(u)\mu(v) = u_p v_q$, so $\mu(u)\mu(v) \geq \alpha$. Thus $M = \begin{pmatrix} u \\ \alpha \end{pmatrix} \begin{pmatrix} v^t & \alpha \end{pmatrix}$.

Next suppose $r(X) = k \geq 1$. Then $X = \sum_{j=1}^{k} Y_j$, where $r(Y_j) = 1$ for all

j (see Lemma 2.1). Let $Z_j = \mu(X)J + Y_j$. Then $r(Z_j) = 1$ by Lemma 3.7. Also $\sum_{j=1}^{k} Z_j = X + \mu(X)J = X$.

Consequently $\sum_{j=1}^{k} \begin{pmatrix} Z_j & \alpha \\ \alpha & \alpha \end{pmatrix} = M$, and each $\begin{pmatrix} Z_j & \alpha \\ \alpha & \alpha \end{pmatrix}$ has rank 1 by the first case, since $\mu(Z_j) \geq \mu(X) \geq \alpha$. Therefore $r(M) \leq k$. Thus $r(M) = k$ by Lemma 3.5. □

Unless otherwise specified, all matrices are $m \times n$ matrices over a chain semiring $\mathbb{K}$, and $min(m, n) > 1$.

Definition 3.5. Rank-1 matrices A, B are said to be *separable* if there is a rank-1 matrix X such that $r(A + X)r(B + X) = 2$. The matrix X is said to *separate A from B*.

Lemma 3.9. If $r(A) = r(B) = 1$ and $r(A + B) = 2$, then A separates A from B.

Lemma 3.10. If $r(A) = 1$ and $\alpha \neq 0$, then αJ and A are separable if and only if A is not a scalar multiple of J.

Proof. Suppose A is not a scalar multiple of J; then $\mu(A) < ||A||$. Now $||A|| = a_{pq}$ for some p and q. By Lemma 3.3, $\mu(A)$ occurs throughout a row or column of A. Say without loss of generality that $a_{pk} = \mu(A)$. Then $k \neq q$. Let $x_{ij} = ||A||$ for all $i \neq p$ and all j. Let $x_{pj} = 0$ for all j. Then $r(X) = 1$ and $r(A + X) = 2$ by Lemmas 3.4 and 3.6. But $r(\alpha J + X) = 1$ by Lemma 3.7. Therefore X separates αJ form A. Lemma 3.7 also implies that βJ can not be separated from αJ. □

In theorem 3.1 below we will show that in fact, every pair of distinct rank-1 matrices is separable unless both are scalar multiples of J. Let $\mathbb{B}$ be the two element subsemiring $\{0, 1\}$ of $\mathbb{K}$, and α be a fixed member of $\mathbb{K}$, other than 1. For each x in $\mathbb{K}$ define $x^\alpha = 0$ if $x \leq \alpha$, and $x^\alpha = 1$ otherwise. Then the mapping $x \to x^\alpha$ is a homomorphism of $\mathbb{K}$ onto $\mathbb{B}$.

Definition 3.6. A mapping $A \to A^\alpha$ of $M_{m,n}(\mathbb{K})$ onto $M_{m,n}(\mathbb{B})$ preserves matrix sums and products and multiplication by scalars. We call A^α the *α-pattern* of A.

Lemma 3.11. Suppose A, B are rank-1 matrices not both scalar multiples of J, and $\alpha = \mu(A)\mu(B)$. If A, B have different α-patterns, then they are separable.

Corollary. Rank-1 matrices with different floors are separable unless both are scalar multiples of J.

Lemma 3.12. If A, B are distinct 2×2 rank-1 matrices, not both scalar multiples of $J_{2,2}$, then they are separable.

Proof. Let $\alpha = \mu(A)$. By Lemma 3.11 and its corollary we may assume $\alpha = \mu(B)$ and $A^\alpha = B^\alpha$. Then A has $k \geq 2$ entries equal to α by Lemma 3.3. By our hypotheses, $k < 4$. Suppose $k = 2$. By Lemma 2.2 we may assume $A = \begin{pmatrix} \alpha & \alpha \\ a & b \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \alpha \\ c & d \end{pmatrix}$ where $\alpha < a \leq b$, $\alpha < cd$ and $a \leq cd$. If $a < cd$ and $a < b$, then $X = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$ separates A from B, since $r(X + A) = 2$ by Lemma 3.4 and $r(X) = r(B + X) = 1$. If $a < cd$ and $a = b$, then $X =$

$\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}$ separates A from B, since $r(X + A) = 1$ while $r(B + X) = 2$ by Lemma 3.4. If $a = cd$, suppose $a = c$. Then $b \neq d$ and we may assume $b < d$. Then $\begin{pmatrix} 1 & 1 \\ b & b \end{pmatrix}$ separates $A = \begin{pmatrix} \alpha & \alpha \\ a & b \end{pmatrix}$ from $B = \begin{pmatrix} \alpha & \alpha \\ a & d \end{pmatrix}$ by Lemma 3.4. Now suppose $a < c$. Then $a = d$. We may assume $b \leq c$; then $\begin{pmatrix} c & a \\ 0 & 0 \end{pmatrix}$ separates $A = \begin{pmatrix} \alpha & \alpha \\ a & b \end{pmatrix}$ from $B = \begin{pmatrix} \alpha & \alpha \\ c & a \end{pmatrix}$ by Lemma 3.4 unless $a = b$. But then $\begin{pmatrix} c & c \\ 0 & 0 \end{pmatrix}$ separates $A = \begin{pmatrix} \alpha & \alpha \\ a & a \end{pmatrix}$ from $B = \begin{pmatrix} \alpha & \alpha \\ c & a \end{pmatrix}$ by Lemma 3.4. Finally suppose $k = 3$. By Lemma 2.2, we may suppose

$A = \begin{pmatrix} \alpha & \alpha \\ \alpha & a \end{pmatrix}$, $B = \begin{pmatrix} \alpha & \alpha \\ \alpha & b \end{pmatrix}$, and $a < b$. But then $\begin{pmatrix} 1 & 1 \\ a & a \end{pmatrix}$ separates A from B by Lemma 3.4. □

Lemma 3.13. Suppose $min(m, n) = 2$. If A, B are distinct rank-1 matrices, not both scalar multiples of J, then A, B are separable.

Theorem 3.1. Distinct rank-1 matrices are separable if and only if at least one of them is not a scalar multiple of J.

Proof. By Lemma 3.10 it is enough to prove that distinct rank-1 matrices A, B are separable if neither is a scalar multiple of J. Let $\alpha = \mu(A)$. Lemma 3.11 and its corollary let us assume that $\mu(B) = \alpha$ and $A^\alpha = B^\alpha$. Lemma 3.13 lets us assume that $m > 2$ and $n > 2$. We may now assume that $A = \begin{pmatrix} M & \alpha \\ \alpha & \alpha \end{pmatrix}$ and $B = \begin{pmatrix} N & \alpha \\ \alpha & \alpha \end{pmatrix}$, where M and N are $k \times l$ rank-1 matrices with floors exceeding α, and $k < m$ or $l < m$. Without loss of generality we may assume $k < m$. If $M = \beta J$ and $N = \gamma J$ with $\beta < \gamma$, then let $x_{ij} = \gamma$ for $j \neq 1$ and $x_{i1} = \beta$.Here X separates A from B. We now assume that M and N are $k \times l$ rank-1 matrices not both of which are scalar multiples of $J_{k,l}$. Lemma 3.13 lets us assume $min(k, l) \geq 2$. Inductively there exists a $k \times l$ rank-1 matrix Y separating M from N. Then $\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}$ separates A from B by Lemma 3.8. □

We consider rank-preserving operators. Let $\mathbb{K}$ be a fixed chain semiring. The set of matrices of rank k in $M_{m,n}(\mathbb{K})$ is denoted R_k.

Lemma 3.14. Let T be a linear operator on $M_{m,n}(\mathbb{K})$ with $min(m, n) > 1$. If T preserves norm and rank 1 but is not injective on R_1, then T decreases the rank of some rank-2 matrix.

Proof. Since T is not injective, $T(A) = T(B)$ for some A, B in R_1 with $A \neq B$. If $A = \alpha J$ and $B = \beta J$, then $\alpha = \beta$ because T preserves norms, contradicting our assumption that $A \neq B$. Therefore by Theorem 3.1 some rank-1 matrix X separates A from B. Say $r(X + A) = 1$ and $r(X + B) = 2$. Then T reduces the rank of $X + B$ from 2 to 1. □

Lemma 3.15. If T is a linear operator on $M_{m,n}(\mathbb{K})$, $min(m, n) > 1$, and T preserves ranks 1 and 2, then T preserves norm.

Proof. Let $A \in M_{m,n}(\mathbb{K})$, $\alpha = \|A\|$, and $\beta = \|T(A)\|$; then $\beta \leq \alpha$, because $T(\sigma X) = \sigma T(X)$ for all $X \in M_{m,n}(\mathbb{K})$. Suppose $\beta \leq \alpha$. Then for some (p, q), $a_{pq} = \alpha$. Let Y be the matrix whose entries are all α except for $y_{pq} = 0$. Then $\alpha J = A + Y$. So $r(A + Y) = 1$ while $r(\beta A + Y) = 2$ by Lemmas

3.4 and 3.6. By the linearity of T and the definition of β, $T(\beta A) = T(A)$, so T reduces the rank of $\beta A + Y$ from 2 to 1, contrary to the hypothesis. □

Lemma 3.16. If T is a linear operator on $M_{m,n}(\mathbb{K})$, $min(m,n) > 1$, T preserves norm, and $A \leq T(A)$, then $T^q(A) = T^{mn-1}(A)$ for all $q \geq mn$.

Let $\Delta = \{E_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Lemma 3.17. Suppose T is a linear operator on $M_{m,n}(\mathbb{K})$ and $min(m,n) > 1$. If T preserves ranks 1 and 2, then T permutes Δ.

Proof. By Lemma 3.15, T preserves norm. Therefore by Lemma 3.14, T is injective on R_1, the $m \times n$ rank-1 matrices over $\mathbb{K}$. Suppose $T(E_{ij})$ is not in Δ for some $(i,\ j)$. Now $T(E_{ij}) = \sum \tau_{uv} E_{uv}$. But $\| T(E_{ij}) \| = 1$, so $\tau_{pq} = 1$ for some $(p,\ q)$. Without loss of generality, we may assume $(p,\ q) = (i,\ j)$, because if P, Q are permutation matrices, then the linear operator $X \to PT(X)Q$ preserves the ranks T preserves (see Lemma 2.2) and permutes Δ if and only if T does. Let $E = E_{ij}$. Then $E \leq T(E)$, so $E \neq T(E) \leq T^2(E) \leq \cdots \leq T^k(E) = T^{k+n}(E)$, where k is the least integer for which equality holds and $n \geq 0$ is arbitraty. By Lemma 3.16 we are assured that k exists and is less than mn. Let $B = T^{k-1}(E)$; then $B \neq T(B)$ but $T(B) = T(T(B))$, despite the fact B, $T(B)$ are both in R_1 and T is injective there. This contradiction implies that T maps Δ into Δ. By injectivity, T permutes Δ. □

Theorem 3.2. Suppose T is a linear operator on the $m \times n$ matrices over a chain semiring and $min(m,n) > 1$. If T preserves ranks 1 and 2, then T is a (U,V)-operator.

Proof. Recall that $\mathbb{B} = \{0,\ 1\}$ is a subsemiring of $\mathbb{K}$. Let $M = M_{m,\ n}(\mathbb{B})$. Lemma 3.17 and linearity imply that T maps M into itself. Let $\tilde{T}$ denote the restriction of T to M. From this definition of rank, the rank $r_{\mathbb{B}}(X)$ of a member X of M is at least $r(X)$, its rank as a member of $M_{m,n}(\mathbb{K})$, because $\mathbb{B} \subset \mathbb{K}$. On the other hand, the mapping that takes a matrix A to its 0-pattern A^0 preserves matrix products. Hence $r_{\mathbb{B}}(X) = r(X)$ for all X in M. Therefore $\tilde{T}$ preserves ranks 1 and 2 over M. By [1, Theorems 3.1, 4.1, 4.2], $\tilde{T}$ is a (U, V)-operator on M. The corresponding matrices $U,\ V$ are also invertible over $M_{m,n}(\mathbb{K})$; in fact, they are just permutation matrices. Let $A \in M_{m,n}(\mathbb{K})$. Then $T(A) = \sum a_{ij}T(E_{ij}) = \sum a_{ij}\tilde{T}(E_{ij})$ as $E_{ij} \in M$. Either (1) $\tilde{T}(E_{ij}) = UE_{ij}V$ for all $i,\ j$ or (2) $m = n$ and $\tilde{T}(E_{ij}) = UE_{ij}^tV$ for all i, j, by the definition of (U, V)-operator the result follows from the linearity of matrix multiplication. □

Theorem 3.3. Suppose T is a linear operator on the $m \times n$ matrices over a chain semiring and $min(m, n) > 1$. Then the following statements are equivalent:

(1) T preserves all ranks;

(2) T preserves ranks 1 and 2;

(3) T is a (U, V)-operator;

(4) T is bijective and preserves rank 1.

Proof. Theorem 3.2 establishes that (2) implies (3). According to Lemma 2.3, (3) implies (4). If T satisfies (4), then the linearity of T and Lemma 2.1

ensure that T does not increase the rank of any matrix. For the same reason, neither does T^{-1}. Therefore (4) implies (1). □

4. Linear Operators That Preserve Zero-Term Rank Of Fuzzy Matrices

Definition 4.1. Let $E_{i,j}$ be the $m \times n$ matrix whose (i,j)th entry is 1 and whose other entries are all zero, which is called a *cell.*
Let J denote the $m \times n$ matrix all of whose entries are 1 and $\triangle = \{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ denote the set of cells.

Definition 4.2. The *zero-term rank* $z(X)$ of a matrix X is the minimum number of lines (rows or columns) needed to cover all the zero entries in X.

Definition 4.3. The *term rank* $t(X)$ of a matrix X is the minimum number of lines (rows or columns) needed to cover all the nonzero entries in X.

Definition 4.4. For any A, $B \in M_{m,n}(\mathbb{F})$, we say A *dominates* B (written $A \geq B$ or $B \leq A$) if $a_{ij} \geq b_{ij}$ for all i, j.

Then we obtain the following result.

Lemma 4.1. For any $A, B \in M_{m,n}(\mathbb{F})$, $A \geq B$ implies that $z(A) \leq z(B)$.

Proof. If $z(B) = k$, then there are k lines which cover all zero entries in B. Since $A \geq B$, this k lines can also cover all zero entries in A. Hence $z(A) \leq k = z(B)$. □

Definition 4.5. If $z(T(X)) = k$ whenever $z(X) = k$, we say T *preserves zero-term rank* k. If T preserves zero-term rank k for every $k \leq min\{m, n\}$, then we say T *preserves zero-term rank*.

Definition 4.6. If $t(T(X)) = k$ whenever $t(X) = k$, we say T *preserves term rank* k. If T preserves term rank k for every $k \leq min\{m, n\}$, then we say T *preserves term rank*.

Which linear operators on $M_{m,n}(\mathbb{F})$ preserve zero-term rank? The operations of permuting rows, permuting columns, and (if $m = n$) transposing the matrices in $M_{m,n}(\mathbb{F})$ are all linear operators that preserve zero-term rank of the matrices on $M_{m,n}(\mathbb{F})$.
If we take a fixed $m \times n$ matrix B in $M_{m,n}(\mathbb{F})$, then its Schur product is defined $B \circ X = [b_{ij}x_{ij}]$ for all X in $M_{m,n}(\mathbb{F})$.

Proposition 4.1. Suppose that T is an operator on $M_{m,n}(\mathbb{F})$ such that $T(X) = B \circ X$, where B is fixed in $M_{m,n}(\mathbb{F})$. Then T is linear.

Proof. For all $\alpha, \beta \in \mathbb{F}$, A, $B \in M_{m,n}(\mathbb{F})$,

$$T(\alpha X + \beta Y) = B \circ (\alpha X + \beta Y) = B \circ (\alpha X) + B \circ (\beta Y)$$
$$= \alpha(B \circ X) + \beta(B \circ Y) = \alpha T(X) + \beta T(Y)$$ □

Proposition 4.2. Suppose that T is a linear operator on $M_{m,n}(\mathbb{F})$ such that $T(X) = B \circ X$, where B is fixed in $M_{m,n}(\mathbb{F})$, none of whose entries is zero in $\mathbb{F}$. Then T preserves zero-term rank.

Proof. It follows the definition of Schur product. □

That these operations and their compositions are the only zero-term rank preservers is one of the consequences of theorem 4.1 below. Such operators are described more formally in the following definition.

Definition 4.7. If P and Q are $m \times m$ and $n \times n$ permutation matrices, respectively and B is an $m \times n$ matrix, none of whose entries is zero, then T is a (P, Q, B)-*operator* if

(1) $T(X) = P(B \circ X)Q$ for all X in $M_{m,n}(\mathbb{F})$ or

(2) $m = n$, and $T(X) = P(B \circ X^t)Q$ for all X in $M_{m,n}(\mathbb{F})$.

From now on we will assume that $2 \leq m \leq n$ for all $m \times n$ matrices, and a mapping T will denote a linear operator on $M_{m,n}(\mathbb{F})$.

Definition 4.8. Let $\mathcal{E} = \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. That is, $\mathcal{E}$ is a set of indices. Define $T' : \mathcal{E} \to \mathcal{E}$ by $T'(i,j) = (u,v)$ whenever $T(E_{ij}) = b_{ij}E_{uv}$ with $0 < b_{ij} \leq 1$.

Lemma 4.2. Suppose T preserves zero-term ranks 0 and 1. Then T maps a cell onto a cell with a scalar multiple and hence T' is a bijection on the set $\mathcal{E}$.

Proof. If $T(E_{ij}) = 0$ for some $E_{ij} \in \triangle$, then we can choose $mn - 1$ cells $E_1, E_2, \cdots, E_{mn-1}$ which are different from E_{ij} such that

$$\begin{aligned} T(J) &= T(E_{ij} + \sum_{h=1}^{mn-1} E_h) \\ &= T(E_{ij}) + T(\sum_{h=1}^{mn-1} E_h) \\ &= 0 + T(\sum_{h=1}^{mn-1} E_h) \\ &= T(\sum_{h=1}^{mn-1} E_h). \end{aligned}$$

But $z(J) = 0$ and $z(\sum_{h=1}^{mn-1} E_h) = 1$. Since T preserves zero-term ranks 0 and 1, we have $z(T(J)) = 0$ and $z(T(\sum_{h=1}^{mn-1} E_h)) = 1$. This is a contradiction because $0 = z(T(J)) = z(T(\sum_{h=1}^{mn-1} E_h)) = 1$. Hence $T(E_{ij})$ dominates at least one cell with a scalar multiple. That is, $T(E_{ij}) \geq b_{ij} E_{uv}$, for some $E_{uv} \in \triangle$ with $0 < b_{ij} \leq 1$.

For some cell $E_{ij} \in \triangle$, suppose $T(E_{ij}) \geq b_{ij} E_{kl} + b'_{ij} E_{uv}$ for some $E_{kl}, E_{uv} \in \triangle$ with $0 < b_{ij}, b'_{ij} \leq 1$. For some cell E_{rs} except for both E_{kl} and E_{uv}, we can choose one cell E_h such that $T(E_h)$ dominates $b_h E_{rs}$ for some $E_{rs} \in \triangle$ with $0 < b_h \leq 1$ because T preserves zero-term rank 0. Since the number of cells except for both E_{kl} and E_{uv} is $mn - 2$, there exist at most $mn - 1$ cells $E_1, E_2, \cdots, E_{mn-1}$ containing E_{ij} such that

$$z(T(\sum_{h=1}^{mn-1} E_h)) = 0. \tag{4.1}$$

But $z(\sum_{h=1}^{mn-1}) = 1$. Since T preserves zero-term rank 1, we have $z(T(\sum_{h=1}^{mn-1} E_h)) = 1$. This contradicts to the equality (4.1). Hence $T(E_{ij}) = b_{ij} E_{uv}$ for all $E_{ij} \in \triangle$. That is, T maps a cell into a cell with a scalar multiple.

Now we show that T' is a bijection on $\mathcal{E}$. If $T'(i,j) = T'(r,s) = (u,v)$ for some different indices (i,j) and (r,s), then we have

$$\begin{aligned} T(J) &= T(\{J - (E_{ij} + E_{rs})\} + (E_{ij} + E_{rs})) \\ &= T(J - (E_{ij} + E_{rs})) + T(E_{ij} + E_{rs}) \\ &= T(J - (E_{ij} + E_{rs})) + T(E_{ij}) + T(E_{rs}) \end{aligned} \tag{4.2}$$

Since $T'(i,j) = T'(r,s) = (u,v)$, we have $T(E_{ij}) = b_{ij} E_{uv}$ and $T(E_{rs}) = b_{rs} E_{uv}$ with $0 < b_{ij}, b_{rs} \leq 1$.

(case 1) $b_{ij} \geq b_{rs}$. By the equality (4.2), we have

$$\begin{aligned} T(J) &= T(J - (E_{ij} + E_{rs})) + T(E_{ij}) + T(E_{rs}) \\ &= T(J - (E_{ij} + E_{rs})) + b_{ij}E_{uv} + b_{rs}E_{uv} \\ &= T(J - (E_{ij} + E_{rs})) + b_{ij}E_{uv} \\ &= T(J - (E_{ij} + E_{rs})) + T(E_{ij}) \\ &= T(J - (E_{ij} + E_{rs}) + E_{ij}) \\ &= T(J - E_{rs}). \end{aligned}$$

(case 2) $b_{ij} < b_{rs}$. By the similar argument of case 1, we have

$$T(J) = T(J - E_{ij}).$$

But $z(J - E_{rs}) = z(J - E_{ij}) = 1$ and $z(J) = 0$. This contradicts that T preserves zero-term ranks 0 and 1. Therefore $T^{'}$ is an injection on $\mathcal{E}$ and so $T^{'}$ is a bijection on $\mathcal{E}$. □

Lemma 4.3. If T preserves zero-term ranks 0 and 1, then T preserves term rank 1.

Proof. Suppose that T does not preserve term rank 1. Then there exist some cells E_{ij} and E_{il} on the same row(or column) such that $T(E_{ij} + E_{il}) = T(E_{ij}) + T(E_{il}) = b_{ij}E_{pq} + b_{il}E_{rs}$ with $p \neq r$ and $q \neq s$, where $T^{'}(i,j) = (p,q)$ and $T^{'}(i,l) = (r,s)$. Since T preserves zero-term ranks 0 and 1, we have that $T^{'}$ is bijective on $\mathcal{E}$ by lemma 4.2. Hence we have $T(J) = B = (b_{uv})_{m\times n}$ for some $B \in M_{m,n}(\mathbb{F})$ and $0 < b_{uv} \leq 1$. Since T preserves zero-term rank 1 and $z(J - E_{ij} - E_{il}) = 1$, we have $z(T(J - E_{ij} - E_{il})) = 1$. Since $T(E_{ij} + E_{il}) = T(E_{ij}) + T(E_{il}) = b_{ij}E_{pq} + b_{il}E_{rs}$, the image of $J - E_{ij} - E_{il}$

has zeros in (p, q) and (r, s) positions and otherwise nonzero entries. Then $z(T(J - E_{ij} - E_{il})) = 2$ because $p \neq r$ and $q \neq s$. This is a contradiction. Hence T preserves term rank 1. □

Lemma 4.4. If T preserves zero-term ranks 0 and 1, then T maps a row of a matrix onto a row with scalar multiple (or column if $m = n$) in $\mathbb{F}$.

Proof. Suppose that T does not map a row into a row with scalar multiple (or column if $m = n$). Then T does not preserve term rank 1. This contradicts to lemma 4.3. Hence T maps a row into a row with scalar multiple (or column if $m = n$). Since T preserves zero-term ranks 0 and 1, we have that T' is bijective on $\mathcal{E}$ by lemma 4.2. Then the bijectivity of T' implies that T maps a row onto a row with scalar multiple (or may be a column if $m = n$). □

Lemma 4.5 For the case $m = n$, suppose that T preserves zero-term ranks 0 and 1. If T maps a row onto a row (or column) with scalar multiples in $\mathbb{F}$, then all rows of a matrix must be mapped some rows (or columns, respectively) with scalar multiples in $\mathbb{F}$.

Proof. Since T preserves zero-term ranks 0 and 1, T' is bijective on $\mathcal{E}$ by lemma 4.2. Let $R_i = \sum_{j=1}^{n} E_{ij}$ and $C^j = \sum_{i=1}^{n} E_{ij}$, where $i, j = 1, 2, \cdots, n$. Suppose T maps a row, say R_1, onto an ith row R_i with scalar multiple B_i and another row, say R_2, onto a jth column C^j with scalar multiple B^j. That is, $T(R_1) = B_i \circ R_i$ and $T(R_2) = B^j \circ C^j$. Then $R_1 + R_2$ has $2n$ cells but $B_i \circ R_i + B^j \circ C^j$ has $2n - 1$ cells. This contradicts to the bijectivity of T' on $\mathcal{E}$. Hence all rows must be mapped some rows (or columns, respectively) with scalar multiple . □

We have the following characterization theorem for zero-term rank preserver on $M_{m,n}(\mathbb{F})$.

Theorem 4.1. Suppose that T is a linear operator on $M_{m,n}(\mathbb{F})$. Then the following statements are equivalent :

(1) T is a (P, Q, B)-operator;

(2) T preserves zero-term rank;

(3) T preserves zero-term ranks 0 and 1.

Proof. (1) $\Rightarrow$ (2): Suppose that T is a (P, Q, B)-operator and the zero-term rank of X is k, that is , $z(X) = k$. Since T is a (P, Q, B)-operator, we have $T(X) = P(B \circ X)Q$ or $m = n$, and $T(X) = P(B \circ X^t)Q$, where P and Q are permutation matrices and B is an $m \times n$ matrix over $\mathbb{F}$, none of whose entries is zero. Hence $z(T(X)) = z(P(B \circ X)Q) = k = z(X)$ or $z(T(X)) = z(P(B \circ X^t)Q) = k = z(X)$. Since k is an arbitrary, we have that T preserves zero-term rank.
(2) $\Rightarrow$ (3): It is clear.
(3) $\Rightarrow$ (1): Suppose that T preserves zero-term ranks 0 and 1. Then T' is a bijection on $\mathcal{E}$ by lemma 4.2. Lemmas 4.4 and 4.5 imply that T maps all rows of a matrix onto rows with scalar multiples or columns onto columns with scalar multiples. Thus, for all $m \times n$ matrix X, $T(X) = P(B \circ X)Q$ or $m = n$, and $T(X) = P(B \circ X^t)Q$ with some permutation matrices P and Q and B is a fixed $m \times n$ matrix over $\mathbb{F}$, none of whose entries is zero. Hence T is a (P, Q, B)-operator. □

Lemma 4.6. For any A, B in $M_{m,n}(\mathbb{F})$, $A \geq B$ implies $T(A) \geq T(B)$.

Proof. By definition of $A \geq B$, we have $a_{ij} \geq b_{ij}$ for all i, j. Using $A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}$ and $B = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} E_{ij}$, we have

$$
\begin{aligned}
T(A) = T(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}) &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} T(E_{ij}) \\
&\geq \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} T(E_{ij}) \\
&= T(\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} E_{ij}) \\
&= T(B)
\end{aligned}
$$

because of linearity and $a_{ij} \geq b_{ij}$. Hence $T(A) \geq T(B)$. □

Definition 4.9. We say that a linear operator T *strongly preserves zero-term rank* k provided that $z(T(A)) = k$ if and only if $z(A) = k$. And a linear operator T *strongly preserves term rank* k provided that $t(T(A)) = k$ if and only if $t(A) = k$.

Lemma 4.7. If T strongly preserves zero-term rank 1, then T preserves zero-term rank 0.

Proof. Suppose that T strongly preserves zero-term rank 1. Since $z(J) \neq 1$, we have $z(T(J)) = 0$ or $z(T(J)) \geq 2$. Suppose $z(T(J)) \geq 2$. Let A be any matrix in $M_{m,n}(\mathbb{F})$. Then $J \geq A$ and so $T(J) \geq T(A)$ by Lemma 4.6. Lemma 4.1 implies $2 \leq z(T(J)) \leq z(T(A))$. Hence $z(T(A)) \geq 2$ for all $A \in M_{m,n}(\mathbb{F})$. For any cell $E_{ij} \in \triangle$, let $A = J - E_{ij}$. Then $z(A) = z(T(J - E_{ij})) = 1$. Since

T strongly preserves zero-term rank 1, $z(T(A)) = z(T(J - E_{ij})) = 1$. This is impossible. Hence $z(T(J)) = 0$. This means that T preserves zero-term rank 0. □

Theorem 4.2. Suppose T is a linear operator on $M_{m,n}(\mathbb{F})$. Then T preserves zero-term rank if and only if it strongly preserves zero-term rank 1.

Proof. Suppose that T strongly preserves zero-term rank 1. Then lemma 4.7 implies that T preserves zero-term rank 0. By theorem 4.1, T preserves zero-term rank.

Conversely, suppose that T preserves zero-term rank. If $z(T(X)) = 1$ and $z(X) \neq 1$, then $z(X) = 0$ or $z(X) \geq 2$. If $z(X) = 0$ (or $z(X) \geq 2$), then $z(T(X)) = 0$ (or $z(T(X)) \geq 2$) by hypothesis. This contradicts to $z(T(X)) = 1$. Hence T strongly preserves zero-term rank 1. □

5. Concluding Remarks

In this thesis, we obtained the results on the linear operator preserving zero-term rank in chapter 4. That is, we had the characterization of linear operators that preserves zero-term rank of fuzzy matrices. It turns out that the linear operator is a (P, Q, B)-operator, which equals term rank preserver. Also, we obtained several kinds of conditions that are equivalent to a (P, Q, B)-operator. We hope that more researches on this topic to generalize our conclusion.

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<국문 초록>

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