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THE CURVATURE OF A REGULAR CURVE UNDER INVERSION

指導教授　玄 進 五

濟州大學校 敎育大學院

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金 順 贇

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指導敎授　玄 進 五

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⟨ Abstract ⟩

THE CURVATURE OF A REGULAR CURVE UNDER INVERSION

Kim, Soon-chan

Mathematics Education Major

Graduate School of Education, Cheju National University

Cheju, Korea

Supervised by professor Hyun, Jin-Oh

In this thesis we derive the formula which is a relation of the curvature of a regular curve under inversion and the one of the given regular curve. We show that if κ and $\bar{\kappa}$ are the curvatures of a unit speed curve α and the inversion curve of α, respectively, then the necessary and sufficient condition for the formula $\bar{\kappa} = \dfrac{\|\alpha(t)\|}{R^2}\kappa$ is that $\|\alpha(t)\| = At + B$ for some constants A and B with $At + B > 0$ for all t. Also, we find all unit speed curves which satisfy the condition $\|\alpha(t)\| = At + B$ with the cases $A = 0$ and $A = 1$ in E^2. Furthermore, we prove that there is no analytic unit speed curve α which satisfies the condition $\|\alpha(t)\| = At (A \neq 0, A \neq 1)$ for all t.

CONTENT

Introduction

In this paper, our study of the curvature will be restrictied to the regular curve in Euclidean space E^3.

In section 1, we introduce the basic concepts of a regular curve in E^3 and the Frenet formulas, a natural instruments which are useful to find the curvature of a regular curve in E^3.

In section 2, we introduce the definition and find some properties of inversion in E^3.

In section 3, we derive the formula which is a relation of the curvature of a regular curve under inversion and the one of the given regular curve. We show that if κ and $\bar{\kappa}$ are the curvatures of a unit speed curve α and the inversion curve of α, repectively, then the necessary and sufficient condition for the formula $\bar{\kappa} = \frac{\|\alpha(t)\|^2}{R^2}\kappa$ is that $\|\alpha(t)\| = At + B$ for some constants A and B with $At + B > 0$ for all t. Furthermore, we prove that if $A = 1$, then α is part of a straight line passing through the orign, and if $A = 0$, then α is part of a circle with center at the origin. Finally, we show that any analytic unit speed curve α does not satisfy the equality $\|\alpha(t)\| = At$.

1. The Curvature of a Regular Curve

In this section, we introduce the basic concepts of the curvature of a regular curve in E^3 and the Frenet formulas, a natural instrument which are useful to find the curvature of a regular curve in E^3.

A map $\alpha : I \longrightarrow E^3$ is called a curve of class C^k if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k.

Definition 1.1. A parametrized differentiable curve $\alpha : I \longrightarrow E^3$ is said to be *regular* if α is of class C^2 and $\dfrac{d\alpha}{dt} \neq 0$ for all $t \in I$.

Definition 1.2. The *unit tangent vector field* to a regular curve $\alpha(t)$ is the vector-valued function

$$T(t) = \frac{\frac{d\alpha}{dt}}{\left\|\frac{d\alpha}{dt}\right\|}, \tag{1.1}$$

where $\left\|\dfrac{d\alpha}{dt}\right\|$ is the length of $\dfrac{d\alpha}{dt}$.

Definition 1.3. The *arc length* of a regular curve α from $t = a$ to $t = b$ is the number

$$\int_a^b \left\| \frac{d\alpha}{dt} \right\| dt. \tag{1.2}$$

Remark. If $\alpha(t)$ is a regular curve and $s = s(t)$ is its arc length, then

$$\begin{aligned} &(a) \quad s = s(t) = \int_0^t \left\| \frac{d\alpha}{dt} \right\| dt; \\ &(b) \quad \frac{ds}{dt} = \left\| \frac{d\alpha}{dt} \right\|. \end{aligned} \tag{1.3}$$

Definition 1.4. A curve $\alpha : I \longrightarrow E^3$ is a *unit speed curve* if $\left\| \frac{d\alpha}{dt} \right\| = 1$.

Let $\alpha : I \longrightarrow E^3$ be a unit speed curve, so $\left\| \frac{d\alpha(s)}{ds} \right\| = 1$ for each s in I. Then $T = \frac{d\alpha}{ds}$ is the unit tangent vector field on α. Since T has constant length 1, its derivative $\frac{dT}{ds} = \frac{d^2\alpha}{ds^2}$ measures the way the curve is turning in E^3. We call $\frac{dT}{ds}$ the *curvature* vector field of α. Differentiation of $\langle T, T \rangle = 1$ gives $2\left\langle \frac{dT}{ds}, T \right\rangle = 0$, so $\frac{dT}{ds}$ is always orthogonal to T, that is, *normal* to α.

The length of the curvature vector field $\frac{dT}{ds}$ gives a numerical measurement of the turning of α. Because of these considerations we are led to make the following definition.

Definition 1.5. Let $\alpha : I \longrightarrow E^3$ be a unit speed curve. Then the real-valued function κ such that

$$\kappa(s) = \left\| \frac{dT(s)}{ds} \right\|, \tag{1.4}$$

for all $s \in I$ is called the *curvature* function of α.

Definition 1.6. Let α be a unit speed curve with $\kappa > 0$, Then the unit-vector field $N = \frac{\frac{dT}{ds}}{\kappa}$ on α tells the *direction* in which α is turning at each point. N is called the *principal normal* vector field of α. The vector field $B = T \times N$ on α is then called the *binormal* vector field of α. The *torsion* of α is the real-valued function $\tau = -\left\langle \frac{dB}{ds}, N \right\rangle$.

Remark. It is clear that $\kappa(s) = 0$ for all s if α is a straight line and $\kappa(s) = \frac{1}{r}$ for all s if α is a circle of radius r.

Lemma 1.7. Let α be a unit speed curve in E^3 with $\kappa > 0$. Then the three vector fields T, N, and B on α are unit vector fields which are mutually orthogonal at each point. We call T, N, B the *Frenet frame field* on α.

Then it is well known the following Frenet formulas.

Proposition 1.8. (Frenet formulas) If $\alpha : I \longrightarrow E^3$ is a unit speed curve with curvature $\kappa > 0$ and tosion τ, then

$$\begin{aligned} \frac{dT}{ds} &= \kappa(s)N(s); \\ \frac{dN}{ds} &= -\kappa(s)T(s) + \tau(s)B(s); \\ \frac{dB}{ds} &= -\tau(s)N(s). \end{aligned} \tag{1.5}$$

Let $\beta(t)$ be a regular curve and let $s(t)$ denote the arc length function. Then $\beta(t) = \alpha(s(t))$, where $\alpha(s)$ is $\beta(t)$ reparametrized by arc length. Note that $\frac{ds}{dt} = \left\| \frac{d\beta}{dt} \right\| > 0$.

Proposition 1.9. If $\beta(t)$ is a regular curve in E^3, then

$$\kappa = \frac{\left\| \frac{d\beta}{dt} \times \frac{d^2\beta}{dt^2} \right\|}{\left\| \frac{d\beta}{dt} \right\|^3}. \tag{1.6}$$

Proof Since $\beta(t) = \alpha(s(t))$ where s is the arc length function of α, we find, using the formulas (1.1) and (1.3), that $\frac{d\beta}{dt} = \frac{d\alpha}{ds}\frac{ds}{dt} = \frac{ds}{dt}T$. From the preceding Proposition 1.8, a second differentiation yields

$$\begin{aligned}\frac{d^2\beta}{dt^2} &= \frac{d^2s}{dt^2}T + \frac{ds}{dt}\frac{dT}{dt} \\ &= \frac{d^2s}{dt^2}T + \left(\frac{ds}{dt}\right)^2\frac{dT}{ds} \\ &= \frac{d^2s}{dt^2}T + \kappa\left(\frac{ds}{dt}\right)^2 N,\end{aligned}$$

and hence

$$\begin{aligned}\frac{d\beta}{dt} \times \frac{d^2\beta}{dt^2} &= \frac{ds}{dt}T \times \left(\frac{d^2s}{dt^2}T + \kappa\left(\frac{ds}{dt}\right)^2 N\right) \\ &= \kappa\left(\frac{ds}{dt}\right)^3 B\end{aligned}$$

since $T \times T = 0$ and $T \times N = B$. Taking norms, we find

$$\kappa\left(\frac{ds}{dt}\right)^3 = \left\|\frac{d\beta}{dt} \times \frac{d^2\beta}{dt^2}\right\|$$

because $\|B\| = 1$, $\kappa \geq 0$, and $\dfrac{ds}{dt} > 0$. Therefore we obtain

$$\kappa = \frac{\left\|\frac{d\beta}{dt} \times \frac{d^2\beta}{dt^2}\right\|}{\left(\frac{ds}{dt}\right)^3} = \frac{\left\|\frac{d\beta}{dt} \times \frac{d^2\beta}{dt^2}\right\|}{\left\|\frac{d\beta}{dt}\right\|^3}.$$

2. Definition and Some Properties of an Inversion

In this section, we define an inversion in E^3 and find some properties of an inversion. Let the sysmbol $(O)_R$ denote the sphere with center O and radius R.

Definition 2.1. Two points P and P' of E^3 are said to be *inverse* with respect to a given sphere $(O)_R$ if

$$OP \cdot OP' = R^2 \tag{2.1}$$

where P, P' are on the same side of O and O, P, P' are collinear.

A sphere $(O)_R$ is called the *sphere of inversion*, and the transformation which sends point P into P' is called an *inversion*. As point P moves on a curve C, its inverse point P' moves on a curve C' which is the inverse curve of C. But the center O of the sphere of inversion has no inverse point because if P is at the center O then $OP = 0$, which means that the relation $OP' = \dfrac{R^2}{OP}$ is meaningless.

From now on, we take the center O as an origin of the coordinate system in E^3, and denote the distance from O to a point $X \in E^3$ by $\|X\|$. Then we have the following properties.

Proposition 2.2. ([8])

(1) A line through O inverts into a line through O.

(2) A line not through O inverts into a circle through O.

(3) A circle through O inverts into a line not through O.

(4) A circle not through O inverts into a circle not through O.

Proposition 2.3. Let $\alpha : (a, b) \longrightarrow E^3$ be a regular curve. Define a mapping $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ by for all $X \in E^3 - \{(0,0,0)\}$

$$f(X) = \frac{R^2 X}{\langle X, X \rangle} = \frac{R^2 X}{\|X\|^2}, \tag{2.2}$$

where $\langle X, X \rangle = X \cdot X$ is the dot product. Then

(1) f is an inversion,

(2) new curve $\bar{\alpha} = f \circ \alpha$ is regular, and

(3) the arc-length $\bar{s}(t)$ of a regular curve segment $\bar{\alpha}$ of α under inversion is given by the formula

$$\bar{s}(t) = R^2 \int_0^t \frac{1}{\|\alpha\|^2} \left\| \frac{d\alpha}{dt} \right\| dt. \tag{2.3}$$

Proof. (3) Since $\alpha(t) \neq 0$ for all $t \in (a, b)$, we have

$$\begin{aligned} \frac{d\bar{\alpha}}{dt} &= \frac{df(\alpha)}{dt} \\ &= \frac{d}{dt}\frac{R^2\alpha}{\|\alpha\|^2} \\ &= \frac{R^2}{\|\alpha\|^2}\frac{d\alpha}{dt} - \frac{2R^2}{\|\alpha\|^4}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \alpha; \end{aligned} \tag{2.4}$$

and so

$$\begin{aligned} \left\|\frac{d\bar{\alpha}}{dt}\right\|^2 &= \left\langle \frac{d\bar{\alpha}}{dt}, \frac{d\bar{\alpha}}{dt} \right\rangle \\ &= \left\langle \frac{R^2}{\|\alpha\|^2}\frac{d\alpha}{dt}, \frac{R^2}{\|\alpha\|^2}\frac{d\alpha}{dt} \right\rangle \\ &= \frac{R^4}{\|\alpha\|^4}\left\langle \frac{d\alpha}{dt}, \frac{d\alpha}{dt} \right\rangle \\ &= \frac{R^4}{\|\alpha\|^4}\left\|\frac{d\alpha}{dt}\right\|^2. \end{aligned} \tag{2.5}$$

By using of (1.3), we get

$$\begin{aligned} \bar{s}(t) &= \int_0^t \left\|\frac{d\bar{\alpha}}{dt}\right\| dt \\ &= R^2 \int_0^t \frac{1}{\|\alpha\|^2} \left\|\frac{d\alpha}{dt}\right\| dt. \end{aligned}$$

3. The Curvature of a Regular Curve under Inversion

In this section, we derive the formula which is a relation of the curvature of a regular curve under inversion and the one of the given regular curve. We show that if κ and $\bar{\kappa}$ are the curvatures of a unit speed curve α and the inversion curve of α, repectively, then the necessary and sufficient condition for the formula $\bar{\kappa} = \dfrac{\|\alpha(t)\|^2}{R^2}\kappa$ is that $\|\alpha(t)\| = At + B$ for some constants A and B with $At + B > 0$ for all t. Furthermore, we prove that if $A = 1$, then α is part of a straight line passing through the orign, and if $A = 0$, then α is part of a circle with center at the origin. Finally, we show that any analytic unit speed curve α does not satisfy the equality $\|\alpha(t)\| = At$.

Lemma 3.1. Let $\alpha : I \longrightarrow E^3$ be a regular curve, and let $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ be an inversion of α. Then, for the new curve $\bar{\alpha} = f(\alpha)$,

$$(1)\quad \begin{aligned} \frac{d^2\bar{\alpha}}{dt^2} =& \frac{R^2}{\|\alpha\|^2}\frac{d^2\alpha}{dt^2} - \frac{4R^2}{\|\alpha\|^4}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \frac{d\alpha}{dt} \\ & - \frac{2R^2}{\|\alpha\|^4}\left(\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\| \frac{d\alpha}{dt} \right\|^2 - \frac{4}{\|\alpha\|^2}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \right)\alpha. \end{aligned} \tag{3.1}$$

(2) $$\left\|\frac{d^2\bar{\alpha}}{dt^2}\right\|^2 = \frac{R^4}{\|\alpha\|^4}\left\|\frac{d^2\alpha}{dt^2}\right\|^2 + \frac{4R^4}{\|\alpha\|^6}\left\|\frac{d\alpha}{dt}\right\|^4 + \frac{4R^4}{\|\alpha\|^6}\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\left\|\frac{d\alpha}{dt}\right\|^2 - \frac{8R^4}{\|\alpha\|^6}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle. \tag{3.2}$$

(3) $$\left\langle\frac{d\bar{\alpha}}{dt},\frac{d^2\bar{\alpha}}{dt^2}\right\rangle = \frac{R^4}{\|\alpha\|^4}\left\langle\frac{d\alpha}{dt},\frac{d^2\alpha}{dt^2}\right\rangle - \frac{2R^4}{\|\alpha\|^6}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle. \tag{3.3}$$

(4) $$\begin{aligned}&\left\|\frac{d\bar{\alpha}}{dt}\times\frac{d^2\bar{\alpha}}{dt^2}\right\|^2\\ &= \frac{R^8}{\|\alpha\|^8}\left\|\frac{d\alpha}{dt}\times\frac{d^2\alpha}{dt^2}\right\|^2 + \frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^6\\ &\quad + \frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^4\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle - \frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle\\ &\quad - \frac{4R^8}{\|\alpha\|^{12}}\left\|\frac{d\alpha}{dt}\right\|^4\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2.\end{aligned} \tag{3.4}$$

Proof. (1) Differentation of (2.4) gives the following ;

$$\begin{aligned}\frac{d^2\bar{\alpha}}{dt^2} &= \frac{R^2\frac{d^2\alpha}{dt^2}\|\alpha\|^2 - 2R^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\frac{d\alpha}{dt}}{\|\alpha\|^4}\\ &\quad - \frac{2R^2\|\alpha\|^4\left[\left(\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle + \left\langle\frac{d\alpha}{dt},\frac{d\alpha}{dt}\right\rangle\right)\alpha + \left\langle\frac{d\alpha}{dt},\alpha\right\rangle\frac{d\alpha}{dt}\right]}{\|\alpha\|^8}\\ &\quad + \frac{8R^2\|\alpha\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\alpha}{\|\alpha\|^8}\end{aligned}$$

$$
\begin{aligned}
&= \frac{R^2}{\|\alpha\|^2}\frac{d^2\alpha}{dt^2} - \frac{2R^2}{\|\alpha\|^4}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \frac{d\alpha}{dt} - \frac{2R^2}{\|\alpha\|^4}\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle \alpha \\
&\quad - \frac{2R^2}{\|\alpha\|^4}\left\| \frac{d\alpha}{dt} \right\|^2 \alpha - \frac{2R^2}{\|\alpha\|^4}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \frac{d\alpha}{dt} + \frac{8R^2}{\|\alpha\|^6}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \alpha \\
&= \frac{R^2}{\|\alpha\|^2}\frac{d^2\alpha}{dt^2} - \frac{4R^2}{\|\alpha\|^4}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \frac{d\alpha}{dt} \\
&\quad - \frac{2R^2}{\|\alpha\|^4}\left(\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\| \frac{d\alpha}{dt} \right\|^2 - \frac{4}{\|\alpha\|^2}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \right) \alpha.
\end{aligned}
$$

(2) From the formula (1), we get

$$
\begin{aligned}
&\left\| \frac{d\bar{\alpha}}{dt} \right\|^2 \\
&= \left\langle \frac{d\bar{\alpha}}{dt}, \frac{d\bar{\alpha}}{dt} \right\rangle \\
&= \frac{R^4}{\|\alpha\|^4}\left\| \frac{d^2\alpha}{dt^2} \right\|^2 + \frac{16R^4}{\|\alpha\|^8}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \left\| \frac{d\alpha}{dt} \right\|^2 \\
&\quad + \frac{4R^4}{\|\alpha\|^8}\left(\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\| \frac{d\alpha}{dt} \right\|^2 - \frac{4}{\|\alpha\|^2}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \right)^2 \|\alpha\|^2 \\
&\quad - \frac{8R^4}{\|\alpha\|^6}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle \\
&\quad + \frac{16R^4}{\|\alpha\|^8}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \left(\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\| \frac{d\alpha}{dt} \right\|^2 - \frac{4}{\|\alpha\|^2}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \right) \\
&\quad - \frac{4R^4}{\|\alpha\|^6}\left(\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\| \frac{d\alpha}{dt} \right\|^2 - \frac{4}{\|\alpha\|^2}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \right) \left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle \\
&= \frac{R^4}{\|\alpha\|^4}\left\| \frac{d^2\alpha}{dt^2} \right\|^2 + \frac{16R^4}{\|\alpha\|^8}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \left\| \frac{d\alpha}{dt} \right\|^2 + \frac{4R^4}{\|\alpha\|^6}\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle^2 \\
&\quad + \frac{4R^4}{\|\alpha\|^6}\left\| \frac{d\alpha}{dt} \right\|^4 + \frac{64R^4}{\|\alpha\|^{10}}\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^4 + \frac{8R^4}{\|\alpha\|^6}\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle \left\| \frac{d\alpha}{dt} \right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{32R^4}{\|\alpha\|^8}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\left\|\frac{d\alpha}{dt}\right\|^2-\frac{32R^4}{\|\alpha\|^8}\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\\
&-\frac{8R^4}{\|\alpha\|^6}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle+\frac{16R^4}{\|\alpha\|^8}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\\
&+\frac{16R^4}{\|\alpha\|^8}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\left\|\frac{d\alpha}{dt}\right\|^2-\frac{64R^4}{\|\alpha\|^{10}}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^4-\frac{4R^4}{\|\alpha\|^6}\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle^2\\
&-\frac{4R^4}{\|\alpha\|^6}\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\left\|\frac{d\alpha}{dt}\right\|^2+\frac{16R^4}{\|\alpha\|^8}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\\
=&\frac{R^4}{\|\alpha\|^4}\left\|\frac{d^2\alpha}{dt^2}\right\|^2+\frac{4R^4}{\|\alpha\|^6}\left\|\frac{d\alpha}{dt}\right\|^4+\frac{4R^4}{\|\alpha\|^6}\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\left\|\frac{d\alpha}{dt}\right\|^2\\
&-\frac{8R^4}{\|\alpha\|^6}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle.
\end{aligned}
$$

(3) Differentiating both sides of (2.5), we have

$$
\begin{aligned}
2\left\langle\frac{d\bar\alpha}{dt},\frac{d^2\bar\alpha}{dt^2}\right\rangle&=\frac{2R^4\left\langle\frac{d\alpha}{dt},\frac{d^2\alpha}{dt^2}\right\rangle\|\alpha\|^4-4R^4\left\|\frac{d\alpha}{dt}\right\|^2\|\alpha\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle}{\|\alpha\|^8}\\
&=\frac{2R^4}{\|\alpha\|^4}\left\langle\frac{d\alpha}{dt},\frac{d^2\alpha}{dt^2}\right\rangle-\frac{4R^4}{\|\alpha\|^6}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle.
\end{aligned}
$$

Hence

$$
\left\langle\frac{d\bar\alpha}{dt},\frac{d^2\bar\alpha}{dt^2}\right\rangle=\frac{R^4}{\|\alpha\|^4}\left\langle\frac{d\alpha}{dt},\frac{d^2\alpha}{dt^2}\right\rangle-\frac{2R^4}{\|\alpha\|^6}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle.
$$

(4) From the formulas (2.5), (3.2), and (3.3), we obtain

$$
\begin{aligned}
&\left\|\frac{d\bar\alpha}{dt}\times\frac{d^2\bar\alpha}{dt^2}\right\|^2\\
=&\left\|\frac{d\bar\alpha}{dt}\right\|^2\left\|\frac{d^2\bar\alpha}{dt^2}\right\|^2-\left\langle\frac{d\bar\alpha}{dt},\frac{d^2\bar\alpha}{dt^2}\right\rangle^2
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{R^8}{\|\alpha\|^8}\left\|\frac{d\alpha}{dt}\right\|^2\left\|\frac{d^2\alpha}{dt^2}\right\|^2+\frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^6+\frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^4\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle\\
&-\frac{8R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle-\frac{R^8}{\|\alpha\|^8}\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle^2\\
&+\frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle\left\langle\frac{d\alpha}{dt},\alpha\right\rangle-\frac{4R^8}{\|\alpha\|^{12}}\left\|\frac{d\alpha}{dt}\right\|^4\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\\
=&\frac{R^8}{\|\alpha\|^8}\left\|\frac{d\alpha}{dt}\right\|^2\left\|\frac{d^2\alpha}{dt^2}\right\|^2-\frac{R^8}{\|\alpha\|^8}\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle^2+\frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^6\\
&+\frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^4\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle-\frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle\\
&-\frac{4R^8}{\|\alpha\|^{12}}\left\|\frac{d\alpha}{dt}\right\|^4\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2.\\
=&\frac{R^8}{\|\alpha\|^8}\left\|\frac{d\alpha}{dt}\times\frac{d^2\alpha}{dt^2}\right\|^2+\frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^6\\
&+\frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^4\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle-\frac{4R^8}{\|\alpha\|^{10}}\left\|\frac{d\alpha}{dt}\right\|^2\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle\\
&-\frac{4R^8}{\|\alpha\|^{12}}\left\|\frac{d\alpha}{dt}\right\|^4\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2.
\end{aligned}
$$

Theorem 3.2. Let $\alpha : I \longrightarrow E^3$ be a regular curve with curvature κ, and let $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ be an inversion of α. Then the curvature $\bar{\kappa}$ of $\bar{\alpha} = f(\alpha)$ under inversion is computed by the following formula

$$
\begin{aligned}
\bar{\kappa}^2 =&\frac{\|\alpha\|^4}{R^4}\kappa^2+\frac{4\|\alpha\|^2}{R^4}+\frac{4}{R^4\|\frac{d\alpha}{dt}\|^2}\left(\|\alpha\|^2\left\langle\frac{d^2\alpha}{dt^2},\alpha\right\rangle-\left\langle\frac{d\alpha}{dt},\alpha\right\rangle^2\right)\\
&-\frac{4\|\alpha\|^2}{R^4\|\frac{d\alpha}{dt}\|^4}\left\langle\frac{d\alpha}{dt},\alpha\right\rangle\left\langle\frac{d^2\alpha}{dt^2},\frac{d\alpha}{dt}\right\rangle.
\end{aligned}
\tag{3.5}
$$

Proof. By using of the formulas (1.6), (2.5), and Lemma 3.1, we have

$$\begin{aligned}
\bar{\kappa}^2 &= \frac{\left\| \frac{d\bar{\alpha}}{dt} \times \frac{d^2\bar{\alpha}}{dt^2} \right\|^2}{\left\| \frac{d\bar{\alpha}}{dt} \right\|^6} \\
&= \frac{\frac{R^8}{\|\alpha\|^8} \left\| \frac{d\alpha}{dt} \times \frac{d^2\alpha}{dt^2} \right\|^2}{\frac{R^{12}}{\|\alpha\|^{12}} \left\| \frac{d\alpha}{dt} \right\|^6} + \frac{\frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^6 + \frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^4 \left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle}{\frac{R^{12}}{\|\alpha\|^{12}} \left\| \frac{d\alpha}{dt} \right\|^6} \\
&\quad - \frac{\frac{4R^8}{\|\alpha\|^{12}} \left\| \frac{d\alpha}{dt} \right\|^4 \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 + \frac{4R^8}{\|\alpha\|^{10}} \left\| \frac{d\alpha}{dt} \right\|^2 \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle}{\frac{R^{12}}{\|\alpha\|^{12}} \left\| \frac{d\alpha}{dt} \right\|^6} \\
&= \frac{\|\alpha\|^4}{R^4} \frac{\left\| \frac{d\alpha}{dt} \times \frac{d^2\alpha}{dt^2} \right\|^2}{\left\| \frac{d\alpha}{dt} \right\|^6} + \frac{4\|\alpha\|^2}{R^4} + \frac{4\|\alpha\|^2 \left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle}{R^4 \left\| \frac{d\alpha}{dt} \right\|^2} - \frac{4\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2}{R^4 \left\| \frac{d\alpha}{dt} \right\|^2} \\
&\quad - \frac{4\|\alpha\|^2 \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle}{R^4 \left\| \frac{d\alpha}{dt} \right\|^4} \\
&= \frac{\|\alpha\|^4}{R^4}\kappa^2 + \frac{4\|\alpha\|^2}{R^4} + \frac{4}{R^4 \left\| \frac{d\alpha}{dt} \right\|^2} \left(\|\alpha\|^2 \left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle - \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 \right) \\
&\quad - \frac{4\|\alpha\|^2}{R^4 \left\| \frac{d\alpha}{dt} \right\|^4} \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle.
\end{aligned}$$

Corollary 3.3. Let α be a unit speed curve with curvature κ. Then the curvature $\bar{\kappa}$ of $\bar{\alpha}$ under inversion is computed by the following;

$$\bar{\kappa}^2 = \frac{\|\alpha\|^4}{R^4}\kappa^2 + \frac{4\|\alpha\|^2}{R^4} + \frac{4}{R^4}\|\alpha\|^2 \left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle - \frac{4}{R^4} \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2. \tag{3.6}$$

Proof. Let α be a unit speed curve. Then $\left\| \frac{d\alpha}{dt} \right\| = 1$; so $\left\| \frac{d\alpha}{dt} \right\|^2 = 1$. Hence $\left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle = 0$ by differentiation of $\left\| \frac{d\alpha}{dt} \right\|^2 = 1$. From the formula (3.5), we get the formula (3.6).

Theorem 3.4. Let $\alpha : (a, b) \longrightarrow E^3$ be a unit speed curve with curvature κ and let $f : E^3 - \{(0,0,0)\} \longrightarrow E^3$ be an inversion. Also, let $\bar{\kappa}$ be the curvature of $\bar{\alpha} = f \circ \alpha$. Then, for any $t \in (a, b)$,

$$\bar{\kappa} = \frac{\|\alpha(t)\|^2}{R^2}\kappa \qquad \text{if and only if} \qquad \|\alpha(t)\| = At + B$$

for some constants A, B with $At + B > 0$ for all t.

Proof. Let $\bar{\kappa} = \dfrac{\|\alpha\|^2}{R^2}\kappa$. Then, by Corollary 3.3,

$$\|\alpha\|^2 + \|\alpha\|^2 \left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle - \left\langle \frac{d\alpha}{dt}, \alpha \right\rangle^2 = 0. \tag{3.7}$$

Put $g(t) = \langle \alpha(t), \alpha(t) \rangle$. Then g is a differentiable real-valued function and $g(t) > 0$ for all t. Differentiating both sides of the formula

$$\langle \alpha(t), \alpha(t) \rangle = g(t),$$

we have

$$\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle = \frac{1}{2}g', \tag{3.8}$$

where g' denotes the derivative of g with respect to t. Since α is a unit speed curve, differentiating both sides of (3.8), we have

$$\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle = \frac{1}{2}g'' - 1. \tag{3.9}$$

Substituting the formulas (3.8) and (3.9) to the formula (3.7), we get the differential equation

$$2gg'' - (g')^2 = 0. \tag{3.10}$$

Case 1 : If $g' = 0$, then there exists a positive constant B such that $g(t) = B$ since $g(t) > 0$ for all t.

Case 2 : If $g' \neq 0$, then, from the formula (3.10),

$$2\frac{g''}{g'} = \frac{g'}{g}.$$

Hence

$$(2\ln|g'|)' = (\ln|g|)';$$

and so

$$\ln(g')^2 = \ln C_1 g,$$

where C_1 is a positive constant. Therefore we obtain

$$(g')^2 = C_1 g.$$

Simplifying this equation, we get

$$\frac{g'}{\sqrt{g}} = \pm\sqrt{C_1}.$$

By integrating both sides of this equation, we obtain

$$\sqrt{g} = \pm\frac{\sqrt{C_1}}{2}t + \frac{C_2}{2},$$

where C_2 is a constant. To get $\|\alpha(t)\| = \sqrt{g(t)} = At + B$, we choose $\pm\frac{\sqrt{C_1}}{2} = A$ and $\frac{C_2}{2} = B$ which are satisfied the inequality $At + B > 0$ for all t. Then we are done.

Conversely, let $\|\alpha(t)\| = At + B$; so $\|\alpha(t)\|^2 = (At + B)^2$. Then, by differentiating both sides of the above equation, we get

$$\left\langle \frac{d\alpha}{dt}, \alpha \right\rangle = A(At + B).$$

By differentiating both sides of the above equation, we have

$$\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle + \left\langle \frac{d\alpha}{dt}, \frac{d\alpha}{dt} \right\rangle = A^2.$$

Since $\left\| \frac{d\alpha}{dt} \right\| = 1$, we have

$$\left\langle \frac{d^2\alpha}{dt^2}, \alpha \right\rangle = A^2 - 1.$$

By Corollary 3.3, we obtain

$$\begin{aligned}\bar{\kappa}^2 &= \frac{\|\alpha\|^4}{R^4}\kappa^2 + \frac{4\|\alpha\|^2}{R^4} + \frac{4}{R^4}\|\alpha\|^2(A^2 - 1) - \frac{4}{R^4}A^2\|\alpha\|^2 \\ &= \frac{\|\alpha\|^4}{R^4}\kappa^2.\end{aligned}$$

Hence our proof is completed.

Proposition 3.5. Let $\alpha(t) = (x(t), y(t))$ be a unit speed curve with $\alpha(t) \neq 0$ for all $t > 0$.

(1) If $\|\alpha(t)\| = B$ for any positive constant B, then α is part of a circle with center at the origin.

(2) If $\|\alpha(t)\| = t + B$ for any constant $B \geq 0$, then α is part of a straight line passing through the origin.

Proof. Let $\alpha(t) = (x(t), y(t))$ be a unit speed curve with $\alpha(t) \neq 0$ for all $t > 0$. Then

$$(x')^2 + (y')^2 = 1, \tag{3.11}$$

where x' and y' are derivatives of x and y with respect to t, repectively.

(1) If $\|\alpha(t)\| = B(B$ is a positive constant), then it is clear that the curve $\alpha(t)$ is part of a circle with center the origin and radius B.

(2) If $\|\alpha(t)\| = t + B$ for any constant $B \geq 0$, then we have

$$x^2 + y^2 = (t + B)^2. \tag{3.12}$$

Differentiation of (3.12) gives

$$xx' + yy' = t + B. \tag{3.13}$$

Case 1: Either $x(t) = 0$ or $y(t) = 0$.

It is trivial from the formula (3.12).

Case 2: If $x(t) \neq 0$ and $y(t) \neq 0$, then, from the formula (3.13), we get

$$y' = -\frac{x}{y}x' + \frac{t+B}{y}.$$

Substituting this formula to (3.11), we have

$$(x')^2 + \left(-\frac{x}{y}x' + \frac{t+B}{y}\right)^2 = 1.$$

Simplyfying this equation, we get

$$(t+B)^2(x')^2 - 2x(t+B)x' + x^2 = 0.$$

Equivalently,

$$\{(t+B)x' - x\}^2 = 0.$$

Hence we have

$$\frac{x'}{x} = \frac{1}{t+B}.$$

By integrating both sides of this equation, we obtain

$$\ln|x(t)| = \ln C|t+B|,$$

where C is a positive constant, and hence

$$x(t) = C|t+B|.$$

From (3.12), we have

$$y(t) = \sqrt{1-C^2}|t+B|$$

where $0 < C < 1$. Therefore we get

$$y = \frac{\sqrt{1-C^2}}{C}x,$$

and hence the curve α is part of a straight line passing through the origin.

Definition 3.6. A function which is represented by power series

$$x(t) = \sum_{n=0}^{\infty} a_n t^n,$$

where a_n's are real numbers, in some open interval containg $t = 0$, is called analytic. The curve $\alpha(t)$ is called analytic if each component function is analytic.

Theorem 3.7. Given number A with $A \neq 1$ and $A \neq 0$, there is no analytic unit speed curve

$$\alpha(t) = (x(t), y(t))$$

locally at $t = 0$ such that $\|\alpha(t)\| = At$.

Proof. Suppose that there is an analytic unit speed curve

$$\alpha(t) = (x(t), y(t))$$

locally at $t = 0$ such that $\|\alpha(t)\| = At$. Without loss of generality, we assume that $y(t) \neq 0$ locally at $t = 0$. Then we have

$$x^2 + y^2 = A^2t^2, \tag{3.14}$$

and

$$(x')^2 + (y')^2 = 1, \tag{3.15}$$

where x' and y' are derivatives of $x(t)$ and $y(t)$ with respect to t, respectively. Differentiating both sides of the equation (3.14), we have

$$x(t)x' + y(t)y' = A^2t.$$

Since $y(t) \neq 0$ locally at $t = 0$, we get

$$y' = -\frac{x}{y}x' + \frac{A^2t}{y}.$$

Substituting y' to the equation (3.15), we have

$$(x')^2 + \left(-\frac{x}{y}x' + \frac{A^2t}{y}\right)^2 = 1,$$

and then

$$A^2t^2\left(x'\right)^2 - 2A^2txx' + A^2t^2\left(A^2-1\right) + x^2 = 0.$$

Hence, finally, we have the following equation :

$$A^2\left(x'\right)^2t^2 - 2A^2\left(xx'\right)t + x^2 + A^2\left(A^2-1\right)t^2 = 0. \tag{3.16}$$

Now, we assume that

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then x satisfies the differential equation (3.16).

To find the coefficients a_n's, we differentiate $x(t)$ with respect to t, and then we have

$$x'(t) = \sum_{n=1}^{\infty} na_n t^{n-1}.$$

Hence we obtain

$$\begin{aligned}\left(x'(t)\right)^2 &= \left(\sum_{n=1}^{\infty} na_n t^{n-1}\right)\left(\sum_{n=1}^{\infty} na_n t^{n-1}\right)\\ &= \sum_{n=0}^{\infty}\left(P_n t^{2n} + Q_n t^{2n+1}\right),\end{aligned}$$

where

$$P_n = \left(\sum_{k=0}^{n-1} 2(n-k)(n+k+2)a_{n-k}a_{n+k+2}\right) + (n+1)^2 a_{n+1}^2,$$

and

$$Q_n = 2\sum_{k=0}^{n}(n-k+1)(n+k+2)a_{n-k+1}a_{n+k+2}.$$

Now,

$$\begin{aligned} x(t)x'(t) &= \left(\sum_{n=0}^{\infty} a_n t^n\right)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) \\ &= \sum_{n=0}^{\infty}\left(R_n t^{2n} + S_n t^{2n+1}\right), \end{aligned}$$

where

$$R_n = \sum_{k=0}^{n}(2n+1)a_{n-k}a_{n+k+1},$$

and

$$S_n = \left(\sum_{k=0}^{n} 2(n+1)a_{n-k}a_{n+k+2}\right) + (n+1)a_{n+1}^2.$$

Also,

$$\begin{aligned} (x(t))^2 &= \left(\sum_{n=0}^{\infty} a_n t^n\right)\left(\sum_{n=0}^{\infty} n a_n t^n\right) \\ &= \sum_{n=0}^{\infty}\left(F_n t^{2n} + G_n t^{2n+1}\right), \end{aligned}$$

where

$$F_n = \left(\sum_{k=0}^{n-1} 2a_{n-k-1}a_{n+k+1}\right) + a_n^2,$$

and

$$G_n = 2\sum_{k=0}^{n} a_{n-k}a_{n+k+1}.$$

By substituting $(x'(t))^2$, $x(t)x'(t)$, and $(x(t))^2$ to the equation (3.16), we have

$$A^2\left(\sum_{n=0}^{\infty}\left(P_n t^{2n}+Q_n t^{2n+1}\right)\right)t^2-2A^2\left(\sum_{n=0}^{\infty}\left(R_n t^{2n}+S_n t^{2n+1}\right)\right)t$$
$$+\left(\sum_{n=0}^{\infty}\left(F_n t^{2n}+G_n t^{2n+1}\right)\right)+A^2(A^2-1)t^2=0,$$

and then

$$A^2\left(\sum_{n=0}^{\infty}Q_n t^{2n+3}\right)+A^2\left(\sum_{n=0}^{\infty}(P_n-2S_n)\,t^{2n+2}\right)$$
$$+\left(\sum_{n=0}^{\infty}\left(G_n-2A^2R_n\right)t^{2n+1}\right)+\left(\sum_{n=0}^{\infty}F_n t^{2n}\right)+A^2\left(A^2-1\right)t^2=0.$$

Gathering terms according to the powers of t and solving each term of t^n, we have the following recursive formulas :

$$F_0=0,$$

$$G_0-2A^2R_0=0,$$

$$A^2P_0-1A^2S_0+A^2(A^2-1)+F_1=0,$$

$$A^2+G_1-2A^2R_1=0,$$

$$A^2P_1-2A^2S_1+F_2=0,$$

$$A^2Q_1+G_2-2A^2R_2=0,$$

$$A^2P_2-2A^2S_2+F_3=0,$$

$$A^2Q_2+G_3-2A^2R_3=0,$$

$$A^2P_3-2A^2S_3+F_4=0,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$A^2 Q_{n-1} + G_n - 2A^2 R_n = 0,$$

$$A^2 P_n - 2A^2 S_n + F_{n+1} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots$$

By a simple calculation, we get

$$a_0^2 = 0 \quad ; \quad a_0 = 0,$$

$$(1 - A^2) a_0 a_1 = 0,$$

$$(1 - A^2)(a_1^2 - A^2) = 0 \quad ; \quad a_1^2 = A^2,$$

$$2(1 - A^2) a_1 a_2 = 0 \quad ; \quad a_2 = 0,$$

$$2(1 - A^2) a_1 a_3 = 0 \quad ; \quad a_3 = 0,$$

$$2(1 - A^2) a_1 a_4 = 0 \quad ; \quad a_4 = 0,$$

$$2(1 - A^2) a_1 a_5 = 0 \quad ; \quad a_5 = 0,$$

$$2(1 - A^2) a_1 a_6 = 0 \quad ; \quad a_6 = 0,$$

$$2(1 - A^2) a_1 a_7 = 0 \quad ; \quad a_7 = 0,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$2(1 - A^2) a_1 a_{2n} = 0 \quad ; \quad a_{2n} = 0,$$

$$2(1 - A^2) a_1 a_{2n+1} = 0 \quad ; \quad a_{2n+1} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots$$

Thus we have

$$x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots$$

$$= At.$$

From the equation (3.14), we have $y(t) = 0$.

This leads to a contradiction.

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〈 초　　록 〉

전위에 의한 정칙 곡선의 곡률

김　　순　　찬

제주대학교　교육대학원　수학교육전공

지도교수　　현　　진　　오

이 논문은 Euclid 공간 E^3에서 주어진 정칙곡선의 곡률 κ를 이용하여 전위에 의한 정칙곡선의 곡률 $\bar{\kappa}$를 구하는 관계식을 유도한다. 단위속도곡선 α와 α의 정칙곡선 $\bar{\alpha}$의 곡률을 각각 κ, $\bar{\kappa}$라고 할 때, $\bar{\kappa} = \frac{\|\alpha(t)\|}{R^2}\kappa$ 일 필요충분조건은 $\|\alpha(t)\| = At + B$(단, A와 B는 모든 t에 대하여 $At + B > 0$을 만족하는 상수)임을 보인다. 또한, E^2 상에서 $A = 0$와 $A = 1$인 경우에 조건 $\|\alpha(t)\| = At + B$를 만족하는 단위속도곡선이 존재함을 밝힌다. 더우기 모든 t에 대하여 조건 $\|\alpha(t)\| = At$ $(A \neq 0, A \neq 1)$을 만족하는 해석적 단위곡선이 존재하지 않음을 보인다.

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감사의 글

본 논문이 완성되기까지 건강이 여의치 않으심에도 불구하고 지도해 주신 현진오 교수님과 특히 연구에 바쁘신 가운데에도 항상 세심한 검토와 조언을 아끼지 않으신 고봉수 교수님, 정승달 교수님 그리고 무지한 저에게 많은 가르침과 격려를 해 주신 수학교육과와 수학과의 모든 교수님께 깊은 감사를 드리며, 함께 강의를 받고 의지하며 협조를 아끼지 않은 김정두 선생님, 양창길 선생님, 고영종 선생님, 김영희 선생님께도 고마운 마음을 전하고 싶습니다.

그리고, 학교의 일과 진행의 어려움 속에서도 석사과정을 무사히 마칠 수 있도록 배려를 해 주신 교장선생님을 비롯한 여러 선생님께 감사 드리며, 주위에서 많은 격려와 용기를 주신 모든 분들께도 감사를 드립니다.

끝으로, 긴 세월동안 언제나 자식 잘 되기를 바라시며 한 평생을 무한한 사랑과 희생으로 살아오신 어머님, 여러 가지 어려운 여건 속에서도 한 번 내색하지 않고 인내와 사랑으로 묵묵히 도와준 소중한 아내, 채정애씨, 하루 하루 건강하고 씩씩하게 자라나는 민성, 민규와 함께 작은 기쁨을 나누고자 합니다.

1996년 7월

김 순 찬 드림