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Spectra of Weighted Shift Operators

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Spectra of Weighted Shift Operators

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<국문초록>

가중 추이 작용소의 여러가지 스펙트럼

본 논문에서는 먼저 힐버트 공간 $l^2(\mathbb{C})$의 일반화 공간인 $l^2_w(\mathbb{C})$와 $l^2_w(H)$ 공간에서 밀림(추이)작용소 (shift operator)의 여러가지 스펙트럼과 성질들을 조사하였다. 여기서 $w = (w_j)$, $w_j > 0$ 이고 H는 힐버트 공간을 나타낸다.

첫째로, $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$ (단, $q > 0$)일 때 좌밀림 작용소의 스펙트럼은 반지름이 $\sqrt{q}$인 폐원판이 됨을 보이고, 그 밖에 근사 점스펙트럼(approximate point spectrum), 점 스펙트럼(point spectrum), 잉여 스펙트럼(residual spectrum), 연속 스펙트럼(continuous spectrum) 그리고 압축 스펙트럼(compression spectrum) 등을 구하였다. 마찬가지로 우밀림 작용소의 여러 가지 스펙트럼에 대해서도 연구하였다.

둘째로, $w = (w_j)$ 가 양의 유계수열일 때 좌 · 우밀림 작용소의 스펙트럼은 반지름이 1인 폐원판임을 보였고, 그 외에 여러가지 스펙트럼을 구하였다.

세째로, $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$ (단, $q > 0$)일 때 $l^2_w(H)$에서 $T = (A_n)$이 $e^{i\theta}T$와 유니타리 동치(unitarily equivalent)임을 보였다.

마지막으로, $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$ (단, $q > 0$) 이고 S는 작용소들의 유계집합 $\{A_n\}$을 대각성분으로 하는 대각 작용소라 하자. 작용소 S가 컴팩트(compact) 작용소되기 위한 필요충분조건은 A_n이 컴팩트 작용소이고 $\lim_{n\to\infty} \|A_n\| = 0$임을 밝혔다.

CONTENTS

1. Introduction

Let $w = (w_0, w_1, \cdots)$ where $w_j > 0$ and $l^2_w(\mathbb{C})$ be the set of all sequences $x = (x_0, x_1, \cdots)$ of complex numbers such that $\sum_{j=0}^{\infty} w_j |x_j|^2 < \infty$. Define an inner product of vectors $x = (x_n)$ and $y = (y_n)$ by $< x,\ y > = \sum_{j=0}^{\infty} w_j x_j \overline{y_j}$. Then we show that $l^2_w(\mathbb{C})$ becomes a Hilbert space(Theorem 3.3). From this result, $l^2(\mathbb{C})$ becomes a Hilbert space. Similarly, let H be a Hilbet space and $l^2_w(H)$ be the set of all sequences $(x_n)_{n=0}^{\infty}$ in H such that $\sum_{n=0}^{\infty} w_n \|x_n\|^2 < \infty$. Then we show that $l^2_w(H)$ becomes a Hilbert space (Theorem 5.1).

In this thesis, we will study various spectra (spectrum, point spectrum, approximate point spectrum, compression spectrum, residual spectrum and continuous spectrum) of shift operators on the spaces $l^2(\mathbb{C})$, $l^2_w(\mathbb{C})$ and $l^2(H)$, respectively and various spectra of weighted shift operators on the space $l^2_w(\mathbb{C})$. Also we will study various spectra (spectrum, point spectrum, approximate point spectrum, compression spectrum, residual spectrum and continuous spectrum) of diagonal operators on the spaces $l^2(\mathbb{C})$ and $l^2_w(\mathbb{C})$, respectively.

The organization of this thesis is as follows. In section 1, we look about the basic properties of various spectra (spectrum, point spectrum, approximate point spectrum, compression spectrum, residual spectrum and continuous spectrum) of a linear bounded operator on a Hilbert space H and relations among them. Also we introduce spectral mapping theorem of spectrum of an operator A in $B(H)$, where $B(H)$ denotes the space of all bounded linear operators on H.

In section 2, we deal with the various spectra of shift operators on $l^2_w(\mathbb{C})$.

First we prove that $l^2_w(\mathbb{C})$ becomes a Hilbert space. According to $w = (w_0, w_1, \cdots)$, we classify the space $l^2_w(\mathbb{C})$ into case 1 and case 2 and then calculate the various spectra of weighted shift operators. From these results, we can get the spectra of shift operators (right shift and left shift) on $l^2(\mathbb{C})$.

In case 1, the condition of $w = (w_0, w_1, \cdots)$ is $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$ where $q > 0$. Then the norm and the various spectra of shift operator on this space are calculated.

In case 2, the space is more generalized. When $w_0, w_1, \cdots$ are positive, increasing and bounded, the various spectra are calculated. In particular, we are interested in the various spectra of weighted shift operator on $l^2_w(\mathbb{C})$.

In section 3, we calculate the spectra of a diagonal operator in $l^2(\mathbb{C})$ and $l^2_w(\mathbb{C})$. In $l^2_w(\mathbb{C})$, we let $w = (w_0, w_1, \cdots) = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots), (q > 1)$.

In section 4, we show that $l^2_w(H)$ becomes a Hilbert space. In particular, we show that a uniformly bounded sequence $T = (A_n)$ of operators in $l^2_w(H)$ is unitarily equivalent to $\widetilde{T} = e^{i\theta}T$ when $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$, $(q > 0)$. And we give a generalized proposition of N. Faour and R. Khalil ([3]). That is, for a diagonal operator S with diagonal $\{A_n\}$ in $l^2_w(H)$, S is compact if and only if A_n is compact and $\lim \|A_n\| = 0$.

2. Basic Properties of Spectra

Let H be a Hilbert space and let $B(H)$ be the set of all bounded linear operators on H. For any $A \in B(H)$, the *spectrum* $\sigma(A)$ of A is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not invertible. $\lambda \in \sigma_p(A)$ iff there exists a unit vector x such that $Ax = \lambda x$, i.e., $\sigma_p(A)$ is the set of all eigenvalues of A. $\sigma_p(A)$ is called the *point spectrum* of A. $\sigma_{ap}(A)$ is the set of complex numbers λ such that $A - \lambda$ is not bounded below. The set $\sigma_{ap}(A)$ is called the *approximate point spectrum* of A. Equivalently, a number λ belongs to $\sigma_{ap}(A)$ if and only if there exists a sequence $\{x_n\}$ of unit vectors such that $\|(A-\lambda)x_n\| \to 0$. The *compression spectrum* of A, denoted by $\sigma_{com}(A)$, is the set of complex numbers λ such that the closure of the range of $A - \lambda$ is a proper subspace of H, i.e., $\sigma_{com}(A) = \{\lambda \in \mathbb{C} : \overline{R(A-\lambda)} \neq H\}$. Thus

$$\begin{aligned}\sigma_{com}(A) &= \{\lambda \in \mathbb{C} : A - \lambda I \quad \text{is a right divisor of zero in } B(H)\} \\ &= \{\lambda \in \mathbb{C} : (A - \lambda I)(H) \quad \text{is not dense in } H\}.\end{aligned}$$

The set of all complex numbers λ such that $A - \lambda I$ is injective but its range is not dense in a Hilbert space H is called the *residual spectrum* of A and denoted by $\sigma_r(A)$. Thus $\sigma_r(A) = \sigma_{com}(A) - \sigma_p(A)$. The *continuous spectrum* of A, denoted by $\sigma_c(A)$, is the set of all complex numbers λ such that $A - \lambda I$ is injective, has dense range, but is singular. Thus $\sigma_c(A) = \sigma(A) - (\sigma_p(A) \cup \sigma_{com}(A))$.

It is obvious from the definitions that $\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A)$, where the terms on the right are mutually disjoint.

Lemma 2.1. ([1]) $\sigma(A) = \sigma_{ap}(A) \cup \sigma_{com}(A)$.

Lemma 2.2. ([1]) *If $A \in B(H)$ is an operator, then $\sigma_p(A^*) = \sigma_{com}(A)^*$ and $\sigma(A^*) = \sigma_{ap}(A^*) \cup \sigma_p(A)^*$.*

Corollary 2.3. $\sigma_p(A) = \sigma_{com}(A^*)^*$ *and* $\sigma_{ap}(A) \cup \sigma_p(A^*)^* = \sigma(A)$.

Theorem 2.4. ([1], [2]) *For each operator $A \in B(H)$, the approximate point spectrum $\sigma_{ap}(A)$ is closed and $\partial\sigma(A) \subset \sigma_{ap}(A)$, where $\partial\sigma(A)$ is a boundary of $\sigma(A)$.*

Theorem 2.5. ([1], [2]) *If A is a normal operator, then $\sigma_{com}(A) = \sigma_p(A)$ and therefore $\sigma(A) = \sigma_{ap}(A)$.*

Theorem 2.6. ([2]) *Let $T \in B(H)$ be any operator. The following conditions are equivalent.*

(1) $\lambda \notin \sigma_{ap}(T)$.
(2) $R(T - \lambda)$ *is closed and* $\dim\ker(T - \lambda) = 0$.
(3) $\lambda \notin \sigma_l(T)$, *the left spectrum of T.*
(4) $\bar{\lambda} \notin \sigma_r(T^*)$, *the right spectrum of T^*.*
(5) $R(T^* - \bar{\lambda}) = H$.

From Theorem 2.6, we can know $\sigma_{ap}(T) = \sigma_l(T) = \sigma_r(T^*)^*$.

Theorem 2.7. ([5]) *If A is an operator and p is a polynomial, then $\sigma_p(p(A)) = p(\sigma_p(A)), \sigma_{ap}(p(A)) = p(\sigma_{ap}(A))$ and $\sigma_{com}(p(A)) = p(\sigma_{com}(A))$. The same equations are true if A is an invertible operator and $p(z) = \frac{1}{z}$ for $z \neq 0$.*

3. Shift Operators on $l^2_w(\mathbb{C})$

Define $l^2(\mathbb{C})$ to be the set of all sequences $x = (x_0, x_1, \cdots)$ of complex numbers such that $\sum_{j=0}^{\infty} |x_j|^2 < \infty$. Then $l^2(\mathbb{C})$ is a vector space over $\mathbb{C}$. For any $x = (x_0, x_1, \cdots) \in l^2(\mathbb{C})$, define a norm $\|\cdot\|$ on $l^2(\mathbb{C})$ by $\|x\| = (\sum_{j=0}^{\infty} |x_j|^2)^{1/2}$. Then $l^2(\mathbb{C})$ becomes a normed space. Define an inner product of vectors $x = (x_n)$ and $y = (y_n)$ by $< x,\ y >= \sum_{j=0}^{\infty} x_j \overline{y_j}$. Then $l^2(\mathbb{C})$ becomes a Hilbert space([2], [3]).

Definition 3.1. ([2], [3]) *Define $S_r : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ by $S_r = (x_0, x_1, \cdots) = (0, x_0, x_1, \cdots)$. The operator S_r is called a right shift operator (or unilateral shift).*

Obviously S_r is linear and $\|S_r x\| = \|x\|,\ x \in l^2(\mathbb{C})$. Thus S_r is an isometry of $l^2(\mathbb{C})$ into $l^2(\mathbb{C})$ and $\|S_r\| = 1$. Also S_r maps $l^2(\mathbb{C})$ onto a proper subspace, i.e., S_r is not surjective. Therefore S_r is not invertible.

Definition 3.2. ([2], [3]) *Define $S_l : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ by $S_l(x_0, x_1, \cdots) = (x_1, x_2, \cdots)$. The operator S_l is called a left shift operator (or backward shift).*

Obviously S_l is linear and $\|S_l\| = 1$, but S_l is not one-to-one. Hence S_l is not invertible.

Let $w = (w_0, w_1, \cdots)$, where $w_j > 0$. Define $l^2_w(\mathbb{C})$ to be the set of all sequences $x = (x_0, x_1, \cdots)$ of complex numbers such that $\sum_{j=0}^{\infty} w_j |x_j|^2 < \infty$. Then $l^2_w(\mathbb{C})$ is a vector space over $\mathbb{C}$. For any $x = (\xi_0, \xi_1, \cdots), y = (\eta_0, \eta_1, \cdots) \in l^2_w(\mathbb{C})$,

(1) $x+y=(\xi_0+\eta_0,\xi_1+\eta_1,\cdots)\in l^2_w(\mathbb{C})$. For

$$\sum_{j=0}^{\infty} w_j|\xi_j+\eta_j|^2 \le \sum_{j=0}^{\infty} w_j(|\xi_j|+|\eta_j|)^2 = \sum_{j=0}^{\infty} w_j(|\xi_j|^2+2|\xi_j||\eta_j|+|\eta_j|^2)$$
$$= \sum_{j=0}^{\infty} w_j|\xi_j|^2 + 2\sum_{j=0}^{\infty} w_j|\xi_j||\eta_j| + \sum_{j=0}^{\infty} w_j|\eta_j|^2 < \infty,$$

which follows from the fact that

$$\sum_{j=0}^{\infty} w_j|\xi_j||\eta_j| = \sum_{j=0}^{\infty}(\sqrt{w_j}|\xi_j|)(\sqrt{w_j}|\eta_j|)$$
$$\le [\sum_{j=0}^{\infty}(\sqrt{w_j}|\xi_j|)^2]^{\frac{1}{2}}[\sum_{j=0}^{\infty}(\sqrt{w_j}|\eta_j|)^2]^{\frac{1}{2}}$$
$$= [\sum_{j=0}^{\infty} w_j|\xi_j|^2]^{\frac{1}{2}}[\sum_{j=0}^{\infty} w_j|\eta_j|^2]^{\frac{1}{2}} < \infty$$

by the Cauchy-Schwarz inequality.

(2) $\alpha x \in l^2_w(\mathbb{C})$ for all $\alpha \in \mathbb{C}$ since $\sum_{j=0}^{\infty} w_j|\alpha\xi_j|^2 = \sum_{j=0}^{\infty} w_j|\alpha|^2|\xi_j|^2 = |\alpha|^2\sum_{j=0}^{\infty} w_j|\xi_j|^2 < \infty$. For any $x=(x_0,x_1,\cdots)\in l^2_w(\mathbb{C})$, define a norm $\|\cdot\|_w$ on $l^2_w(\mathbb{C})$ by $\|x\|_w = (\sum_{j=0}^{\infty} w_j|x_j|^2)^{1/2}$. Then $l^2_w(\mathbb{C})$ is clearly a normed space.

Theorem 3.3. *Let $w=(w_0,w_1,\cdots)$ where $w_j>0$ and let $l^2_w(\mathbb{C})$ be the set of all sequences $x=(x_0,x_1,\cdots)$ of complex numbers such that $\sum_{j=0}^{\infty} w_j|x_j|^2<\infty$. Define an inner product of vectors $x=(x_n)$ and $y=(y_n)$ by $<x,\ y>=\sum_{j=0}^{\infty} w_jx_j\overline{y_j}$. Then $l^2_w(\mathbb{C})$ becomes a Hilbert space.*

Proof. Clearly $l^2_w(\mathbb{C})$ is a normed space. Let (z_n) be any Cauchy sequence in the space $l^2_w(\mathbb{C})$ where $z_n=(\alpha_0^{(n)},\alpha_1^{(n)},\alpha_2^{(n)},\cdots)$. Then for any $\varepsilon>0$, there exists an $N\in\mathbb{N}$ such that for all $m,n\ge N$,

$$\|z_n-z_m\|_w = (\sum_{j=0}^{\infty} w_j|\alpha_j^{(n)}-\alpha_j^{(m)}|^2)^{\frac{1}{2}} < \varepsilon. \qquad (3.1)$$

It follows that for every $j = 0, 1, 2, \cdots$, we have $|(\alpha_j^{(n)} - \alpha_j^{(m)})\sqrt{w_j}| < \varepsilon$ $(n, m \geq N)$. That is, $|\alpha_j^{(n)} - \alpha_j^{(m)}| < \varepsilon w_j^{-\frac{1}{2}}$ $(n, m \geq N)$. For each j, $(\alpha_j^{(m)})_{m=0}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$. Let $\beta_j = \lim_{m\to\infty} \alpha_j^{(m)}$. Using this limits, we define $z = (\beta_0, \beta_1, \beta_2, \cdots)$ and show that $z \in l_w^2(\mathbb{C})$ and $\|z_n - z\|_w \to 0$ as $n \to \infty$. From (3.1), we have for all $n, m \geq N$,

$$(\sum_{j=0}^{k} w_j |\alpha_j^{(n)} - \alpha_j^{(m)}|^2)^{\frac{1}{2}} < \varepsilon \quad (k = 0, 1, 2, 3, \cdots).$$

Letting $m \to \infty$, we obtain for $n \geq N$,

$$(\sum_{j=0}^{k} w_j |\alpha_j^{(n)} - \beta_j|^2)^{\frac{1}{2}} \leq \varepsilon \quad (k = 0, 1, 2, 3, \cdots).$$

Letting $k \to \infty$, then for $n \geq N$,

$$\|z_n - z\|_w = (\sum_{j=0}^{\infty} w_j |\alpha_j^{(n)} - \beta_j|^2)^{\frac{1}{2}} \leq \varepsilon.$$

This implies that $z_n \to z$ and $z_n - z = (\alpha_j^{(n)} - \beta_j) \in l_w^2(\mathbb{C})$. Since $z_n \in l_w^2(\mathbb{C})$, we have $z = (z - z_n) + z_n \in l_w^2(\mathbb{C})$. Thus $l_w^2(\mathbb{C})$ is a Banach space.

It suffices to show that the norm satisfies the parallelogram law: for any $x = (x_0, x_1, \cdots)$, $y = (y_0, y_1, \cdots) \in l_w^2(\mathbb{C})$,

$$\begin{aligned}\|x + y\|_w^2 + \|x - y\|_w^2 &= \sum_{j=0}^{\infty} w_j |x_j + y_j|^2 + \sum_{j=0}^{\infty} w_j |x_j - y_j|^2 \\ &= 2\sum_{j=0}^{\infty} w_j |x_j|^2 + 2\sum_{j=0}^{\infty} w_j |y_j|^2 \\ &= 2\|x\|_w^2 + 2\|y\|_w^2.\end{aligned}$$

Hence $l_w^2(\mathbb{C})$ is a Hilbert space. □

Case I. $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$ **where** $q > 0$

Lemma 3.4. *Let $q > 0$ be given and $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$. Let $l^2_w(\mathbb{C})$ be the set of all sequences $x = (x_0, x_1, \cdots)$ of complex numbers such that $\sum_{n=0}^{\infty} |x_n|^2 q^{-n} < \infty$. Define an inner product of vectors $x = (x_n)$ and $y = (y_n)$ by $< x,\ y >= \sum_{n=0}^{\infty} q^{-n} x_n \overline{y_n}$. Then $l^2_w(\mathbb{C})$ becomes a Hilbert space.*

Lemma 3.5. *Let $q > 0$ be given and $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$. Let A be the left shift operator on $l^2_w(\mathbb{C})$. Then A is a bounded linear operator with $\|A\| = \sqrt{q}$ and $A^* = qV$ where V is a right shift operator on $l^2_w(\mathbb{C})$. Moreover, $V^* = \frac{1}{q}A$.*

Proof. Clearly A is a linear operator. If $x_0 = (0, q, 0, 0, \cdots)$, then $\|Ax_0\|^2_w = \|(q, 0, \cdots)\|^2_w = q^2$ and $\|x_0\|^2_w = q^2 \frac{1}{q} = q$. Thus $\|Ax_0\|^2_w = q\|x_0\|^2_w$ and so $\sqrt{q} \leq \|A\|$. Also for all $x = (x_n)_{n=0}^{\infty} \in l^2_w(\mathbb{C})$,

$$\begin{aligned} \|Ax\|^2_w = \|(x_1, x_2, \cdots)\|^2_w &= \sum_{j=0}^{\infty} q^{-j} |x_{j+1}|^2 \\ &= q \sum_{j=0}^{\infty} q^{-(j+1)} |x_{j+1}|^2 = q \sum_{j=1}^{\infty} q^{-j} |x_j|^2 \\ &\leq q \sum_{j=0}^{\infty} q^{-j} |x_j|^2 = q\|(x_0, x_1, \cdots)\|^2_w = q\|x\|^2_w. \end{aligned}$$

Therefore A is a bounded linear operator with $\|A\| = \sqrt{q}$.

For any $x = (x_0, x_1, \cdots)$ and $y = (y_0, y_1, \cdots)$ in $l^2_w(\mathbb{C})$,

$$\begin{aligned} < A^*x,\ y > = < x, Ay >&= x_0\overline{y_1} + \frac{1}{q} x_1 \overline{y_2} + \frac{1}{q^2} x_2 \overline{y_3} + \cdots \\ &= q(\frac{1}{q} x_0 \overline{y_1} + \frac{1}{q^2} x_1 \overline{y_2} + \frac{1}{q^3} x_2 \overline{y_3} + \cdots) \\ &= < q(0, x_0, x_1, x_2, \cdots), (y_0, y_1, y_2, \cdots) > . \end{aligned}$$

Since this holds for all $y = (y_n) \in l^2_w(\mathbb{C})$, $A^*x = A^*(x_0, x_1, \cdots) = q(0, x_0, x_1, x_2, \cdots) = qV(x_0, x_1, \cdots)$ for any $x = (x_n) \in l^2_w(\mathbb{C})$. Hence $A^* = qV$. □

Theorem 3.6. *Let $q > 0$ be given and $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$. Let A be the left shift operator on $l_w^2(\mathbb{C})$. Then the followings hold.*

(1) $\sigma_p(A) = \{\lambda \in \mathbb{C} : |\lambda| < \sqrt{q}\}$, $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{q}\}$ *and* $\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{q}\}$.

(2) $\sigma_{com}(A) = \phi$.

(3) $\sigma_r(A) = \phi$ *and* $\sigma_c(A) = \{\lambda \in \mathbb{C} : |\lambda| = \sqrt{q}\}$.

(4) *For* $|\lambda| < \sqrt{q}$, $\ker(A - \lambda)$ *is the one-dimensional space spanned by the vector* $(1, \lambda, \lambda^2, \cdots)$.

Proof. (1) Suppose that $Ax = \lambda x$ for $x = (x_n) \in l_w^2(\mathbb{C})$. Then $(x_1, x_2, \cdots) = (\lambda x_0, \lambda x_1, \cdots)$, that is, $x_{n+1} = \lambda x_n$ for $n = 0, 1, 2, \cdots$. If $x_0 = 0$, then $x = 0$ in $l_w^2(\mathbb{C})$. Let $\lambda \in \sigma_p(A)$. Then there exists a non-zero element $x = (x_n)$ in $l_w^2(\mathbb{C})$ such that $Ax = \lambda x$. By the above fact, $x_0 \neq 0$. Thus $\|\lambda x\|_w = \|Ax\|_w < \sqrt{q}\|x\|_w$ and so $\sigma_p(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| < \sqrt{q}\}$.
Let $|\lambda| < \sqrt{q}$ and $x = (\sqrt{q}, \lambda\sqrt{q}, \lambda^2\sqrt{q}, \cdots)$. Then

$$\begin{aligned}\|x\|_w^2 &= q + q^{-1}(\lambda^2 q) + q^{-2}(\lambda^4 q) + q^{-3}(\lambda^6 q) + \cdots \\ &= q + \lambda^2 + \frac{\lambda^4}{q} + \frac{\lambda^6}{q^2} + \cdots = \frac{q^2}{q - \lambda^2} < \infty\end{aligned}$$

and $Ax = \lambda x$, that is, $\lambda \in \sigma_p(A)$. Thus we have $\sigma_p(A) = \{\lambda \in \mathbb{C} : |\lambda| < \sqrt{q}\}$. It is well known that $\sigma_p(A) \subseteq \sigma_{ap}(A) \subseteq \sigma(A)$. Since $\sigma(A)$ is a closed subset of $\mathbb{C}$ and $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{q}\}$, $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{q}\}$.
Since $\partial\sigma(A) \subseteq \sigma_{ap}(A)$ and $\sigma_p(A) \subseteq \sigma_{ap}(A)$, $\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{q}\}$.

(2) It is well known that $\sigma_{com}(A) = (\sigma_p(A^*))^* = (\sigma_p(qV))^* = (q\sigma_p(V))^* = q\sigma_p(V)^*$. In fact, $\lambda \in \sigma_p(qV) \Leftrightarrow \exists\, x \neq 0$ such that $(qV)x = \lambda x \Leftrightarrow \exists\, x \neq 0$ such that $Vx = \frac{\lambda}{q}x \Leftrightarrow \frac{\lambda}{q} \in \sigma_p(V) \Leftrightarrow \lambda \in q\sigma_p(V)$. It suffices to show that $\sigma_p(V) = \phi$. Suppose that $Vx = \lambda x$ for some $x = (x_n)$ and $\lambda \neq 0$. Then

$(0, x_0, x_1, \cdots) = (\lambda x_0, \lambda x_1, \lambda x_2, \cdots)$, that is, $\lambda x_0 = 0$ and $\lambda x_{n+1} = x_n$ for all $n = 0, 1, 2, \cdots$. Thus $x = 0$ in $l^2_w(\mathbb{C})$. Since $\|Vx\|_w = \|(0, x_0, x_1, \cdots)\|_w = \frac{1}{\sqrt{q}}\|x\|_w$ for $x = (x_n)$ in $l^2_w(\mathbb{C})$, $\ker V = (0)$ and so $\lambda = 0 \notin \sigma_p(V)$. Hence $\sigma_p(V) = \phi$ and so $\sigma_{com}(A) = \phi$.

(3) It comes from $\sigma_r(A) = \sigma_{com}(A) \setminus \sigma_p(A) = \phi$ and $\sigma_c(A) = \sigma(A) \setminus (\sigma_{com}(A) \cup \sigma_p(A)) = \{\lambda \in \mathbb{C} : |\lambda| = \sqrt{q}\}$. □

Corollary 3.7. *Let S_l be the left shift operator on $l^2(\mathbb{C})$. Then we have the followings.*

(1) $\sigma_p(S_l) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, $\sigma(S_l) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ *and* $\sigma_{ap}(S_l) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

(2) $\sigma_{com}(S_l) = \sigma_r(S_l) = \phi$ *and* $\sigma_c(S_l) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

(3) *For* $|\lambda| < 1$, $\ker(S_l - \lambda)$ *is the one-dimensional space spanned by the vector* $(1, \lambda, \lambda^2, \cdots)$.

Theorem 3.8. *Let $q > 0$ be given and $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$. Let V be the right shift operator on $l^2_w(\mathbb{C})$. Then we have the followings.*

(1) $\sigma_p(V) = \phi$, $\sigma_{com}(V) = \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{\sqrt{q}}\}$, *and* $\sigma(V) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{\sqrt{q}}\}$.

(2) $\sigma_r(V) = \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{\sqrt{q}}\}$.

(3) $\sigma_{ap}(V) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{\sqrt{q}}\} = \sigma_c(V)$.

(4) *For* $|\lambda| < \frac{1}{\sqrt{q}}$, $R(V - \lambda)$ *is closed and* $\dim R(V - \lambda)^\perp = 1$.

Proof. (1) For any $x = (x_n) \in l^2_w(\mathbb{C})$,

$$\|Vx\|^2_w = \|(0, x_0, x_1, \cdots)\|^2_w = \frac{1}{q}|x_0|^2 + \frac{1}{q^2}|x_1|^2 + \cdots = \frac{1}{q}\|x\|^2_w,$$

and so $\|Vx\|_w = \frac{1}{\sqrt{q}}\|x\|_w$. Thus V is a bounded linear operator with $\|V\| = \frac{1}{\sqrt{q}}$. Suppose that $Vx = \lambda x$ for some $x = (x_n) \in l^2_w(\mathbb{C})$ and $\lambda \neq 0$. Then $(0, x_0, x_1, \cdots) = (\lambda x_0, \lambda x_1, \lambda x_2, \cdots)$, that is, $\lambda x_0 = 0$ and $\lambda x_{n+1} = x_n$ for $n = 0, 1, 2, \cdots$. Thus $x = 0$ and so $\lambda \notin \sigma_p(V)$. Since $\|Vx\|_w = \frac{1}{\sqrt{q}}\|x\|_w$ for any $x = (x_n)$ in $l^2_w(\mathbb{C})$, $\ker V = (0)$ and so $\lambda = 0 \notin \sigma_p(V)$. Therefore $\sigma_p(V) = \phi$. Since $V^* = \frac{1}{q}A$ where A is the left shift operator on $l^2_w(\mathbb{C})$, by Theorem 2.7, $\sigma_{com}(V) = (\sigma_p(V^*))^* = (\sigma_p(\frac{1}{q}A))^* = (\frac{1}{q}\sigma_p(A))^* = \frac{1}{q}(\sigma_p(A))^*$. Since $\sigma_p(A) = \{\lambda \in \mathbb{C} : |\lambda| < \sqrt{q}\}$, we have $\sigma_{com}(V) = \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{\sqrt{q}}\}$. Since $\sigma_{com}(V) \subset \sigma(V) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{\sqrt{q}}\}$ and $\sigma(V)$ is a closed subset of $\mathbb{C}$, $\sigma(V) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{\sqrt{q}}\}$.

(2) It comes from $\sigma_r(V) = \sigma_{com}(V) \setminus \sigma_p(V) = \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{\sqrt{q}}\}$.

(3) Clearly $\{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{\sqrt{q}}\} = \partial\sigma(V) \subset \sigma_{ap}(V) \subset \sigma(V)$. Let $|\lambda| < \frac{1}{\sqrt{q}}$. Then $\|(V-\lambda)x\|_w \geq |\|Vx\|_w - \|\lambda x\|_w| = (\frac{1}{\sqrt{q}} - |\lambda|)\|x\|_w$ and $V - \lambda$ is bounded below. Thus $\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{\sqrt{q}}\} \not\subseteq \sigma_{ap}(V)$. Therefore $\sigma_{ap}(V) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{\sqrt{q}}\}$. Also $\sigma_c(V) = \sigma(V) \setminus (\sigma_{com}(V) \cup \sigma_p(V)) = \partial\sigma(V)$.

(4) For $|\lambda| < \frac{1}{\sqrt{q}}$, $R(V - \lambda)$ is closed from Theorem 2.6. Since $\ker(V^* - \overline{\lambda})$ is one-dimensional space, $\dim R(V - \lambda)^{\perp} = \dim\ker(V^* - \overline{\lambda}) = 1$. □

Corollary 3.9. *Let S_r be the right shift operator on $l^2(\mathbb{C})$. Then we have the followings.*

(1) $\sigma_p(S_r) = \phi$, $\sigma_{com}(S_r) = \sigma_r(S_r) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ *and* $\sigma(S_r) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

(2) $\sigma_{ap}(S_r) = \sigma_c(S_r) = \partial\sigma(S_r) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Case II. General Case

Theorem 3.10. *Suppose* $0 < w_0 \leq w_1 \leq w_2 \leq \cdots$ *such that* $r = \sup w_n < \infty$. *Let* A *be the left shift operator on* $l_w^2(\mathbb{C})$. *Then we have the followings.*

(1) $\|A\| = 1$.

(2) $\sigma_p(A) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ *and* $\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

(3) $\sigma_{com}(A) = \phi$.

(4) $\sigma_r(A) = \phi$ *and* $\sigma_c(A) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Proof. (1) Note that

$$\|A\| = \sup_{\|x\|_w=1} \|Ax\|_w = \sup_{\|x\|_w=1} (w_0|x_1|^2 + w_1|x_2|^2 + \cdots)^{1/2}$$
$$\leq \sup_{\|x\|_w=1} (w_1|x_1|^2 + w_2|x_2|^2 + \cdots)^{1/2} \leq \sup_{\|x\|_w=1} \|x\|_w = 1.$$

Thus A is bounded. Let $x_k = (\xi_j^{(k)}) \in l_w^2(\mathbb{C})$ where $\xi_j^{(k)} = (1/\sqrt{w_j})\delta_{jk}$. Then $\|x_k\|_w = 1$ for any $k = 0, 1, 2, \cdots$ and $\|Ax_k\|_w^2 = \frac{w_{k-1}}{w_k}(\leq 1) \to 1$ as $k \to \infty$. Hence $\|A\| = 1$.

(2) Suppose that $Ax = \lambda x$ for $x = (x_n) \in l_w^2(\mathbb{C})$. Then $(x_1, x_2, \cdots) = (\lambda x_0, \lambda x_1, \cdots)$, that is, $x_{n+1} = \lambda x_n$ for $n = 0, 1, 2, \cdots$. If $x_0 = 0$, then $x = 0$ in $l_w^2(\mathbb{C})$. Let $\lambda \in \sigma_p(A)$. Then there exists a non-zero element $x = (x_n)$ in $l_w^2(\mathbb{C})$ such that $Ax = \lambda x$. By the above fact, $x_0 \neq 0$. Thus $|\lambda|\|x\|_w = \|\lambda x\|_w = \|Ax\|_w < \|A\|\|x\|_w$ and so $\sigma_p(A) \in \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Let $|\lambda| < 1$ and $x = (1, \lambda, \lambda^2, \cdots)$. Then $\|x\|_w^2 = w_0 + w_1|\lambda|^2 + w_2|\lambda|^4 + w_3|\lambda|^6 + \cdots < \infty$ and so $x \in l_w^2(\mathbb{C})$. Also $Ax = \lambda x$, that is, $\lambda \in \sigma_p(A)$. We have $\sigma_p(A) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. It is well known that $\sigma_p(A) \subseteq \sigma_{ap}(A) \subseteq \sigma(A)$. Since $\sigma(A)$ is a closed subset of $\mathbb{C}$ and $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Since $\partial\sigma(A) \subseteq \sigma_{ap}(A)$, $\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

(3) For any $x = (x_0, x_1, \cdots)$, $y = (y_0, y_1, \cdots) \in l^2_w(\mathbb{C})$,

$$\begin{aligned} < Ax, y > &= < (x_1, x_2, \cdots), (y_0, y_1, \cdots) > \\ &= w_1 x_1 (\frac{w_0}{w_1}\overline{y_0}) + w_2 x_2 (\frac{w_1}{w_2}\overline{y_2}) + w_3 x_3 (\frac{w_2}{w_3}\overline{y_3}) + \cdots \\ &= < (x_0, x_1, x_2, \cdots), (0, \frac{w_0}{w_1} y_0, \frac{w_1}{w_2} y_1, \cdots) > . \end{aligned}$$

Since this holds for any $x = (x_n) \in l^2_w(\mathbb{C})$, $A^*(x_0, x_1, \cdots) = (0, \frac{w_0}{w_1}x_0, \frac{w_1}{w_2}x_1, \cdots)$. Suppose that $A^*x = \lambda x$ for $x = (x_n)$ and $\lambda \neq 0$. Then $(0, \frac{w_0}{w_1}x_0, \frac{w_1}{w_2}x_1, \cdots) = (\lambda x_0, \lambda x_1, \lambda x_2, \cdots)$, that is, $\lambda x_0 = 0$ and $\lambda x_{n+1} = \frac{w_n}{w_{n+1}} x_n$ for all $n = 0, 1, 2, \cdots$. Thus $x = 0$ in $l^2_w(\mathbb{C})$ and so $0 \neq \lambda \notin \sigma_p(A^*)$. Also $\ker A^* = (0)$ and so $\lambda = 0 \notin \sigma_p(A^*)$. Thus $\sigma_p(A^*) = \phi$. Hence $\sigma_{com}(A) = \sigma_p(A^*)^* = \phi$.

(4) From $\sigma_{com}(A) = \phi$, $\sigma_r(A) = \sigma_{com}(A) \setminus \sigma_p(A) = \phi$ and $\sigma_c(A) = \sigma(A) \setminus (\sigma_{com}(A) \cup \sigma_p(A)) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. □

Corollary 3.11. *Let S_l be the left shift operator on $l^2(\mathbb{C})$. Then we have the followings.*

(1) $\sigma_p(S_l) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, $\sigma(S_l) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ *and* $\sigma_{ap}(S_l) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

(2) $\sigma_{com}(S_l) = \sigma_r(S_l) = \phi$ *and* $\sigma_c(S_l) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Theorem 3.12. *Suppose $w_0 \geq w_1 \geq w_2 \geq \cdots$ such that $r = \inf w_n > 0$. Let V be the right shift operator on $l^2_w(\mathbb{C})$. Then the followings hold.*

(1) $\|V\| = 1$.

(2) $\sigma_p(V) = \phi$.

(3) $\sigma_{ap}(V) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

(4) $\sigma_{com}(V) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, *and* $\sigma(V) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

(5) $\sigma_c(V) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ *and* $\sigma_r(V) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

(6) *If* $|\lambda| < 1$, *then* $R(V - \lambda)$ *is closed and* $\dim R(V - \lambda)^{\perp} = 1$.

Proof. (1) Note that

$$\|Vx\|_w^2 = \|(0, x_0, x_1, x_2, \cdots)\|^2 = w_1|x_0|^2 + w_2|x_1|^2 + \cdots$$
$$\leq w_0|x_0|^2 + w_1|x_1|^2 + w_2|x_2|^2 + \cdots = \|x\|_w^2.$$

Thus V is bounded. Let $x_k = (\xi_j^{(k)}) \in l_w^2(\mathbb{C})$ where $\xi_j^{(k)} = (1/\sqrt{w_j})\delta_{jk}$. Then $\|x_k\|_w = 1$ for any $k = 0, 1, 2, \cdots$ and $\|Vx_k\|_w^2 = \frac{w_{k+1}}{w_k}(\leq 1) \to 1$ as $k \to \infty$. Hence $\|V\| = 1$.

(2) Suppose $x = (x_0, x_1, \cdots) \in l^2(\mathbb{C})$ and $\lambda \neq 0$. If $Vx = \lambda x$, then $0 = \lambda x_0, x_0 = \lambda x_1, \cdots$. Since $\lambda \neq 0, x = 0$ and so $\lambda \notin \sigma_p(V)$. Also $\ker V = (0)$ and so $\lambda = 0 \notin \sigma_p(V)$. Therefore $\sigma_p(V) = \phi$. Since $\|V\| = 1$, $\sigma(V) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

(3) Let $|\lambda| < 1$. Then there exists a real number q such that $|\lambda| < q < 1$. Since $\lim \frac{w_{n+1}}{w_n} = 1$, there exists a positive integer N such that $q < \frac{w_{n+1}}{w_n} \leq 1$ for any $n \geq N$. Thus for any $x = (0, \cdots, 0, x_N, x_{N+1}, \cdots) \in l_w^2(\mathbb{C})$,

$$\|Vx\|_w^2 = w_1|x_0|^2 + w_2|x_1|^2 + \cdots$$
$$> w_1|x_0|^2 + w_2|x_1|^2 + \cdots + w_N|x_{N-1}|^2 + qw_N|x_N|^2 + \cdots$$
$$= q\|x\|_w^2$$

and so $\|Vx\|_w^2 \geq q\|x\|_w^2$. Therefore $\|(V - \lambda)x\|_w \geq \|Vx\|_w - |\lambda|\|x\|_w \geq (\sqrt{q} - |\lambda|)\|x\|_w$ for any $x = (x_n) \in l_w^2(\mathbb{C})$. Since $\sqrt{q} - |\lambda| > 0$, $V - \lambda$ is bounded below and so $\lambda \notin \sigma_{ap}(V)$. Since $\sigma_{ap}(V) \subset \sigma(V)$ and $\partial\sigma(V) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \subset \sigma_{ap}(V)$, $\sigma_{ap}(V) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

(4) For any $x = (x_n),\ y = (y_n) \in l_w^2(\mathbb{C})$,

$$< Vx, y > = w_1 x_0 \overline{y_1} + w_2 x_1 \overline{y_2} + \cdots$$
$$= w_0 x_0 \overline{\frac{w_1}{w_0} y_1} + w_1 x_1 \overline{\frac{w_2}{w_1} y_2} + \cdots$$
$$= < (x_0, x_1, \cdots), (\frac{w_1}{w_0} y_1,\ \frac{w_2}{w_1} y_2, \cdots) > .$$

Thus $V^*(y_0, y_1, \cdots) = (\frac{w_1}{w_0}y_1, \frac{w_2}{w_1}y_2, \cdots)$. Let $|\lambda| < 1$ and choose $x = (1, \frac{w_0}{w_1}\lambda, \frac{w_0}{w_2}\lambda^2, \cdots)$. Then

$$\begin{aligned}\|x\|_w^2 &= w_0 + w_1|\frac{w_0}{w_1}\lambda|^2 + w_2|\frac{w_0}{w_2}\lambda^2|^2 + \cdots \\ &= |w_0|^2(\frac{1}{w_0} + \frac{1}{w_1}|\lambda|^2 + \frac{1}{w_2}|\lambda|^4 + \cdots) \\ &\leq \frac{|w_0|^2}{r}(1 + |\lambda|^2 + |\lambda|^4 + \cdots) < \infty\end{aligned}$$

and

$$\begin{aligned}V^*x &= V^*(1, \frac{w_0}{w_1}\lambda, \frac{w_0}{w_2}\lambda^2, \cdots) = (\frac{w_1}{w_0}\cdot\frac{w_0}{w_1}\lambda, \frac{w_2}{w_1}\cdot\frac{w_0}{w_2}\lambda^2, \cdots) \\ &= (\lambda, \frac{w_0}{w_1}\lambda^2, \cdots) = \lambda x.\end{aligned}$$

Therefore $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_p(V^*)$. We know that $\sigma_p(V^*) \subset \sigma(V^*) = \sigma(V)^* = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Suppose that there exists $\lambda \in \sigma_p(V^*)$ with $|\lambda| = 1$. From $V^*x = (\frac{w_1}{w_0}x_1, \frac{w_2}{w_1}x_2, \cdots) = (\lambda x_0, \lambda x_1, \cdots) = \lambda x$, it is clear that if $x_0 = 0$, then $x = 0$ and so x_0 must not be zero. But

$$\begin{aligned}\|V^*x\|_w^2 &= w_0|\frac{w_1}{w_0}x_1|^2 + w_1|\frac{w_2}{w_1}x_2|^2 + \cdots \\ &= \frac{w_1}{w_0}\cdot w_1|x_1|^2 + \frac{w_2}{w_1}\cdot w_2|x_2|^2 + \cdots \\ &\leq w_1|x_1|^2 + w_2|x_2|^2 + \cdots < \|x\|_w^2.\end{aligned}$$

This is a contradiction to the fact $\|V^*x\|_w = |\lambda|\|x\|_w = \|x\|_w$. Thus $\lambda \notin \sigma_p(V^*)$ when $|\lambda| = 1$ and so $\sigma_{com}(V) = \sigma_p(V^*)^* = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Since $\sigma_{com}(T) \subseteq \sigma(T)$ for any operator T, $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq \sigma(V)$ and from (1), $\sigma(V) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Since $\sigma(V)$ is a closed subset of $\mathbb{C}$, $\sigma(V) = \{\lambda : |\lambda| \leq 1\}$.

(5) It comes from $\sigma_r(V) = \sigma_{com}(V) - \sigma_p(V) = \sigma_{com}(V) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\sigma_c(V) = \sigma(V) - (\sigma_{com}(V) \cup \sigma_p(V)) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

(6) For $|\lambda| < 1$, $R(V-\lambda)$ is closed by Theorem 2.6 and $\dim R(V-\lambda)^{\perp} = \dim\ker(V^* - \overline{\lambda}) = 1$ since $\ker(V^* - \lambda)$ is one-dimensional space. □

Corollary 3.13. *Let S_r be the right shift operator on $l^2(\mathbb{C})$. Then the followings hold.*

(1) $\sigma_p(S_r) = \phi$, $\sigma_{com}(S_r) = \sigma_r(S_r) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\sigma(S_r) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

(2) $\sigma_{ap}(S_r) = \sigma_c(S_r) = \partial\sigma(S_r) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Theorem 3.14. *Suppose that $0 < w_0 \leq w_1 \leq w_2 \leq \cdots \leq u = \sup w_n < \infty$ and $0 < |\alpha_0| \leq |\alpha_1| \leq \cdots \leq r = \sup|\alpha_n| < \infty$. Define the operator $A : l^2_w(\mathbb{C}) \to l^2_w(\mathbb{C})$ by $A(x_0, x_1, x_2, \cdots) = (\alpha_1 x_1, \alpha_2 x_2, \cdots)$. Then we have the followings.*

(1) $\sigma_p(A) = \{\lambda \in \mathbb{C} : |\lambda| < r\}$.

(2) $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$.

(3) $\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$.

(4) $\sigma_{com}(A) = \phi$.

(5) $\sigma_r(A) = \phi$ and $\sigma_c(A) = \{\lambda \in \mathbb{C} : |\lambda| = r\}$.

Proof. Since $\|Ax\|^2_w = \|(\alpha_1 x_1, \alpha_2 x_2, \cdots)\|^2_w = w_0|\alpha_1 x_1|^2 + w_1|\alpha_2 x_2|^2 + \cdots \leq r^2\|x\|^2_w$, $\|A\| \leq r$. Thus A is a bounded linear operator with $\|A\| = r$. Thus $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$.

(1) Suppose that $Ax = \lambda x$ for $x = (x_n) \in l^2_w(\mathbb{C})$. Then $(\alpha_1 x_1, \alpha_2 x_2, \cdots) = (\lambda x_0, \lambda x_1, \cdots)$, that is, $\lambda x_n = \alpha_{n+1} x_{n+1}$ for all $n = 0, 1, 2, \cdots$. If $x_0 = 0$, then $x = 0$ in $l^2_w(\mathbb{C})$. Let $\lambda \in \sigma_p(A)$. Then there exists a nonzero element $x = (x_n)$ in $l^2_w(\mathbb{C})$ such that $Ax = \lambda x$. From the above fact, $x_0 \neq 0$. Thus $|\lambda|\|x\|_w = \|Ax\|_w < r\|x\|_w$ and so $|\lambda| < r$. We have $\sigma_p(A) \subset \{\lambda \in \mathbb{C} : |\lambda| <$

$r\}$. Let $|\lambda| < r$. Choose $x = (1, \frac{\lambda}{\alpha_1}, \frac{\lambda^2}{\alpha_1\alpha_2}, \frac{\lambda^3}{\alpha_1\alpha_2\alpha_3}, \cdots)$. Since

$$\|x\|_w^2 = w_0 + w_1|\frac{\lambda}{\alpha_1}|^2 + w_2|\frac{\lambda^2}{\alpha_1\alpha_2}|^2 + \cdots$$
$$\leq u(1 + |\frac{\lambda}{\alpha_1}|^2 + |\frac{\lambda^2}{\alpha_1\alpha_2}|^2 + \cdots) = u\sum_{n=0}^{\infty} a_n < \infty$$

and

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty}\frac{|\frac{\lambda^{n+1}}{\alpha_1\cdots\alpha_{n+1}}|^2}{|\frac{\lambda^n}{\alpha_1\cdots\alpha_n}|^2} = \lim_{n\to\infty}\frac{|\lambda|^2}{|\alpha_{n+1}|^2} = \frac{|\lambda|^2}{|r|^2} < 1,$$

$x \in l_w^2(\mathbb{C})$ and $Ax = A(1, \frac{\lambda}{\alpha_1}, \frac{\lambda^2}{\alpha_1\alpha_2}, \cdots) = (\alpha_1\frac{\lambda}{\alpha_1}, \alpha_2\frac{\lambda^2}{\alpha_1\alpha_2}, \cdots) = (\lambda, \frac{\lambda^2}{\alpha_1}, \frac{\lambda^3}{\alpha_1\alpha_2}, \cdots) = \lambda x$. Thus $\lambda \in \sigma_p(A)$ and $\sigma_p(A) = \{\lambda \in \mathbb{C} : |\lambda| < r\}$.

(2) Since $\{\lambda \in \mathbb{C} : |\lambda| < r\} = \sigma_p(A) \subseteq \sigma(A) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$ and $\sigma(A)$ is closed, $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$.

(3) By (2), $\partial\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| = r\}$. Since $\partial\sigma(T) \subseteq \sigma_{ap}(T)$ and $\sigma_p(T) \subseteq \sigma_{ap}(T)$ for any operator T, $\sigma_{ap}(A) = \{\lambda : |\lambda| \leq r\}$.

(4) For any $x = (x_n)$, $y = (y_n) \in l_w^2(\mathbb{C})$,

$$< x, Ay > = < (x_0, x_1, \cdots), (\alpha_1 y_1, \alpha_2 y_2, \cdots) >$$
$$= w_0 x_0 \overline{\alpha_1 y_1} + w_1 x_1 \overline{\alpha_2 y_2} + \cdots$$
$$= w_1\frac{w_0}{w_1}\overline{\alpha_1}x_0\overline{y_1} + w_2\frac{w_1}{w_2}\overline{\alpha_2}x_1\overline{y_2} + \cdots$$
$$= < (0, \frac{w_0}{w_1}\overline{\alpha_1}x_0, \frac{w_1}{w_2}\overline{\alpha_2}x_1, \cdots), (y_0, y_1, y_2, \cdots) > .$$

Thus

$$A^*(x_0, x_1, \cdots) = (0, \frac{w_0}{w_1}\overline{\alpha_1}x_0, \frac{w_1}{w_2}\overline{\alpha_2}x_1, \cdots).$$

Suppose that $A^*x = \lambda x$ for $x \in l_w^2(\mathbb{C})$ and $\lambda \neq 0$. Then $(0, \frac{w_0}{w_1}\overline{\alpha_1}x_0, \frac{w_1}{w_2}\overline{\alpha_2}x_1, \cdots) = (\lambda x_0, \lambda x_1, \cdots)$ and so $x = 0$. Thus $0 \neq \lambda \notin \sigma_p(A^*)$. If $\lambda = 0$, then

$A^*x = 0$ and so $x = 0$. Hence $\lambda = 0 \notin \sigma_p(A^*)$. Therefore $\sigma_p(A^*) = \phi$. Also $\sigma_{com}(A) = \sigma_p(A^*)^* = \phi$.

(5) It comes from $\sigma_r(A) = \sigma_{com}(A) - \sigma_p(A) = \phi$ and $\sigma_c(A) = \sigma(A) - (\sigma_{com}(A) \cup \sigma_p(A)) = \{\lambda \in \mathbb{C} : |\lambda| = r\}$. □

From this theorem we obtain Theorem 3.10.

Corollary 3.15. *Let (α_n) be a monotone increasing sequence in a real field $\mathbb{R}$ such that $\alpha_0 > 0$ and $r = \sup \alpha_n < \infty$. Let $A : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ be the operator defined by $A(x_0, x_1, \cdots) = (\alpha_1 x_1, \alpha_2 x_2, \cdots)$. Then the followings hold.*

(1) $\sigma_p(A) = \{\lambda \in \mathbb{C} : |\lambda| < r\}$, $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$ *and* $\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$.

(2) $\sigma_{com}(A) = \sigma_r(A) = \phi$ *and* $\sigma_c(A) = \{\lambda \in \mathbb{C} : |\lambda| = r\}$.

Corollary 3.16. *Let S_l be a left shift operator on $l^2(\mathbb{C})$. Then $\sigma_p(S_l) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, $\sigma(S_l) = \{\lambda : |\lambda| \leq 1\} = \sigma_{ap}(S_l)$, $\sigma_{com}(S_l) = \sigma_r(S_l) = \phi$ and $\sigma_c(S_l) = \partial\sigma(S_l)$.*

Lemma 3.17. *Suppose that $0 < |\alpha_0| \leq |\alpha_1| \leq \cdots$ such that $r = \sup |\alpha_n| < \infty$. Define the operator $S : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ by $S(x_0, x_1, \cdots) = (0, \alpha_0 x_0, \alpha_1 x_1, \cdots)$. Then its adjoint S^* is given by $S^*(x_0, x_1, \cdots) = (\overline{\alpha_0} x_1, \overline{\alpha_1} x_2, \cdots)$.*

Proof. Note that

$$\begin{aligned} < x, Sy > &= < (x_0, x_1, \cdots), (0, \alpha_0 y_0, \alpha_1 y_1, \cdots) > \\ &= x_1 \overline{\alpha_0 y_0} + x_2 \overline{\alpha_1 y_1} + \cdots \\ &= < (\overline{\alpha_0} x_1, \overline{\alpha_1} x_2, \cdots), (y_0, y_1, \cdots) >, \end{aligned}$$

for any $x = (x_n)$ and $y = (y_n)$ in $l^2(\mathbb{C})$. Since this holds for every $y = (y_n)$ in $l^2(\mathbb{C})$, $S^*(x_0, x_1, \cdots) = (\overline{\alpha_0}x_1, \overline{\alpha_1}x_2, \cdots)$. □

Theorem 3.18. *Suppose that $0 < |\alpha_0| \leq |\alpha_1| \leq \cdots$ such that $r = \sup |\alpha_n| < \infty$. Define the operator $S : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ by $S(x_0, x_1, \cdots) = (0, \alpha_0 x_0, \alpha_1 x_1, \cdots)$. Then the followings hold.*

(1) $\sigma_p(S) = \phi$.

(2) $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$.

(3) $\sigma_{ap}(S) = \{\lambda \in \mathbb{C} : |\lambda| = r\}$.

(4) $\sigma_{com}(S) = \{\lambda \in \mathbb{C} : |\lambda| < r\}$.

(5) $\sigma_r(S) = \{\lambda \in \mathbb{C} : |\lambda| < r\}$ *and* $\sigma_c(S) = \{\lambda \in \mathbb{C} : |\lambda| = r\}$.

(6) *For* $|\lambda| < r$, $R(S - \lambda)$ *is closed and* $\dim R(S - \lambda)^\perp = 1$.

Proof. Note that $\|Sx\|^2 = \|(0, \alpha_0 x_0, \alpha_1 x_1, \cdots)\|^2 = |\alpha_0|^2|x_0|^2 + |\alpha_1|^2|x_1|^2 + \cdots \leq r^2(|x_0|^2 + |x_1|^2 + \cdots) = r^2\|x\|^2$ and $\|Sx\|^2 = \|S(0, 1, 0, \cdots)\|^2 = \|(0, 0, \alpha_1, 0, \cdots)\|^2 = |\alpha_1|^2\|x\|^2$. Then $\|S(0, 1, 0, \cdots)\| = |\alpha_1|$. So $\|Se_k\| = |\alpha_k|$ for $k = 0, 1, 2, \cdots$, where $e_k = (0, \cdots, 0, 1, 0, \cdots) \in l^2(\mathbb{C})$. Thus $\|Se_0\| \leq \|Se_1\| \leq \cdots < r$. Hence $\|S\| = r$.

(1) Suppose that $Sx = \lambda x$ for $x = (x_n) \in l^2(\mathbb{C})$ and $\lambda \neq 0$. Then $(0, \alpha_0 x_0, \alpha_1 x_1, \cdots) = (\lambda x_0, \lambda x_1, \cdots)$, that is, $\lambda x_0 = 0$ and $\lambda x_{n+1} = \alpha_n x_n$ for all $n = 0, 1, 2, \cdots$. We have $x = 0$ and so $\lambda \notin \sigma_p(S)$. Also $\ker S = (0)$ and so $\lambda = 0 \notin \sigma_p(S)$. Hence $\sigma_p(S) = \phi$.

(2) Let $|\lambda| < r$ and put $x = (1, \frac{\lambda}{\alpha_0}, \frac{\lambda^2}{\alpha_0\alpha_1}, \cdots)$. Then $\|x\|^2 = 1 + |\frac{\lambda}{\alpha_0}|^2 + |\frac{\lambda^2}{\alpha_0\alpha_1}|^2 + \cdots = \sum_{n=0}^{\infty} a_n < \infty$ since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|\frac{\lambda^{n+1}}{\alpha_0\alpha_1\cdots\alpha_n}|^2}{|\frac{\lambda^n}{\alpha_0\cdots\alpha_{n-1}}|^2} = \lim_{n\to\infty}\frac{|\lambda|^2}{|\alpha_n|^2} = |\lambda|^2\frac{1}{r^2} < 1.$$

Since $S^*(y_n) = (\overline{\alpha_0}y_1, \overline{\alpha_1}y_2, \cdots)$,

$$S^*x = (\overline{\alpha_0}\frac{\lambda}{\overline{\alpha_0}}, \overline{\alpha_1}\frac{\lambda^2}{\overline{\alpha_0\alpha_1}}, \cdots) = \lambda(1, \frac{\lambda}{\overline{\alpha_0}}, \frac{\lambda^2}{\overline{\alpha_0\alpha_1}}, \cdots) = \lambda x.$$

Thus $\lambda \in \sigma_p(S^*)$ and $x \in \ker(S^* - \lambda)$. Therefore $\{\lambda \in \mathbb{C} : |\lambda| < r\} \subseteq \sigma_p(S^*) \subseteq \sigma(S^*) = \sigma(S)^* \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$. Since $\sigma(S)$ is closed, $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$.

(3) Since $\partial\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| = r\}$ and $\partial\sigma(S) \subseteq \sigma_{ap}(S) \subseteq \sigma(S)$, $\{\lambda \in \mathbb{C} : |\lambda| = r\} \subseteq \sigma_{ap}(S) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$. Let $|\lambda| < r$. Since $|\alpha_0| \leq |\alpha_1| \leq \cdots$ and $r = \lim|\alpha_n|$, there exists $N \in \mathbb{N}$ such that $|\lambda| < |\alpha_N|$. Since $\|Sx\|^2 = \|(0, \cdots, 0, \alpha_N x_N, \alpha_{N+1}x_{N+1}, \cdots)\|^2 \geq |\alpha_N|^2(|x_N|^2 + |x_{N+1}|^2 + \cdots) = |\alpha_N|^2\|x\|^2$, for any $x = (0, \cdots, 0, x_N, x_{N+1}, \cdots) \in l^2(\mathbb{C})$, $\|(S - \lambda)x\| \geq \|Sx\| - |\lambda|\|x\| \geq (|\alpha_N| - |\lambda|)\|x\|$. Thus $S - \lambda$ is bounded below and so $\lambda \notin \sigma_{ap}(S)$. Hence $\sigma_{ap}(S) = \{\lambda \in \mathbb{C} : |\lambda| = r\}$.

(4) For each $x = (x_n),\ y = (y_n) \in l^2(\mathbb{C})$,

$$\begin{aligned} < x,\ S^*y > &= < Sx, y > = < (0, \alpha_0 x_0, \alpha_1 x_1, \cdots), (y_0, y_1, \cdots) > \\ &= \alpha_0 x_0 \overline{y_1} + \alpha_1 x_1 \overline{y_2} + \cdots = x_0 \overline{\overline{\alpha_0} y_1} + x_1 \overline{\overline{\alpha_1} y_2} + \cdots \\ &= < (x_0, x_1, \cdots), (\overline{\alpha_0} y_1, \overline{\alpha_1} y_2, \cdots) > . \end{aligned}$$

Thus $S^*(y_0, y_1, \cdots) = (\overline{\alpha_0}y_1, \overline{\alpha_1}y_2, \cdots)$ for all $y = (y_n) \in l^2(\mathbb{C})$ and so S^* is a weighted left shift operator on $l^2(\mathbb{C})$. From Theorem 3.14, $\sigma_p(S^*) = \{\lambda \in \mathbb{C} : |\lambda| < r\}$. Hence $\sigma_p(S^*)^* = \{\lambda \in \mathbb{C} : |\lambda| < r\}$ and $\sigma_{com}(S) = \sigma_p(S^*)^* = \{\lambda \in \mathbb{C} : |\lambda| < r\}$.

(5) Note that $\sigma_r(S) = \sigma_{com}(S) - \sigma_p(S) = \{\lambda \in \mathbb{C} : |\lambda| < r\}$. Also $\sigma_c(S) = \sigma(S) - (\sigma_p(S) \cup \sigma_{com}(S)) = \{\lambda \in \mathbb{C} : |\lambda| = r\}$.

(6) For $|\lambda| < 1$, $R(S - \lambda)$ is closed by Theorem 2.6, and $\dim R(S - \lambda)^\perp = \dim\ker(S^* - \overline{\lambda}) = 1$ since $\ker(S^* - \lambda)$ is one-dimensional space. □

4. Diagonal Operators

Suppose that H is a Hilbert space and that $\{e_j\}$ is a family of vectors that constitute an orthonormal basis for H. An operator A is called a *diagonal operator* if Ae_j is a scalar multiple of e_j, say $Ae_j = \alpha_j e_j$ for each j. The family $\{\alpha_j\}$ is called the diagonal of A.

Theorem 4.1. ([5]) *A family $\{\alpha_j\}$ is the diagonal of a diagonal operator iff it is bounded. If it is bounded, then the equations $Ae_j = \alpha_j e_j$ uniquely determine an operator A, and $\|A\| = \sup_j |\alpha_j|$.*

Proof. If A is a diagonal operator with $Ae_j = \alpha_j e_j$, then $|\alpha_j| = \|\alpha_j e_j\| = \|Ae_j\| \leq \|A\|\|e_j\| = \|A\|$. So $\{\alpha_j\}$ is bounded and $\sup_j |\alpha_j| \leq \|A\|$. By Parseval's equality,

$$\|Ax\|^2 = \|\sum_j \alpha_j \xi_j e_j\|^2 = \sum_j |\alpha_j \xi_j|^2 \leq (\sup_j |\alpha_j|)^2 \sum_j |\xi_j|^2$$
$$= (\sup_j |\alpha_j|)^2 \|\sum_j \xi_j e_j\|^2 = (\sup_j |\alpha_j|)^2 \|x\|^2.$$

Thus $\|A\| \leq \sup_j |\alpha_j|$ and so $\|A\| = \sup_j |\alpha_j|$.

Conversely, given a bounded family $\{\alpha_j\}$, define A by $A(x_0, x_1, \cdots) = (\alpha_0 x_0, \alpha_1 x_1, \cdots)$. Then $A(\sum_{j=0}^{\infty} x_j e_j) = \sum_{j=0}^{\infty} \alpha_j x_j e_j$ and

$$\|A(\sum_{j=0}^{\infty} x_j e_j)\|^2 = \|\sum_{j=0}^{\infty} \alpha_j x_j e_j\|^2 = \sum_{j=0}^{\infty} |\alpha_j x_j|^2$$
$$\leq (\sup_j |\alpha_j|)^2 \sum_{j=0}^{\infty} |x_j|^2 = (\sup_j |\alpha_j|)^2 \|x\|^2 < \infty.$$

Hence $\|A\| < \infty$ and A is a diagonal operator since $Ae_j = \alpha_j e_j$ for each j. Clearly the diagonal of A is exactly the sequence $\{\alpha_j\}$. □

Lemma 4.2. ([5]) *If* $\{\alpha_n\}$ *is a sequence of complex scalars such that* $\sum_n |\alpha_n \xi_n|^2 < \infty$ *whenever* $\sum_n |\xi_n|^2 < \infty$, *then* $\{\alpha_n\}$ *is bounded.*

Proof. If $\{\alpha_n\}$ is not bounded, then $|\alpha_n|$ takes arbitrarily large values. Without loss of generality, we can assume that $|\alpha_n| \geq n$. If $\xi_n = \frac{1}{\alpha_n}$ $(n = 1, 2, \cdots)$, then $\sum_n |\xi_n|^2 \leq \sum_n \frac{1}{n^2} < \infty$ but $\sum_n |\alpha_n \xi_n|^2 (= \infty)$ diverges. This contradicts to the hypothesis. Hence $\{\alpha_n\}$ is a bounded sequence. □

The set of all bounded sequences $\{\alpha_n\}$ of complex numbers is an algebra (pointwise operations), with unit ($|\alpha_n| = 1$ for all n), a conjugation ($\{\alpha_n\} \to \{\alpha_n^*\}$) and a norm ($\|\{\alpha_n\}\| = \sup_n |\alpha_n|$). A bounded sequence $\{\alpha_n\}$ is said to be invertible if it has an inverse in this algebra, i.e., if there exists a bounded sequence $\{\beta_n\}$ such that $\alpha_n \beta_n = 1$ for all n. $\{\alpha_n\}$ is said to be bounded away from zero if there exists a positive number δ such that $|\alpha_n| \geq \delta$ for all n.

Lemma 4.3. ([5]) $\{\alpha_n\}$ *is invertible if and only if* $\{\alpha_n\}$ *is bounded away from zero.*

Proof. If $\{\alpha_n\}$ is invertible, then there exists a bounded sequence $\{b_n\}$ such that $\alpha_n b_n = 1$ for all n. Thus $|\alpha_n| = \frac{1}{|b_n|} \geq \frac{1}{\sup_n |b_n|} = \delta > 0$.

Conversely, suppose that there exists $\delta > 0$ such that $|\alpha_n| \geq \delta$ for all n. Then $\alpha_n \neq 0$ for all n and so there exists $\{\frac{1}{\alpha_n}\}$ such that $\frac{1}{\alpha_n}\alpha_n = 1$ for all n. Hence $\{\alpha_n\}$ is invertible. □

If H is a Hilbert space with an orthonormal basis $\{e_n\}$, then the correspondence $\{\alpha_n\} \longmapsto A$ where A is the operator on H such that $Ae_n = \alpha_n e_n$ for all n is an isomorphism (an embedding) of the sequence algebra into the algebra of operators on H. The correspondence preserves not only the familiar algebraic operations but also conjugation, i.e., if $\{\alpha_n\} \to A$, then $\{\alpha_n^*\} \to A^*$.

The correspondence preserves the norm also, i.e., $\|\{\alpha_n\}\| = \sup_n |\alpha_n| = \|A\|$ where $Ae_n = \alpha_n e_n$ for all n.

Theorem 4.4. *A diagonal operator A with diagonal $\{\alpha_n\}$ is an invertible operator if and only if the sequence $\{\alpha_n\}$ is an invertible sequence.*

Proof. If $\{\alpha_n\}$ is invertible, there exists a bounded sequence $\{\beta_n\}$ such that $\alpha_n\beta_n = 1$ for all n. Let B be the diagonal operator with diagonal $\{\beta_n\}$, i.e., $Be_n = \beta_n e_n$ for all n. Then $AB(\sum_j \xi_j e_j) = A(\sum_j \xi_j \beta_j e_j) = \sum_j \xi_j \beta_j \alpha_j e_j = \sum_j \xi_j e_j$ and similarly $BA(\sum_j \xi_j e_j) = \sum_j \xi_j e_j$. Thus B is an inverse of A and so A is an invertible operator.

Conversely, if A is invertible, then $A^{-1}(\alpha_n e_n) = e_n$ so that $A^{-1}e_n = \frac{1}{\alpha_n} e_n$ for all n. Since $\|A^{-1}e_n\| \leq \|A^{-1}\|$, the sequence $\{\frac{1}{\alpha_n}\}$ is bounded, and hence the sequence $\{\alpha_n\}$ is invertible. □

We note that a diagonal operator A is a normal operator.

Theorem 4.5. *Define $A : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ by $A(x_0, x_1, \cdots) = (\alpha_0 x_0, \alpha_1 x_1, \cdots)$. Suppose that $\{|\alpha_n|\}$ is bounded with $\sup|\alpha_n| = r < \infty$. Then the followings hold.*

(1) *$\sigma(A)$ is closure of $\{\alpha_n\}$.*

(2) *$\sigma_p(A) = \sigma_{com}(A) = \{\alpha_n\}$ and $\sigma_{ap}(A)$ is the closure of $\{\alpha_n\}$.*

(3) *$\sigma_r(A) = \phi$ and $\sigma_c(A) = \overline{\{\alpha_n\}} - \{\alpha_n\}$.*

Proof. Note that $\|A\| = r$.

(1) By Theorem 4.4, we note that $A - \lambda$ is invertible if and only if $\{\alpha_n - \lambda\}$ is invertible since $|\alpha_n - \lambda| < |\alpha_n| + |\lambda| < \infty$ and $\{\alpha_n - \lambda\}$ is the diagonal of $A - \lambda$. By Lemma 4.3, we know that $\{\alpha_n - \lambda\}$ is invertible if and only if $\{\alpha_n - \lambda\}$ is bounded away from zero. Then $\{\alpha_n - \lambda\}$ is bounded away

from 0 if and only if $|\alpha_n - \lambda| \nrightarrow 0$ as $n \to \infty$ if and only if 0 is not a limit point of $\{\alpha_n - \lambda\}$ if and only if λ is not a limit point of $\{\alpha_n\}$ if and only if $\lambda \notin \overline{\{\alpha_n\}}$. Consequently, $A - \lambda$ is not invertible if and only if $\lambda \in \overline{\{\alpha_n\}}$. Hence $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda \neq \text{ invertible}\} = \overline{\{\alpha_n\}}$.

(2) Let $\{e_j\}$ be an orthonormal basis for $l^2(\mathbb{C})$ such that $Ae_j = \alpha_j e_j$ for all j. Then α_j is an eigenvalue of A and so $\{\alpha_n\} \subseteq \sigma_p(A)$. Conversely, if $Ax = \lambda x$ for some nonzero x, then $((\alpha_0 - \lambda)x_0, (\alpha_1 - \lambda)x_1, \cdots) = (0, 0, \cdots)$. Since $x \neq 0$, at least one of x_i is not zero, and so at least one of $\alpha_j - \lambda$ is zero. Thus $\sigma_p(A) = \{\alpha_n\}$. Since A is normal, by Lemma 2.5, $\sigma_{com}(A) = \sigma_p(A) = \{\alpha_n\}$. Now, since $\sigma_r(A) = \sigma_{com}(A) - \sigma_p(A)$, $\sigma_r(A) = \phi$. Since $\sigma_p(A) \subset \sigma_{ap}(A) \subset \sigma(A)$, $\{\alpha_n\} \subset \sigma_{ap}(A) \subset \overline{\{\alpha_n\}}$. Since $\sigma_{ap}(A)$ is closed, $\sigma_{ap}(A) = \overline{\{\alpha_n\}} = \sigma(A)$.

(3) Since $\sigma_c(A) = \sigma(A) - (\sigma_p(A) \cup \sigma_{com}(A))$ and $\sigma_p(A) = \sigma_{com}(A)$, $\sigma_c(A) = \overline{\{\alpha_n\}} - \{\alpha_n\}$. □

Lemma 4.6. *Let* (α_n) *be a bounded sequence of complex field* $\mathbb{C}$. *Define* $T : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ *by* $T(x_0, x_1, x_2, \cdots) = (\alpha_0 x_0, \alpha_1 x_1, \frac{\alpha_2 x_2}{2}, \frac{\alpha_3 x_3}{3}, \cdots)$. *Then* T *and* T^* *are compact linear operators.*

Proof. Let M be a positive number such that $|\alpha_n| \leq M$ for all n. Define the operator $T_n : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ by

$$T_n(x_0, x_1, x_2, x_3, \cdots) = (\alpha_0 x_0, \alpha_1 x_1, \frac{\alpha_2 x_2}{2}, \frac{\alpha_3 x_3}{3}, \cdots, \frac{\alpha_n x_n}{n}, 0, \cdots).$$

Then T_n is a bounded linear operator of finite rank and so T_n is compact.

For any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\frac{M}{N+1} < \varepsilon$. Thus if $n > N$,

$$\|(T - T_n)x\|^2 = \sum_{k=n+1}^{\infty} |\frac{\alpha_k x_k}{k}|^2 \le \sum_{k=n+1}^{\infty} (\frac{M}{k})^2 |x_k|^2 = M^2 \sum_{k=n+1}^{\infty} \frac{|x_k|^2}{k^2}$$

$$\le \frac{M^2}{(n+1)^2} \sum_{k=n+1}^{\infty} |x_k|^2 \le (\frac{M}{n+1})^2 \|x\|^2 < \varepsilon^2 \|x\|^2$$

for all $x = (x_n) \in l^2(\mathbb{C})$. Thus $\|T - T_n\| \to 0$ as $n \to \infty$ and so T is compact. By [4], T^* is compact. □

From Theorem 4.5, we have the following result.

Corollary 4.7. *Let (α_n) be a bounded sequence of complex field $\mathbb{C}$. Define $T : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ by $T(x_0, x_1, x_2, \cdots) = (\alpha_0 x_0, \alpha_1 x_1, \frac{\alpha_2 x_2}{2}, \frac{\alpha_3 x_3}{3}, \cdots)$. Then $\sigma_p(T) = \{\alpha_0, \frac{\alpha_n}{n} : n = 1, 2, \cdots\}$, $\sigma(T) = \{0\} \cup \sigma_p(T)$, and $\sigma_{ap}(T) = \sigma(T)$, $\sigma_r(T) = \phi$, $\sigma_{com}(T) = \sigma_p(T)$ and $\sigma_c(T) = \{0\}$.*

Proof. Let $\lambda \in \sigma_p(T)$. Then there exists a non-zero elements $x = (x_n)$ in $l^2(\mathbb{C})$ such that $Tx = \lambda x$. Thus $(\alpha_0 x_0, \alpha_1 x_1, \frac{\alpha_2 x_2}{2}, \frac{\alpha_3 x_3}{3}, \cdots) = (\lambda x_0, \lambda x_1, \lambda x_2, \lambda x_3, \cdots)$, that is, $\alpha_0 x_0 = \lambda x_0$, $\frac{\alpha_n x_n}{n} = \lambda x_n$ for all $n = 1, 2, \cdots$. So $\lambda = \frac{\alpha_m}{m}$ for some $m \in \mathbb{N}$ or $\lambda = \alpha_0$. Therefore we have $\sigma_p(T) \subset \{\alpha_0, \frac{\alpha_n}{n} : n = 1, 2, \cdots\}$. Let $\lambda = \frac{\alpha_m}{m}$ or α_0. If $e_m = (0, \cdots, 0, 1, 0, \cdots)$, then $x \in l^2(\mathbb{C})$ and $Te_m = \lambda e_m$. So $\lambda \in \sigma_p(T)$ and $\{\alpha_0, \frac{\alpha_n}{n} : n = 1, 2, \cdots\} \subset \sigma_p(T)$. Thus $\sigma_p(T) = \{\alpha_0, \frac{\alpha_n}{n} : n = 1, 2, \cdots\}$. Since T is compact, $\sigma(T) - \{0\} = \sigma_p(T)$ and $0 \in \sigma(T)$. We have $\sigma(T) = \{0\} \cup \sigma_p(T)$. Since $\partial\sigma(T) \subset \sigma_{ap}(T) \subset \sigma(T)$, $\sigma_p(T) \subseteq \sigma_{ap}(T)$ and $\sigma_{ap}(T)$ is a closed subset of $\mathbb{C}$, $\sigma_{ap}(T) = \sigma(T)$. We know that $T^*(x_0, x_1, x_2, \cdots) = (\overline{\alpha_0} x_0, \overline{\alpha_1} x_1, \frac{\overline{\alpha_2}}{2} x_2, \cdots)$. Thus $\sigma_p(T^*)^* = \{\overline{\alpha_0}, \frac{\overline{\alpha_n}}{n} : n = 1, 2, \cdots\}^* = \{\alpha_0, \frac{\alpha_n}{n} : n = 1, 2, \cdots\} = \sigma_p(T)$. Hence $\sigma_{com}(T) = \sigma_p(T)$, $\sigma_r(T) = \sigma_{com}(T) - \sigma_p(T) = \phi$ and $\sigma_c(T) = \sigma(T) - (\sigma_p(T) \cup \sigma_{com}(T)) = \{0\}$. □

Theorem 4.8. *Suppose* $\{|\alpha_n|\}$ *is bounded and* $\sup_j |\alpha_j| = r < \infty$, *and let* $w = (w_0, w_1, \cdots) = (1, \frac{1}{q}\frac{1}{q^2}, \cdots)$ $(q > 1)$. *Let* D *be a diagonal operator such that* $D : l^2_w(\mathbb{C}) \to l^2_w(\mathbb{C})$ *defined by* $D(x_0, x_1, \cdots) = (\alpha_0 x_0, \alpha_1 x_1, \cdots)$. *Then we have the followings.*

(1) $\sigma(D) = \overline{\{\alpha_n\}} = \sigma_{ap}(D)$.

(2) $\sigma_p(D) = \{\alpha_n\} = \sigma_{com}(D)$.

(3) $\sigma_r(D) = \sigma_{com}(D) - \sigma_p(D) = \phi$ *and* $\sigma_c(D) = \overline{\{\alpha_n\}} - \{\alpha_n\}$.

Proof. Since

$$\|Dx\|_w^2 = \sum_{j=0}^{\infty} w_j |\alpha_j x_j|^2 = \sum_{j=0}^{\infty} \frac{1}{q^j} |\alpha_j x_j|^2$$
$$\leq (\sup_j |\alpha_j|) \sum_{j=0}^{\infty} \frac{1}{q^j} |x_j|^2 = r^2 \|x\|_w^2 < \infty,$$

$\|D\| \leq r = \sup_j |\alpha_j|$. Note that $\|e_i\|_w = \frac{1}{\sqrt{q^i}}$, $\|\sqrt{q^i} e_i\|_w = 1$ for $e_i = (0, \cdots, 0, 1, 0, \cdots) \in l^2_w(\mathbb{C})$. Thus

$$\|D(\sqrt{q^i} e_i)\|_w = \|D(0, \cdots, 0, \sqrt{q^i}, 0, \cdots)\|_w = \|(0, \cdots, 0, \alpha_i \sqrt{q^i}, 0, \cdots)\|_w$$
$$= (\frac{1}{q^i} |\alpha_i \sqrt{q^i}|^2)^{1/2} = |\alpha_i|.$$

Hence $\|D\| = \sup_{\|x\|_w = 1} \|Dx\|_w = \sup_i |\alpha_i| = r$.

(1) By Theorem 4.4, $D - \lambda$ is invertible if and only if $\{\alpha_n - \lambda\}$ is invertible. That is, $\{\alpha_n - \lambda\}$ is invertible if and only if $\{\alpha_n - \lambda\}$ is bounded away from zero if and only if $|\alpha_n - \lambda| \nrightarrow 0$ as $n \to \infty$ if and only if λ is not a limit point of $\{\alpha_n\}$ if and only if $\lambda \notin \overline{\{\alpha_n\}}$. So $\overline{\{\alpha_n\}} = \sigma(D)$. For any $x = (x_n)$, $y = (y_n) \in l^2_w(\mathbb{C})$,

$$< Dx, y > = < (\alpha_0 x_0, \alpha_1 x_1, \cdots), (y_0, y_1, \cdots) >$$
$$= \alpha_0 x_0 \overline{y_0} + \frac{1}{q} \alpha_1 x_1 \overline{y_1} + \cdots$$
$$= < (x_0, x_1, \cdots), (\overline{\alpha_0} y_0, \overline{\alpha_1} y_1, \cdots) > = < x, D^* y >$$

where D^* is a diagonal operator with diagonal $\{\overline{\alpha_n}\}$. Thus

$$\begin{aligned} DD^*x &= D(\overline{\alpha_0}x_0, \overline{\alpha_1}x_1, \cdots) = (\alpha_0\overline{\alpha_0}x_0, \alpha_1\overline{\alpha_1}x_1, \cdots) \\ &= (|\alpha_0|^2x_0, |\alpha_1|^2x_1, \cdots) = (\overline{\alpha_0}\alpha_0x_0, \overline{\alpha_1}\alpha_1x_1, \cdots) = D^*Dx. \end{aligned}$$

Hence $DD^* = D^*D$, i.e., D is a normal operator, and so $\sigma_{ap}(D) = \sigma(D) = \overline{\{\alpha_n\}}$.

(2) Suppose that $\{\psi_j\}_{j=0}^{\infty}$ is an orthonormal basis for $l_w^2(\mathbb{C})$. Then $\psi_k = (0, 0, \cdots, 0, \sqrt{q^k}, 0, \cdots)$ and $D\psi_k = (0, \cdots, 0, \alpha_k\sqrt{q^k}, 0, \cdots) = \alpha_k(0, \cdots, 0, \sqrt{q^k}, 0, \cdots) = \alpha_k\psi_k$. Thus $\alpha_k = \lambda$ is an eigenvalue of D. Also if $Dx = \lambda x$ $(x \neq 0)$, then $((\alpha_0 - \lambda)x_0, (\alpha_1 - \lambda)x_1, \cdots) = (0, 0, \cdots)$. Since $x \neq 0$, at least one $x_i \neq 0$. So at least one $\alpha_i - \lambda = 0$. Then $\lambda = \alpha_i$ for some $\alpha_i \in \{\alpha_n\}$. Thus $Dx = \lambda x$ $(x \neq 0)$ if and only if $\lambda = \alpha_k$ for some $\alpha_k \in \{\alpha_n\}$. Therefore we have $\sigma_p(D) = \{\alpha_n\} = \sigma_{com}(D)$, and $\sigma_r(D) = \sigma_{com}(D) - \sigma_p(D) = \phi$. Also $\sigma_c(D) = \sigma(D) - (\sigma_p(D) \cup \sigma_{com}(D)) = \overline{\{\alpha_n\}} - \{\alpha_n\}$. □

Corollary 4.9. *Define $A : l^2(\mathbb{C}) \to l^2(\mathbb{C})$ is defined by $A(x_0, x_1, \cdots) = (\alpha_0x_0, \alpha_1x_1, \cdots)$. Suppose that $\{|\alpha_n|\}$ is bounded with $\sup_j |\alpha_j| = r < \infty$. Then we have the followings.*

(1) *$\sigma(A)$ is closure of $\{\alpha_n\}$.*

(2) *$\sigma_p(A) = \sigma_{com}(A) = \{\alpha_n\}$ and $\sigma_{ap}(A)$ is the closure of $\{\alpha_n\}$.*

5. Weighted Operator Shift on the $l^2_w(H)$ Space

Let H be a Hilbet space and let $l^2(H)$ be the set of all sequences $(x_n)_{n=0}^{\infty}$ in H such that $\sum_{n=0}^{\infty}\|x_n\|^2 < \infty$. Then we show that $l^2(H)$ becomes a Hilbert space. Suppose that (A_n) is a uniformly bounded sequence for all $A_n \in B(H)$. The unilateral shift operator associated with (A_n) is an operator $T : l^2(H) \to l^2(H)$ defined by $T(x_0, x_1, \cdots) = (0, A_0x_0, A_1x_1, \cdots)$. We write $T = (A_n)$ for this operator T. Then

(1) the norm of T is given by $\|T\| = \sup_n \|A_n\|$, and

(2) the adjoint operator of T is given by $T^* : l^2(H) \to l^2(H)$, $T^*(x_0, x_1, \cdots) = (A_0^*x_1, A_1^*x_2, \cdots)$.

For (1), note that

$$\begin{aligned}\|T\| &= \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \|(0, A_0x_0, A_1x_1, \cdots)\| \\ &= \sup_{\|x\|=1} (\sum_{j=0}^{\infty} \|A_jx_j\|^2)^{\frac{1}{2}} \le \sup_{\|x\|=1} \left(\sup_n \|A_n\|^2 \sum_{j=0}^{\infty} \|x_j\|^2\right)^{\frac{1}{2}} \\ &= \sup_n \|A_n\| \cdot \sup_{\|x\|=1} (\sum_{j=0}^{\infty} \|x_j\|^2)^{\frac{1}{2}} = \sup_n \|A_n\|,\end{aligned}$$

that is, $\|T\| \le \sup_n \|A_n\| < \infty$. For all x with $\|x\| = 1$, $z = (0, \cdots, 0, x, 0, \cdots) \in l^2(H)$, and $\|A_kx\| = \|(0, \cdots, 0, A_kx, 0, \cdots)\| \le \|Tz\|$ for any k. Thus for any k, $\|A_k\| = \sup_{\|x\|=1} \|A_kx\| \le \sup_{\|z\|=1} \|Tz\| = \|T\|$ and so $\sup_k \|A_k\| \le \|T\|$. Therefore $\|T\| = \sup_n \|A_n\|$.

For (2), we know that

$$\begin{aligned}< Tx, y > &=< (0, A_0x_0, A_1x_1, \cdots), (y_0, y_1, \cdots) > \\ &=< x_0, A_0^*y_1 > + < x_1, A_1^*y_2 > + \cdots \\ &=< (x_0, x_1, \cdots), (A_0^*y_1, A_1^*y_2, \cdots) >,\end{aligned}$$

for each $x = (x_n)$, $y = (y_n) \in l^2(H)$. Since this holds for any $x = (x_n) \in l^2(H)$, $T^*(y_0, y_1, \cdots) = (A_0^* y_1, A_1^* y_2, \cdots)$, for all $y = (y_n) \in l^2(H)$. □

Theorem 5.1. *Let $w = (w_0, w_1, \cdots)$ where $w_j > 0$ and let $l_w^2(H)$ be the set of all sequences $x = (x_0, x_1, \cdots)$ of vectors in H such that $\sum_{j=0}^{\infty} w_j \|x_j\|^2 < \infty$. Define an inner product of vectors $x = (x_n)$ and $y = (y_n)$ by $< x, y >= \sum_{j=0}^{\infty} w_j < x_j, y_j >$. Then $l_w^2(H)$ becomes a Hilbert space.*

Proof. Clearly $l_w^2(H)$ is a normed space. Let (z_n) be any Cauchy sequence in the space $l_w^2(H)$ where $z_n = (\alpha_0^{(n)}, \alpha_1^{(n)}, \alpha_2^{(n)}, \cdots)$. Then for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\|z_n - z_m\|_w = (\sum_{j=0}^{\infty} w_j \|\alpha_j^{(n)} - \alpha_j^{(m)}\|^2)^{\frac{1}{2}} < \varepsilon, \tag{5.1}$$

for all $m, n \geq N$. It follows that for every $j = 0, 1, 2, \cdots$, we have $\|(\alpha_j^{(n)} - \alpha_j^{(m)})\sqrt{w_j}\| < \varepsilon$, for all $m, n \geq N$. That is, for all $m, n \geq N, \|\alpha_j^{(n)} - \alpha_j^{(m)}\| < \varepsilon w_j^{-\frac{1}{2}}$. For each j, $(\alpha_j^{(m)})_{m=0}^{\infty}$ is a Cauchy sequence in H. Let $\beta_j = \lim_{m \to \infty} \alpha_j^{(m)}$. Using this limits, we define $z = (\beta_0, \beta_1, \beta_2, \cdots)$ and show that $z \in l_w^2(H)$ and $\|z_n - z\|_w \to 0$ as $n \to \infty$. From (5.1), we have for all $n, m \geq N$,

$$(\sum_{j=0}^{k} w_j \|\alpha_j^{(n)} - \alpha_j^{(m)}\|^2)^{\frac{1}{2}} < \varepsilon, \quad (k = 0, 1, 2, 3, \cdots).$$

Letting $m \to \infty$, we obtain for $n \geq N$,

$$(\sum_{j=0}^{k} w_j \|\alpha_j^{(n)} - \beta_j\|^2)^{\frac{1}{2}} \leq \varepsilon, \quad (k = 0, 1, 2, 3, \cdots).$$

Letting $k \to \infty$, then for $n \geq N$,

$$\|z_n - z\|_w = (\sum_{j=0}^{\infty} w_j \|\alpha_j^{(n)} - \beta_j\|^2)^{\frac{1}{2}} \leq \varepsilon.$$

This implies that $z_n \to z$ and $z_n - z = (\alpha_j^{(n)} - \beta_j) \in l_w^2(H)$. Since $z_n \in l_w^2(H)$, we have $z = (z - z_n) + z_n \in l_w^2(H)$. Therefore $l_w^2(H)$ is a Banach space. It suffices to show that the norm satisfies the parallelogram law: for any $x = (x_0, x_1, \cdots),\ y = (y_0, y_1, \cdots) \in l_w^2(H)$,

$$\begin{aligned}
\|x+y\|_w^2 + \|x-y\|_w^2 &= \sum_{j=0}^{\infty} w_j \|x_j + y_j\|^2 + \sum_{j=0}^{\infty} w_j \|x_j - y_j\|^2 \\
&= 2\sum_{j=0}^{\infty} w_j \|x_j\|^2 + 2\sum_{j=0}^{\infty} w_j \|y_j\|^2 \\
&= 2\|x\|_w^2 + 2\|y\|_w^2.
\end{aligned}$$

Hence $l_w^2(H)$ is a Hilbert space. □

Lemma 5.2. *Let $q > 0$ be given and $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$. Define an operator $T : l_w^2(H) \to l_w^2(H)$ by $T(x_0, x_1, \cdots) = (0, A_0 x_0, A_1 x_1, \cdots)$, where (A_n) is a uniformly bounded sequence in $B(H)$. Then*

(1) $\|T\| = \frac{1}{\sqrt{q}} \sup_k \|A_k\|$ *and*

(2) $T^*(y_0, y_1, \cdots) = \frac{1}{q}(A_0^* y_1, A_1^* y_2, \cdots)$ *for* $y = (y_n) \in l_w^2(H)$.

Proof. (1) Note that

$$\begin{aligned}
\|Tx\|_w^2 &= \|(0, A_0 x_0, A_1 x_1, \cdots)\|_w^2 \\
&= \frac{1}{q}(\|A_0 x_0\|^2 + \frac{1}{q}\|A_1 x_1\|^2 + \cdots) \\
&\le \frac{1}{q}(c^2\|x_0\|^2 + \frac{1}{q}c^2\|x_1\|^2 + \cdots) \\
&= \frac{1}{q}c^2\|x\|_w^2,
\end{aligned}$$

where $c = \sup_n \|A_n\|$. Thus

$$\|T\| = \sup_{\|x\|_w = 1} \|Tx\|_w \le \sup_{\|x\|_w = 1} (\frac{1}{\sqrt{q}} c\|x\|_w) = \frac{1}{\sqrt{q}} c = \frac{1}{\sqrt{q}} \sup_n \|A_n\|.$$

For each $x \in H$ with $\|x\| = 1$, let $z = (0, \cdots, 0, q^{\frac{k}{2}}x, 0, \cdots)$. Then $\|z\|_w^2 = \frac{1}{q^k}\|q^{\frac{k}{2}}x\|^2 = \frac{1}{q^k}q^k\|x\|^2 = 1$ and $\|Tz\|_w^2 = \|(0, \cdots, 0, A_k(q^{\frac{k}{2}}x), 0, \cdots)\|_w^2 = \frac{1}{q^{k+1}}\|A_k(q^{\frac{k}{2}}x)\|^2 = \frac{1}{q^{k+1}}q^k\|A_kx\|^2 = \frac{1}{q}\|A_kx\|^2$, i.e., $\|Tz\|_w = \frac{1}{\sqrt{q}}\|A_kx\|$. Thus $\|T\| = \sup_{\|y\|_w=1}\|Ty\|_w \geq \sup_{\|z\|_w=1}\frac{1}{\sqrt{q}}\|A_kx\| \geq \frac{1}{\sqrt{q}}\sup_{\|x\|=1}\|A_kx\| = \frac{1}{\sqrt{q}}\|A_k\|$. Hence $\frac{1}{\sqrt{q}}\sup_n\|A_n\| \leq \|T\|$.

(2) For all $x = (x_n),\ y = (y_n) \in l_w^2(H)$,

$$\begin{aligned} < Tx, y > &=< (0, A_0x_0, A_1x_1, \cdots), (y_0, y_1, \cdots) > \\ &= \frac{1}{q} < x_0, A_0^*y_1 > + \frac{1}{q^2} < x_1, A_1^*y_2 > + \cdots \\ &= \frac{1}{q} < (x_0, x_1, x_2, \cdots), (A_0^*y_1, A_1^*y_2, \cdots) > . \end{aligned}$$

Since this holds for any $x = (x_n) \in l_w^2(H)$, $T^*(y_0, y_1, \cdots) = \frac{1}{q}(A_0^*y_1, A_1^*y_2, \cdots)$ for all $y = (y_0, y_1, \cdots) \in l_w^2(H)$. □

Proposition 5.3. *Let $q > 0$ be given and $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$. Let $\{U_n\}_{n=0}^\infty$ be a sequence of unitary operators on H. Then $T = (A_n)$ on $l_w^2(H)$ is unitarily equivalent to the weighted shift operator with the weight sequence $\{U_{n+1}^*A_nU_n\}_{n=0}^\infty$.*

Proof. Let $Ux = (U_0x_0, U_1x_1, \cdots)$ for all $x = (x_n) \in l_w^2(H)$. Then

$$\begin{aligned} \|Ux\|_w = \|(U_0x_0, U_1x_1, \cdots)\|_w &= \sqrt{\sum_{n=0}^\infty \frac{1}{q^n}\|U_nx_n\|^2} \\ &= \sqrt{\sum_{n=0}^\infty \frac{1}{q^n}\|x_n\|^2} = \|x\|_w. \end{aligned}$$

Since for all $x = (x_n),\ y = (y_n) \in l_w^2(H)$,

$$\begin{aligned} < Ux,\ y > &= < (U_0x_0, U_1x_1, \cdots),\ (y_0, y_1, \cdots) > \\ &= < U_0x_0, y_0 > + \frac{1}{q} < U_1x_1, y_1 > + \cdots \\ &= < x_0, U_0^*y_0 > + \frac{1}{q} < x_1, U_1^*y_1 > + \cdots \\ &= < (x_0, x_1, \cdots),\ (U_0^*y_0, U_1^*y_1, \cdots) >, \end{aligned}$$

we have $U^*(y_0, y_1, \cdots) = (U_0^*y_0, U_1^*y_1, \cdots)$ and $U^*Uy = U^*(U_0y_0, U_1y_1, \cdots)$ $= (U_0^*U_0y_0, U_1^*U_1y_1, \cdots) = (y_0, y_1, \cdots) = y$, for all $y \in l_w^2(H)$. Then U is a unitary operator on $l_w^2(H)$. Hence

$$\begin{aligned} U^*TUx = U^*TU(x_0, x_1, \cdots) &= U^*T(U_0x_0, U_1x_1, \cdots) \\ &= U^*(0, A_0U_0x_0, A_1U_1x_1, \cdots) \\ &= (0, U_1^*A_0U_0x_0, U_2^*A_1U_1x_1, \cdots, U_{n+1}^*A_nU_nx_n, \cdots), \end{aligned}$$

for all $x \in l_w^2(H)$. □

Corollary 5.4. *Suppose that* $\{U_n\}_{n=0}^\infty$ *is a sequence of unitary operators on* H. *Then* $T = (A_n) : l^2(H) \to l^2(H)$ *the unilateral shift operator associated with* (A_n) *is unitarily equivalent to the weighted shift operator with the weight sequence* $\{U_{n+1}^*A_nU_n\}_{n=0}^\infty$.

Proposition 5.5. *Let* $q > 0$ *be given and* $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$. *Suppose that* $T = (A_n) : l_w^2(H) \to l_w^2(H)$ *is the unilateral shift operator associated with* (A_n) *and that* A_n *is invertible for each* n. *Then* T *is unitarily equivalent to a weighted shift operator* $\widetilde{T}$ *with a weight sequence* $\{B_n\}_{n=0}^\infty$ *of positive operators.*

Proof. For each n, let $A_n = W_n|A_n|$ be the polar decomposition of A_n. Since A_n is invertible, W_n is unitary. From Proposition 5.3, T is unitarily equivalent to the weighted shift operator with weight $U_{n+1}^*W_n|A_n|U_n$. Choose

$U_0 = I, U_n = W_{n-1}U_{n-1}$ for $n \geq 1$ and set $B_n = U_{n+1}^* W_n|A_n|U_n$ for $n \geq 0$. We claim that B_n is a positive operator for $n \geq 0$.

Case 1. $n = 0$. For each $x \in H$, $< B_0x,\ x > = < U_1^* W_0|A_0|U_0x,\ x > = < (W_0U_0)^* W_0|A_0|U_0x,\ x > = < U_0^* W_0^* W_0|A_0|U_0x,\ x > = < |A_0|x, x > \geq 0$.

Case 2. $n > 0$. For each $y \in H$, $< B_ny,\ y > = < U_{n+1}^* W_n|A_n|U_ny,\ y > = < (W_nU_n)^* W_n|A_n|U_ny,\ y > = < U_n^* W_n^* W_n|A_n|U_ny,\ y > = < U_n^*|A_n|U_n y,\ y > = < |A_n|(U_ny),\ (U_ny) > \geq 0$.

Hence T is unitarily equivalent to a weighted shift operator $\widetilde{T} = (B_n)$ with $\{B_n\}_{n=0}^{\infty}$ sequence of positive operators. □

Corollary 5.6. *Suppose that $T = (A_n) : l^2(H) \to l^2(H)$ is the unilateral shift operator associated with (A_n) and that A_n is invertible for each n. Then T is unitarily equivalent to an operator weighted shift $\widetilde{T}$ with weight sequence $\{B_n\}_{n=0}^{\infty}$ of positive operators.*

Proposition 5.7. *Let $q > 0$ be given and $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$. Let $T = (A_n)$ on $l_w^2(H)$. Then T is unitarily equivalent to $\widetilde{T} = e^{i\theta}T$.*

Proof. Define the unitary opetator U on $l_w^2(H)$ by $U(x_0, x_1, \cdots) = (x_0, e^{i\theta}x_1, e^{2i\theta}x_2, \cdots)$. Then

$$\begin{aligned}\|Ux\|_w &= \|(x_0, e^{i\theta}x_1, e^{2i\theta}x_2, \cdots)\|_w \\ &= (\|x_0\|^2 + \frac{1}{q}\|e^{i\theta}x_1\|^2 + \frac{1}{q^2}\|e^{2i\theta}x_2\|^2 + \cdots)^{\frac{1}{2}} \\ &= \|x\|_w (< \infty)\end{aligned}$$

and

$$\begin{aligned} < Ux, y > &= < (x_0, e^{i\theta}x_1, e^{2i\theta}x_2, \cdots), \ (y_0, y_1, y_2, \cdots) > \\ &= < x_0, y_0 > + \frac{1}{q} < x_1, \ e^{-i\theta}y_1 > + \frac{1}{q^2} < x_2, \ e^{-2i\theta}y_2 > + \cdots \\ &= < (x_0, x_1, x_2, \cdots), \ (y_0, e^{-i\theta}y_1, e^{-2i\theta}y_2, \cdots) >, \end{aligned}$$

for all $x = (x_n)$, $y = (y_n) \in l^2_w(H)$. Therefore $U^*(y_0, y_1, y_2, \cdots) = (y_0, e^{-i\theta}y_1, e^{-2i\theta}y_2, \cdots)$ and $U^*Uy = U^*(y_0, e^{i\theta}y_1, e^{2i\theta}y_2, \cdots) = (y_0, e^{-i\theta}e^{i\theta}y_1, e^{-2i\theta}e^{2i\theta}y_2, \cdots) = (y_0, y_1, y_2, \cdots)$ for all $y = (y_n) \in l^2_w(H)$. So U is a unitary operator on $l^2_w(H)$ and

$$\begin{aligned} U^*\widetilde{T}Ux &= U^*e^{i\theta}TU(x_0, x_1, \cdots) \\ &= U^*e^{i\theta}T(x_0, e^{i\theta}x_1, e^{2i\theta}x_2, \cdots) \\ &= U^*e^{i\theta}(0, A_0x_0, A_1e^{i\theta}x_1, A_2e^{2i\theta}x_2, \cdots) \\ &= U^*(0, e^{i\theta}A_0x_0, e^{2i\theta}A_1x_1, \cdots) \\ &= (0, e^{-i\theta}e^{i\theta}A_0x_0, e^{-2i\theta}e^{2i\theta}A_1x_1, \cdots) \\ &= (0, A_0x_0, A_1x_1, \cdots) = Tx, \end{aligned}$$

for all $x = (x_n) \in l^2_w(H)$. Hence $U^*\widetilde{T}U = T$. □

Corollary 5.8. *Let $T = (A_n) : l^2(H) \to l^2(H)$ be the unilateral shift operator associated with (A_n). Then T is unitarily equivalent to $\widetilde{T} = e^{i\theta}T$.*

Proposition 5.9. *Let $q > 0$ be given and $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$. Suppose that $\{A_n\}_{n=0}^{\infty}$ is a sequence of uniformly bounded operators on H such that S is a diagonal operator on $l^2_w(H)$ defined by $S(x_0, x_1, \cdots) = (A_0x_0, A_1x_1, \cdots)$. Then S is a compact operator if and only if A_n is a compact operator and $\lim_{n\to\infty} \|A_n\| = 0$.*

Proof. Suppose A_n is a compact operator and $\|A_n\| \to 0$ as $n \to \infty$. Let $S_n(x_0, x_1, \cdots) = (A_0x_0, A_1x_1, \cdots, A_{n-1}x_{n-1}, 0, \cdots)$. Then S_n is a compact operator for each n, since S_n is finite rank. Since $\|A_n\| \to 0$ as $n \to \infty$, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|A_n\| < \varepsilon$ for any $n \geq N$. Thus for all $n \geq N$,

$$\begin{aligned}
\|S - S_n\| &= \sup_{\|x\|_w = 1} \|(S - S_n)x\|_w \\
&= \sup_{\|x\|_w = 1} \|(0, \cdots, 0, A_nx_n, A_{n+1}x_{n+1}, \cdots)\|_w \\
&= \sup_{\|x\|_w = 1} (\frac{1}{q^n}\|A_nx_n\|^2 + \frac{1}{q^{n+1}}\|A_{n+1}x_{n+1}\|^2 + \cdots)^{\frac{1}{2}} \\
&\leq \sup_{\|x\|_w = 1} (\frac{1}{q^n}\|A_n\|^2\|x_n\|^2 + \frac{1}{q^{n+1}}\|A_{n+1}\|^2\|x_{n+1}\|^2 + \cdots)^{\frac{1}{2}} \\
&\leq \sup_{\|x\|_w = 1} (\varepsilon^2\frac{1}{q^n}\|x_n\|^2 + \varepsilon^2\frac{1}{q^{n+1}}\|x_{n+1}\|^2 + \cdots)^{\frac{1}{2}} \\
&\leq \varepsilon^2 \sup_{\|x\|_w = 1} \|x\|_w = \varepsilon^2.
\end{aligned}$$

That is, $\|S - S_n\| \longrightarrow 0$ as $n \to \infty$. Hence S is a compact operator since S_n is compact.

Conversely, suppose that S is a compact operator. For each $i \geq 0$ let $\{x_n^{(i)}\}_{n=0}^{\infty}$ be a sequence in H such that $x_n^{(i)} \to 0$ weakly as $n \to \infty$ and $y_n^{(i)} = (0, \cdots, 0, x_n^{(i)}, 0, \cdots)$ where $x_n^{(i)}$ is the i-th coordinate. Note that

$$\begin{aligned}
< y_n^{(i)}, y > &=< (0, \cdots, 0, x_n^{(i)}, 0, \cdots), (y_0, y_1, \cdots, y_i, \cdots) > \\
&= \frac{1}{q^i} < x_n^{(i)}, y_i > \to \frac{1}{q^i} < 0, y_i >=< 0, y_i >,
\end{aligned}$$

for each $y = (y_n) \in l_w^2(H)$. Thus $< y_n^{(i)},\ y > \to < 0,\ y >$ and then $y_n^{(i)} \to 0$ weakly as $n \to \infty$. Since S is a compact operator, $\{Sy_n^{(i)}\}$ has a convergent subsequence $\{Sy_m^{(i)}\}$ with $\|y_m^{(i)}\| = 1$ such that $\|Sy_m^{(i)} - l\| \longrightarrow 0$ as

$m \to \infty$. Then $\|(0,\cdots,0,A_i x_m^{(i)},0,\cdots) - l\| \longrightarrow 0$ as $m \to \infty$ and so $\frac{1}{\sqrt{q^i}}\|A_i x_m^{(i)}\| \longrightarrow \|l\|$ as $m \to \infty$. That is, $\|A_i \frac{x_m^{(i)}}{\sqrt{q^i}}\|$ converges as $m \to \infty$ with $\|\frac{x_m^{(i)}}{\sqrt{q^i}}\| = 1$. Thus A_i is a compact operator. Since $\{A_i\}_{i=0}^{\infty}$ is a sequence of compact operators on H, a sequence $\{\frac{f_i}{\sqrt{q^i}}\}_{i=0}^{\infty}$ of unit vectors can be chosen such that $\|A_i\| = \|A_i \frac{f_i}{\sqrt{q^i}}\|$. In fact, $\|A_i f_i\| \leq \|A_i\|$ with $\|f_i\| = 1$. Suppose that there is no x such that $\|A_i\| = \|A_i x\|$, i.e.,

$$\|A_i x\| < \|A_i\|. \tag{5.2}$$

For all i, there exists a sequence of unit vectors f_{i_n} such that $\|A_i f_{i_n}\| > \|A_i\| - \frac{1}{n}$. Since A_i is a compact operator, $\{A_i f_{n_k}\}$ converges with $\|f_{n_k}\| = 1$ and $\lim_{k\to\infty} A_i f_{n_k} = A_i(\lim_{k\to\infty} f_{n_k}) = A_i f_i$ with $\|f_i\| = 1$. If $\|A_i f_{n_k}\| > \|A_i\| - \frac{1}{n_k}$, then $\|A_i f_i\| = \|\lim_{k\to\infty} A_i f_{n_k}\| \geq \|A_i\| - \lim_{k\to\infty} \frac{1}{n_k} = \|A_i\|$. This is a contradiction to (5.2). Let $g_n = (0,\cdots,0,f_n,0,\cdots)$. Since $| < g_n, y > | = |\frac{1}{q^n} < f_n, y_n > | \leq \frac{1}{q^n}\|f_n\|\|y_n\| = \frac{1}{\sqrt{q^n}}\|y_n\| \to 0$ as $n \to \infty$, $g_n \to 0$ weakly. Thus $\|Sg_n\|_w = < Sg_n, Sg_n >^{\frac{1}{2}} = < g_n, S^*Sg_n >^{\frac{1}{2}} \longrightarrow 0$. Therefore $\|Sg_n\|_w \longrightarrow 0$. That is, $Sg_n \to 0$ strongly. Also

$$\begin{aligned}
\|Sg_n\|_w &= < Sg_n, Sg_n >^{\frac{1}{2}} \\
&= < S(0,\cdots,0,f_n,0,\cdots), S(0,\cdots,0,f_n,0,\cdots) >^{\frac{1}{2}} \\
&= < (0,\cdots,0,A_n f_n,0,\cdots), (0,\cdots,0,A_n f_n,0,\cdots) >^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{q^n}} < A_n f_n, A_n f_n >^{\frac{1}{2}} = \frac{1}{\sqrt{q^n}}\|A_n f_n\| \\
&= \|A_n \frac{f_n}{\sqrt{q^n}}\| = \|A_n\| \longrightarrow 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

□

Corollary 5.10. *Suppose that* $\{A_n\}_{n=0}^{\infty}$ *is a sequence of uniformly bounded operators on* H *and* S *is a diagonal operator defined on* $l^2(H)$ *such that*

$S(x_0, x_1, \cdots) = (A_0 x_0, A_1 x_1, \cdots)$. *Then* S *is a compact operator if and only if* A_n *is a compact operator and* $\lim_{n\to\infty} \|A_n\| = 0$.

Corollary 5.11. *Let* $T = (A_n) : l^2(H) \to l^2(H)$ *be the unilateral shift operator associated with* (A_n). *Then* T *is a compact operator if and only if* A_n *is a compact operator and* $\lim_{n\to\infty} \|A_n\| = 0$.

Proof. The operator $(T^*T)^{\frac{1}{2}}$ is a diagonal operator and T is compact if and only if $(T^*T)^{\frac{1}{2}}$ is compact. □

Lemma 5.12. *Suppose that* S *is an operator defined in Corollary 5.10. Then the followings hold.*

(1) $\|S\| = \sup_j \|A_j\|$.

(2) $\sigma_p(S) = \cup_k \sigma_p(A_k)$.

(3) $\sigma(S) = \overline{\cup_k \sigma(A_k)}$.

(4) $S^* = (A_n^*)$.

(5) *If for all* n, A_n *is normal, then* $\sigma_{com}(S) = \sigma_p(S) = \cup_k \sigma_p(A_k)$ *and* $\sigma_r(S) = \phi$.

Proof. (1) Note that

$$\begin{aligned} \|S\| &= \sup_{\|x\|=1} \|Sx\| = \sup_{\|x\|=1} \|(A_0x_0, A_1x_1, \cdots)\| \\ &= \sup_{\|x\|=1} (\sum_{n=0}^{\infty} \|A_n x_n\|^2)^{\frac{1}{2}} \le \sup_{\|x\|=1} (\sum_{n=0}^{\infty} \sup_j \|A_j\|^2 \|x_n\|^2)^{\frac{1}{2}} \\ &= \sup_j \|A_j\| \cdot \sup_{\|x\|=1} (\sum_{n=0}^{\infty} \|x_n\|^2)^{\frac{1}{2}} = \sup_j \|A_j\|. \end{aligned}$$

For any x with $\|x\| = 1, z = (0, \cdots, 0, x, 0, \cdots) \in l^2(H)$, $\|A_k x\| = \|(0, \cdots, 0, A_k x, 0, \cdots)\| \le \|Sx\|$. Thus $\sup_{\|x\|=1} \|A_k x\| \le \sup_{\|z\|=1} \|Sz\| = \|S\|$, i.e., for all k, $\|A_k\| \le \|S\|$. Therefore $\sup_k \|A_k\| \le \|S\|$ and so $\|S\| = \sup_j \|A_j\|$.

(2) If $Sx = \lambda x$ for some nonzero $x \in l^2(H)$, then $(A_0x_0, A_1x_1, \cdots) = (\lambda x_0, \lambda x_1, \cdots)$, i.e., $((A_0 - \lambda)x_0, (A_1 - \lambda)x_1, \cdots) = (0, 0, \cdots)$. Since $x \neq 0$, $x_0, x_1, \cdots$ are not all zero. That is, there exist at least one nonzero x_i, say x_k. Thus $x_k \neq 0$ and $(A_k - \lambda)x_k = 0$. Therefore $\lambda \in \sigma_p(A_k)$ for some k and so we have $\sigma_p(S) \subset \cup_k \sigma_p(A_k)$. Conversely, let $\lambda \in \cup_k \sigma_p(A_k)$. Then there exists a nonzero $x_k \in H$ such that $A_k x_k = \lambda x_k$. Let $x = (0, \cdots, 0, x_k, 0, \cdots) \in l^2(H)$. Then $Sx = S(0, \cdots, 0, x_k, 0, \cdots) = (0, \cdots, 0, A_k x_k, 0, \cdots) = (0, \cdots, 0, \lambda x_k, 0, \cdots) = \lambda(0, \cdots, 0, x_k, 0, \cdots) = \lambda x$. Thus $\lambda \in \sigma_p(S)$ and we have $\cup_k \sigma_p(A_k) \subset \sigma_p(S)$. Hence $\sigma_p(S) = \cup_k \sigma_p(A_k)$.

(3) We show that $S - \lambda$ is not invertible iff $A_k - \lambda$ is not invertible for some k. If $A_k - \lambda$ is invertible for all k, then $(A_i - \lambda)^{-1}(A_i - \lambda)x_i = x_i$ for all i and for all $x = (x_n) \in l^2(H)$. Thus $x = ((A_0 - \lambda)^{-1}(A_0 - \lambda)x_0, (A_1 - \lambda)^{-1}(A_1 - \lambda)x_1, \cdots) = V(S - \lambda)x$, where $V(y_0, y_1, \cdots) = ((A_0 - \lambda)^{-1}y_0, (A_1 - \lambda)^{-1}y_1, \cdots)$. This means $S - \lambda$ is invertible. Hence $\sigma(S) \subseteq \overline{\cup_k \sigma(A_k)}$. Conversely, suppose that $\lambda \notin \sigma(S)$, i.e., $S - \lambda$ is invertible. So there exists $(S - \lambda)^{-1}$ such that $(S - \lambda)^{-1}(S - \lambda) = I$. For each $x = (x_n) \in l^2(H)$, $(S - \lambda)^{-1}(S - \lambda)x = (S - \lambda)^{-1}((A_0 - \lambda)x_0, (A_1 - \lambda)x_1, \cdots) = (x_0, x_1, \cdots)$. Then $(S - \lambda)$ must be the form $(S - \lambda)^{-1}(x_0, x_1, \cdots) = ((A_0 - \lambda)^{-1}x_0, (A_1 - \lambda)^{-1}x_1, \cdots)$. This means that $(A_n - \lambda)$ is invertible for all $n \geq 0$. Thus $\lambda \notin \cup_k \sigma(A_k)$. Then $\cup_k \sigma(A_k) \subseteq \sigma(S)$. Hence $\overline{\cup_k \sigma(A_k)} \subseteq \overline{\sigma(S)} = \sigma(S)$.

(4) Note that for all $x = (x_n)$, $y = (y_n) \in l^2(H)$,

$$
\begin{aligned}
< Sx, y > &= < (A_0x_0, A_1x_1, \cdots), (y_0, y_1, \cdots) > \\
&= < A_0x_0, y_0 > + < A_1x_1, y_1 > + \cdots \\
&= < (x_0, x_1, \cdots), (A_0^* y_0, A_1^* y_1, \cdots) > .
\end{aligned}
$$

Thus $S^* y = (A_0^* y_0, A_1^* y_1, \cdots)$ for all $y = (y_0, y_1, \cdots) \in l^2(H)$.

(5) Since $S^*(x_0, x_1, \cdots) = (A_0^* x_0, A_1^* x_1, \cdots)$ for any $x = (x_n) \in l^2(H)$,

$$S^*Sx = S^*(A_0x_0, A_1x_1, \cdots) = (A_0^*A_0x_0, A_1^*A_1x_1, \cdots) \quad \text{and}$$

$$SS^*x = S(A_0^*x_0, A_1^*x_1, \cdots) = (A_0A_0^*x_0, A_1A_1^*x_1, \cdots).$$

Thus if for all n, A_n is normal, then S is normal. We know that if an operator A is normal, then $\sigma_{com}(A) = \sigma_p(A)$. Thus if for all n, A_n is normal, then $\sigma_{com}(S) = \sigma_p(S) = \cup_k \sigma_p(A_k)$, $\sigma_{ap}(S) = \sigma(S) = \overline{\cup_k \sigma(A_k)}$ and $\sigma_r(S) = \sigma_{com}(S) - \sigma_p(S) = \sigma_p(S) - \sigma_p(S) = \phi$. □

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< Abstract >

Spectra of Weighted Shift Operators

In this thesis, we deal with various spectra of shift operators on the space $l^2(\mathbb{C}), l^2_w(\mathbb{C})$ and $l^2_w(H)$. Here $w = (w_j)$, $w_j > 0$ and H denotes a Hilbert space.

The main results are as follows.

(1) Let $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$ $(q > 0)$. Then the spectrum of left shift operator is the closed disk with radius $\sqrt{q}$ and center 0. Also we calculate the approximate point spectrum, the point spectrum, the residual spectrum, the continuous spectrum and the compression spectrum of the left shift operator and the right shift operator, respectively.

(2) Let $w = (w_j)$ be a bounded sequence of positive numbers. Then the spectrum of right (left) shift operator on $l^2_w(\mathbb{C})$ is the closed disk with radius 1 and center 0. Also we calculate several spectra of these operators.

(3) $T = (A_n)$ is uniformly equivalent to $e^{i\theta}T$ on $l^2_w(H)$ whenever $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$ $(q > 0)$.

(4) Let $w = (1, \frac{1}{q}, \frac{1}{q^2}, \cdots)$ $(q > 0)$ and let S be a diagonal operator with diagonal $\{A_n\}$ where $\{A_n\}$ is uniformly bounded sequence of a bounded operator A_n. Then S is compact iff A_n is compact and $\lim_{n\to\infty} \|A_n\| = 0$.

감 사 의 글

먼저 지금까지 인도하시고 도와주신 하나님께 감사와 영광을 돌려 드립니다. 본 논문이 완성되기까지 부족한 저에게 세심한 지도와 가르침과 격려를 아끼지 않으신 양영오 교수님께 깊은 감사를 드립니다. 자세한 검토와 귀한 조언을 해주신 김도현 교수님과 고윤희 교수님께도 감사를 드립니다. 또한, 논문을 섬세하게 검토해 주시고 부족한 부분을 수정,편집하도록 도와주신 윤용식 교수님께도 감사의 마음을 전합니다. 대학원을 다니는 동안 많은 가르침을 주시고 "하면 된다"는 신념을 갖도록 용기를 주신 송석준 교수님과 여러 교수님들께도 감사의 마음을 전하고 싶습니다. 4학기 동안 서로 배우고 의지하며, 저에게 많은 도움을 준 대학원 동기 최희봉, 김희선 그리고 이진아님들께도 깊은 감사와 사랑의 마음을 전합니다. 무엇보다도 헌신적으로 저를 지원해주시고 변함없이 사랑해주신 부모님께도 감사를 드립니다. 끝으로, 저를 위해 기도하며 끊임없이 격려해주신 UBF의 여러 선·후배님들에게도 감사를 드립니다.

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