碩士學位論文

Second Order Semi – linear Elliptic Boundary Value Problems and Differential Inequalities

濟州大學校 大學院

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Second Order Semi-linear Elliptic Boundary Value Problems and Differential Inequalities

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<국문초록>

이계 반선형인 타원형 경계치 문제와 미분 부등식

본 논문에서는 이미 연구된 이계 선형·반선형인 타원형 경계치 문제의 해의 존재성에 관한 정리들을 소개하고 이를 이용해서 경계가 미분 가능한 정의역 Ω에서 정의된 이계 반선형인 타원형 문제

$$\Delta u(x) + b(x) \cdot \nabla u(x) + f(u(x)) = 0, \qquad x \in \Omega$$

$$u(x) = 0, \qquad x \in \partial\Omega.$$

의 양수인 해의 존재성을 조사한다. 해의 존재성을 밝히기 위해 상계-하계 방법(Upper solution-Lower solution Method) 및 적분 부등식을 이용한다.

CONTENTS

1. Introduction

In this thesis we study the existence of solutions of the second order semi-linear elliptic boundary value problem of the form

$$\begin{aligned} &\Delta u(x) + b(x) \cdot \nabla u(x) + f(u(x)) = 0 \quad \text{for} \quad x \in \Omega \\ &u(x) = 0 \quad \text{for} \quad x \in \partial\Omega. \end{aligned} \tag{I}$$

In Section 2 we introduce basic notations, definitions of some function spaces which are Banach spaces. The definitions of partial differential equations and its order is also mentioned in this Section.

In Section 3, Maximum principles are represented, and we show that the solvability of general second order linear elliptic boundary value problem is equivalent to that of boundary value problem for Laplace equation. We organize theorems on existence and uniqueness of solutions of second order linear elliptic boundary value problems.

In Section 4 we see that the semi-linear elliptic boundary value problem is converted into an equivalent operator equation, and by using Leray-Shauder continuation theorem. We have theorem about the solvability of second order semi-linear elliptic boundary value problems under a Nagumo condition. We also introduce notions of lower and upper solutions of the problems and get an existence result by using them.

In Section 5 we discuss existence of solutions of the semi-linear elliptic boundary value problem of the form (I) by using the result of Section 4 and the integral inequality which is proved in Section 5. At first we show that there exists a solution of (I) where $f(u) > 0$, f is bounded. Then we discuss the existence of solutions of (I) in case that f is unbounded. We can see that see that the problem (I) has positive solutions whenever $0 < f(u) \leq \alpha u + r$, where α and r are suitable constants. Moreover, if $f(0) > 0$ and $0 \leq f(u) \leq \alpha u + r$ on a sutable finite interval, our problem (I) has positive solutions.

The existence and multiplicity of positive solutions to the equation

$$\begin{aligned} &\Delta u(x) + b(x) \cdot \nabla u(x) + \lambda f(x, u(x)) = 0 \quad \text{for} \quad x \in \Omega \\ &u(x) = 0 \quad \text{for} \quad x \in \partial\Omega. \end{aligned} \tag{II}$$

where $f(x, u) > 0$, $\dfrac{d}{du} f(x, u) > 0$, are discussed by Choe and Hernandez ([10]) under suitable assumptions.

We prove in this paper the existence of positive solutions of the problem of the form (I) without assumption of $\dfrac{df}{du} > 0$, and also give another existence theorem in case $f(0) = 0$.

2. Definitions and Preliminaries

We introduce some notations and definitions that we shall use in this paper.

$\mathbf{R}^n$ denotes the Euclidean space, $x = (x_1, x_2, \cdots, x_n)$ denotes an arbitrary point of $\mathbf{R}^n$, $x_i \in \mathbf{R}^1$, $i = 1, 2, \cdots, n$, with the norm $|x| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$.

We say an open connected set a domain.

Ω denotes a bounded domain of $\mathbf{R}^n$, $\partial\Omega$ denotes the boundary of Ω and $\overline{\Omega}$ the closure of Ω.

Let k be a non-negative integer. $C^k(\Omega)$ denotes the set of functions which has all continuous partial derivatives of order $\leq k$ in Ω. $C^k(\overline{\Omega})$ also denotes the set of functions which has all continuous partial derivatives of order $\leq k$ in Ω such that all the derivatives can be extended continuously to the closure $\overline{\Omega}$. $C^0(\Omega)$, $C^0(\overline{\Omega})$ are written by simply $C(\Omega)$, $C(\overline{\Omega})$, respectively. We denote $C_0^k(\Omega)$ the set of functions u in $C^k(\Omega)$ such that u and its all partial derivatives have compact support.

For $u \in C(\overline{\Omega})$ we define the supremum norm

$$\|u\|_{C^0} = \sup_{x \in \overline{\Omega}} |u(x)|.$$

$C^\infty(\overline{\Omega})$ denotes a set of function u which belongs to class $C^k(\overline{\Omega})$ for all $k \geq 1$. $C_0^\infty(\overline{\Omega})$ is the set of functions u of class $C^\infty(\overline{\Omega})$ such that u and its all partial derivatives have compact support.

Definition 2.1. *Let x_0 be a point in $\mathbf{R}^n$ and f a function defined on a bounded subset Ω of $\mathbf{R}^n$ containing x_0. If $0 < \alpha < 1$, we say that f is Hölder* *continuous with exponent α (or, α-Hölder continuous) at x_0 if the quantity*

$$\sup_{\substack{x \in \Omega \\ x \neq x_0}} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha}$$

is finite. We call f (uniformly) Hölder continuous with exponent α in Ω if the quantity

$$\sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite. And f is called locally Hölder continuous with exponent α in Ω if f is uniformly Hölder continuous with exponent α on compact subsets of Ω.

We denote the quantity

$$H_\alpha^\Omega(f) = \sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

and call it Hölder constant for f if it is finite.

We define the Hölder space $C^{k,\alpha}(\overline{\Omega})$ (respectively, $C^{k,\alpha}(\Omega)$) as a subspace of $C^k(\overline{\Omega})$ (respectively, $C^k(\Omega)$) consisting of functions whose k-th order partial derivatives are uniformly Hölder continuous (respectively, locally Hölder continuous) with exponent α in Ω. For simplicity we write $C^{0,\alpha}(\overline{\Omega}) = C^\alpha(\overline{\Omega})$, $C^{0,\alpha}(\Omega) = C^\alpha(\Omega)$.

$\partial\Omega \in C^{2,\alpha}$ means that for every $x \in \partial\Omega$ there exists a neighborhood N of x such that $\partial\Omega \cap N$ can be represented in the form $x_i = h(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$, for some i, where h belongs to class $C^{2,\alpha}$.

We define the norms;

For $f \in C^\alpha(\overline{\Omega})$,

$$\|f\|_\alpha^{\overline{\Omega}} = \|f(x)\|_{C^0} + H_\alpha^{\overline{\Omega}}(f).$$

For $f \in C^{1,\alpha}(\overline{\Omega})$,

$$\|f\|_{1,\alpha}^{\overline{\Omega}} = \|f\|_{\alpha}^{\overline{\Omega}} + \|\partial f\|_{\alpha}^{\overline{\Omega}},$$

where $\partial f = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$. In case f is a scalar function we denote $\partial f = \nabla f$.

For $f \in C^{2,\alpha}(\overline{\Omega})$,

$$\|f\|_{2,\alpha}^{\overline{\Omega}} = \|f\|_{1,\alpha}^{\overline{\Omega}} + \|\partial^2 f\|_{\alpha}^{\overline{\Omega}},$$

where $\partial^2 f = \partial(\partial f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}.$

Remark. We note that

$$\left(C^{\alpha}(\overline{\Omega}), \|\cdot\|_{\alpha}^{\overline{\Omega}}\right), \quad \left(C^{1,\alpha}(\overline{\Omega}), \|\cdot\|_{1,\alpha}^{\overline{\Omega}}\right), \quad \left(C^{2,\alpha}(\overline{\Omega}), \|\cdot\|_{2,\alpha}^{\overline{\Omega}}\right)$$

are all Banach spaces.([2])

For $p \geq 1$, $L^p(\Omega)$ is the set of function f which is Lebesgue-measurable in Ω such that $\int_{\Omega} |f|^p dx < \infty$. For $f \in L^p(\Omega)$, we define

$$\|f\|_{L^p} = \left(\int_{\Omega} |f|^p dx\right)^{\frac{1}{p}}.$$

Then $(L^p(\Omega), \|\cdot\|_{L^p})$ is a Banach space.([8])

Definition 2.2. *A function u is twice weakly differentiable over $\overline{\Omega}$ if there exist integrable functions v_k, w_{jk} for $j, k = 1, 2, \cdots, n$*

$$\int_{\overline{\Omega}} v_k \varphi dx = -\int_{\overline{\Omega}} u \varphi_{x_k}, \quad \int_{\overline{\Omega}} w_{jk} \varphi dx = \int_{\overline{\Omega}} u \varphi_{x_j x_k} dx$$

holds for all $\varphi \in C_0^\infty(\overline{\Omega})$. *The integrable functions are* v_k, w_{jk}, $j, k = 1, 2, \cdots, n$ *considered as generalized first and second derivatives of* u, *respectively.*

The space of twice weakly differentiable functions in Ω is denoted by $W^2(\Omega)$. In the space $W^2(\Omega)$ we shall write the weak-derivatives as u_{x_k}, $u_{x_j x_k}$ without distinguishing from the usual concept of first and second derivatives.

We also define the space $W^{2,p}(\Omega)$ by

$$W^{2,p}(\Omega) = \{u \in W^2(\Omega) | u, u_{x_k}, u_{x_k x_j} \in L^p(\Omega) \text{ for all } j, k = 1, 2, \cdots, n\}.$$

$W^{2,p}(\Omega)$ is a Banach space with the norm defined by ([7])

$$\|u\|_{W^{2,p}} = \Big[\int_\Omega (|u|^p + \sum_{k=1}^n |u_{x_k}|^p + \sum_{j,k=1}^n |u_{x_j x_k}|^p) dx\Big]^{\frac{1}{p}}.$$

Definition 2.3. *A partial differential equation for a function* $u = u(x)$, $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$ *defined on a subset of* $\mathbf{R}^n$, $n \geq 2$, *is a relation of the form*

$$F(x_1, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, \cdots) = 0 \qquad (2.1)$$

where F *is a given function of the independent variables* $x_1, \cdots, x_n$ *and of the unknown function* u *and of its finite number of partial derivatives.*

We call u *a solution of the equation* (2.1) *if it has all the partial derivatives appearing in the equation in some region in* $\mathbf{R}^n$ *and* $u(x_1, \cdots, x_n)$ *and its partial derivatives satisfy the relation* (2.1) *in the region.*

Definition 2.4. *The order of a partial differential equation is the order of the highest derivative that occurs. A partial differential equation is said to be linear if it is of the first degree in the unknown function and its derivatives, with coefficients only depending on the independent variables* $x_1, \cdots, x_n$.

3. The Second Order Linear Elliptic Boundary Value Problems

Definition 3.1. *A second order linear partial differential equation for a function $u = u(x_1, \cdots, x_n)$ in a domain $\Omega \subset \mathbf{R}^n$ has the form*

$$\sum_{j,k=1}^{n} a_{jk}(x) u_{x_j x_k} + \sum_{j=1}^{n} a_j(x) u_{x_j} + a(x) u = f(x) \tag{3.1}$$

where the coefficients a_{jk}, a_j, a and f are given functions of $x = (x_1, \cdots, x_n)$, $j, k = 1, \cdots, n$.

We call the differential equation (3.1) elliptic if for all nonzero $\xi = (\xi_1, \cdots, \xi_n)$ in $\mathbf{R}^n$, and for all $x = (x_1, \cdots, x_n) \in \Omega$ the inequality holds

$$\sum_{j,k=1}^{n} a_{jk}(x) \xi_j \xi_k > 0.$$

We introduce differential operators L and P as in (3.1)

$$Lu = \sum_{j,k=1}^{n} a_{jk}(x) u_{x_j x_k} + \sum_{j=1}^{n} a_j(x) u_{x_j} \tag{3.2a}$$

and

$$Pu = Lu + au. \tag{3.2b}$$

Theorem 3.2 (Maximum principle). ([1]) *Let Ω be a bounded domain in $\mathbf{R}^n$ and assume that the coefficients a_{jk}, a_j, a and f are all continuous in $\overline{\Omega}$, $j, k = 1, \cdots, n$, and that the equation*

$$Pu = Lu + au = f \tag{3.3}$$

is elliptic. Suppose that $a(x) \leq 0$ throughout $\overline{\Omega}$, and let $Pu(x) \leq 0$ (respectively, $Pu(x) \geq 0$) in $\overline{\Omega}$, then every non-constant solution of (3.3) attains its negative minimum (respectively, positive maximum), if it exists, *on the boundary of Ω not in Ω. Furthermore, if u achieves its negative minimum(respectively, positive maximum) at $p \in \partial\Omega$, then every outward directional derivative of u at p is negative(respectively,positive), unless u is identically equal to a constant in $\overline{\Omega}$.*

Proof. cf. Theorem 8.1 and Theorem 8.6 in [1]. □

Corollary 3.3. ([1]) *Let u_1 and u_2 be solutions of*

$$Lu + au = f$$

in Ω, with $u_i = \phi_i$ on $\partial\Omega$, $i = 1, 2$. Then if $a \leq 0$ in $\overline{\Omega}$,

$$\max_{x\in\Omega} |u_1(x) - u_2(x)| \leq \max_{x\in\partial\Omega} |\phi_1(x) - \phi_2(x)|$$

Proof. cf. Corollary 8.2 in [1]. □

Theorem 3.4. ([1]) *The solution of the boundary value problem*

$$\begin{cases} Lu + au = f & \text{in} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases}$$

is unique, if it exists, when $a \leq 0$ in $\overline{\Omega}$.

Proof. By Corollary 3.3. □

Definition 3.5. *Let Ω be a bounded domain in $\mathbf{R}^n$ and a second linear partial differential equation be defined by (3.3) in Ω*

$$Pu(x) = (Lu + au)(x) = f(x).$$

We call the differential operator P uniformly elliptic in Ω if there exists a positive number m such that for all $\xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n$ and for all $x \in \Omega$ the inequality

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq m|\xi|^2 \tag{3.4}$$

holds.

As an important case of uniformly elliptic operator we take

$$\Delta u = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}.$$

We call the differential operator Δ Laplace operator, and the equation $\Delta u = 0$ the Laplace equation.

Definition 3.6. *Let the equation given in (3.1) be uniformly elliptic, and let the coefficients a_{jk}, a_j, a be continuous functions in the bounded domain Ω of $\mathbf{R}^n$. Then the Dirichlet problem is finding a function u which is continuous on $\overline{\Omega}$, twice differentiable in Ω and satisfies the equation $Pu = f$ in Ω and coincides with g on $\partial\Omega$ when the right hand side function f of (3.3) and the boundary condition g are given arbitrarily.*

We call such a function u a solution of the Dirichlet problem for given functions f and g.

Theorem 3.7. ([2], [4]) *The following statement $\mathcal{A}$ and $\mathcal{B}$ are equivalent.*

$\mathcal{A}$. *For every bounded domain Ω in $\mathbf{R}^n$ with boundary $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$, if P defined in (3.2a, b) is a uniformly elliptic differential operator whose coefficients are all in $C^{\alpha}(\overline{\Omega})$ and if $(f, g) \in C^{\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega)$ is arbitrary, then there exists a function $u \in C^{2,\alpha}(\overline{\Omega})$ which solves the Dirichlet problem $Pu = f$ in Ω, $u = g$ on $\partial\Omega$.*

$\mathcal{B}$. *For every bounded domain* Ω *with* $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$, *if* $(f, g) \in C^{\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega)$ *is arbitrary, there exists a function* $u \in C^{2,\alpha}(\overline{\Omega})$ *which solves the Dirichlet problem* $\Delta u = f$ *in* Ω, $u = g$ *on* $\partial\Omega$, *where* Δ *is the Laplace operator.*

Proof. $\mathcal{A} \Rightarrow \mathcal{B}$ is obvious. For the reverse case, see Lemma 1.1, pp 111 in [4].
□

Definition 3.8. *Let* Ω *be a domain in* $\mathbf{R}^n$ *and* $u \in C^2(\Omega)$. *The function* u *is called harmonic in* Ω *if*

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$$

is satisfied in Ω.

Definition 3.9. *Let* $y \in \Omega$ *be fixed. We define the (normalized) fundamental solution with a pole* y *of Laplace's equation by*

$$\Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} \dfrac{1}{n(2-n)\omega_n}|x-y|^{2-n} & \text{when } n > 2, \\[2ex] \dfrac{1}{2\pi}\log|x-y| & \text{when } n = 2, \end{cases}$$

where ω_n *denotes the volume of the unit ball in* $\mathbf{R}^n$.

Remark. The function $\Gamma(x-y)$ satisfies the Laplace equation for $x \neq y$, but becomes infinite for $x = y$.

Lemma 3.10. ([2], [3]) *Let* Ω *be a domain for which the divergence theorem holds and let* ν *denote the unit outward normal to* $\partial\Omega$. *Then for* u, $v \in C^2(\Omega)$ *we obtain Green's first identity:*

$$\int_{\Omega} v\Delta u dx + \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} dS$$

and Green's second identity

$$\int_{\Omega}(v\Delta u - u\Delta v)\,dx = \int_{\partial\Omega}\left(v\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial\nu}\right)dS$$

where dS indicates the $(n-1)$-dimensional area element in $\partial\Omega$.

Lemma 3.11. ([2], [3]) *Let $u \in C^2(\overline{\Omega})$ and y be a point of Ω, and let $v(x) = \Gamma(|x-y|)$ be the fundamental solution with pole y of Δ. Then we have*

$$u(y) = \int_{\partial\Omega}\left(u\frac{\partial u}{\partial\nu} - v\frac{\partial u}{\partial\nu}\right)dS + \int_{\Omega} v\Delta u\,dx$$

which is called Green's representation formula.

Definition 3.12. *Let $h \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ with $\Delta h = 0$ in Ω and let for a point $y \in \Omega$*

$$G = G(x,y) = \Gamma(x-y) + h(x),$$

where Γ is the fundamental solution of the Laplace equation. If $G = 0$ on $\partial\Omega$, then we call the function G the Green's function for the domain (or the Green's function of the first kind for Ω).

Remark. The Green's function is unique. For to construct the Green's function $G(x,y)$ for a fixed point $y \in \Omega$ we need to find a solution $h \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ of the Dirichlet problem

$$\begin{cases} \Delta h(x) = 0, & x \in \Omega \\ h(x) = -\Gamma(x-y), & x \in \partial\Omega. \end{cases}$$

We note that the solution is unique by the maximum principle. Therefore, the Green's function is also unique, if it exists.

Lemma 3.13. ([2], [3]) *Let $G(x,y) = \Gamma(x,y) + h(x)$ be the Green's function for a domain Ω, where $\Gamma(x,y)$ is the fundamental solution with pole $y \in \Omega$ and $h \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ is a harmonic function in Ω. Then for $u \in C^2(\overline{\Omega})$*

$$u(y) = \int_{\partial\Omega} u \frac{\partial G}{\partial \nu} dS + \int_{\Omega} G \Delta u dx.$$

Lemma 3.14. ([2], [3]) *Let $B = B(0,a) = \{x : |x| < a\}$. Then the Green's function for B exists and for $y \in B$*

$$H(x,y) = \frac{\partial G(x,y)}{\partial \nu} = \frac{1}{na\omega_n} \frac{a^2 - |y|^2}{|x-y|^2}$$

and a harmonic function $u \in C^2(\overline{\Omega})$ can be expressed by

$$u(y) = \int_{|x|=a} H(x,y) u(x) dS. \tag{3.5}$$

We call (3.5) Poisson's integral formula.

Theorem 3.15. ([2], [3]) *Let $B = B(0,a)$ and f be a continuous function on ∂B. Then the function defined by*

$$u(x) = \begin{cases} \dfrac{a^2 - |x|^2}{na\omega_n} \displaystyle\int_{\partial B} \frac{f(y)}{|x-y|^n} dS_y & \text{for } x \in B \\ f(x) & \text{for } x \in \partial B \end{cases}$$

belongs to $C^2(B) \cap C^o(\overline{B})$ and satisfies $\Delta u = 0$ in B.

Proof. cf. pp 106 in [3]. □

Theorem 3.16. ([2], [3]) *A $C^0(\Omega)$-function u is harmonic if and only if for every ball $B_R(y) = B(y,R)$ strictly contained in Ω it satisfies the mean value property*

$$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u dS, \quad \text{or} \quad u(y) = \frac{1}{\omega_n R^n} \int_{B_R} u dx.$$

Theorem 3.17(Convergence theorem). ([2], [3]) *The limit of a uniformly convergent sequence of harmonic functions is harmonic.*

Definition 3.18. *A $C^0(\Omega)$ function u is called subharmonic (superharmonic) in Ω if for every ball B with $\overline{B} \subset \Omega$ and for every function h harmonic in B satisfying $u \leq h\,(u \geq h)$ on ∂B, we also have $u \leq h\,(h \geq h)$ in B.*

Definition 3.19. *Let Ω be bounded and φ be a bounded function on $\partial\Omega$. A $C^0(\overline{\Omega})$ subharmonic function u is called a subfunction relative to φ if it satisfies $u \leq \varphi$ on $\partial\Omega$.*

Similarly a $C^0(\overline{\Omega})$ superharmonic function is called a superfunction relative to φ if it satisfies $u \geq \varphi$ on $\partial\Omega$.

Theorem 3.20. ([2], [3]) *Let S_φ denote the set of all subfunction relative to φ. Then the function $u(x) = \sup_{v \in S_\varphi} v(x)$ is harmonic in Ω.*

Proof. cf. pp 111 in [3].

Definition 3.21. *We call u defined in the above theorem the Perron solution of the classical Dirichlet problem*

$$\Delta u = 0 \quad in \quad \Omega, \qquad u = \varphi \quad on \quad \partial\Omega.$$

Remark. ([2]) If the Dirichlet problem is solvable, its solution is identical with Perron solution.

Definition 3.22. *Let ξ be a point of $\partial\Omega$. Then a $C^0(\overline{\Omega})$ function $w = w_\xi$ is called a barrier(function) at ξ relative to Ω if*

(i) w is superharmonic in Ω

(ii) $w > 0$ in $\overline{\Omega}\backslash\{\xi\}$; $w(\xi) = 0$.

We call w a local barrier at $\xi \in \partial\Omega$ if there is a neighborhood N of ξ such that w satisfies the above definition in $\overline{\Omega} \cap N$.

Definition 3.23. *A boundary point is called regular (with respect to the Laplace operator) if there exists a barrier at that point.*

Lemma 3.24. ([2], [3]) *Let u be the harmonic function defined in Ω as in Theorem 3.20. If ξ is a regular boundary point of Ω and φ is continuous at ξ, then $u(x) \to \varphi(\xi)$ as $x \to \xi$.*

Theorem 3.25. ([2], [3]) *The classical Dirichlet problem in a bounded domain Ω*

$$\Delta u = 0 \quad in \quad \Omega, \qquad u = \varphi \quad on \quad \partial\Omega$$

is solvable for arbitrary continuous boundary data φ if and only if the boundary points are all regular.

Proof. See pp.115, [3].

Definition 3.26. *For an integrable function f on a domain $\Omega \subset \mathbf{R}^n$ the Newtonian potential f is the function w defined on $\mathbf{R}^n$ by*

$$w(x) = \int_\Omega \Gamma(x-y)f(y)dy$$

where $\Gamma(x-y)$ is the fundamental solution of Laplace equation defined in Definition 3.9.

Lemma 3.27. ([2]) *Let f be bounded and locally Hölder continuous with exponent $0 < \alpha \leq 1$, and let w be Newtonian potential of f. Then $\Delta w = f$ in Ω.*

Theorem 3.28. ([2]) *Let Ω be a bounded domain and suppose that each point of $\partial\Omega$ is regular (with respect to Laplacian). Then if f is a bounded, locally Hölder continuous function in Ω the Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega \end{cases}$$

is uniquely solvable for any continuous boundary value φ.

Theorem 3.29. ([2]) *Let Ω be a bounded domain. For $f \in C^{\alpha}(\overline{\Omega})$ the Dirichlet boundary value problem*

$$\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.

Theorem 3.30. ([1]) *Let Ω be a bounded domain. Let $P = L + a$ be the uniformly elliptic differential operator defined in (3.2b) with $a \leq 0$ and let the coefficients belong to $C^{\alpha}(\Omega)$, $0 < \alpha < 1$. Then for $f \in C^{\alpha}(\overline{\Omega})$ the Dirichlet boundary value problem*

$$Pu = Lu + au = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.

Proof. cf. Theorem 8.9 in [1]

Theorem 3.31. ([1]) *For every bounded domain $\Omega \subset \mathbf{R}^n$ with boundary $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$. Let $P = L + a$ be the uniformly elliptic differential operator defined by (3.2b) with $a \leq 0$ and let the coefficients belong to*

$C^\alpha(\overline{\Omega})$. *Then for each* $(f, g) \in C^\alpha(\overline{\Omega}) \times C^0(\partial\Omega)$ *there exists a unique solution* $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ *to the boundary value problem*

$$\begin{cases} Pu = f & \text{in} \quad \Omega \\ \ \ u = g & \text{on} \quad \partial\Omega. \end{cases} \tag{3.6}$$

Proof. See Theorem 8.10 in [1]. □

Theorem 3.32. ([4]) *For every bounded domain* $\Omega \subset \mathbf{R}^n$ *with boundary* $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$. *Let* $P = L + a$ *be the uniformly elliptic differential operator defined by* (3.2*b*) *with* $a \leq 0$ *and let the coefficients belong to* $C^\alpha(\Omega)$. *Then the problem* (3.6) *has a unique solution in* $C^{2,\alpha}(\overline{\Omega})$ *for all* $f \in C^\alpha(\overline{\Omega})$ *and* $g \in C^{2,\alpha}(\partial\Omega)$.

Proof. cf. Theorem 1.3, pp 115, [4]. □

4. The Second Order Semilinear Elliptic Boundary Value problems and Differential Inequalities

Definition 4.1. *A second order quasilinear partial differential equation for a function* $u = u(x_1, \cdots, x_n)$ *in a domain* $\Omega \subset \mathbf{R}^n$ *has the form*

$$\sum_{j,k=1}^{n} A_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} = H$$

where the coefficients A_{jk}, H *are functions of the independent variables* x_i *and of the unknown function* u *and its first derivatives* u_{x_i}, $i = 1, 2, \cdots, n$.

We call the differential operator $\sum_{j,k=1}^{n} A_{jk} \frac{\partial^2}{\partial x_j \partial x_k}$ *elliptic if for all* $x \in \Omega$, $u \in \mathbf{R}$, $p \in \mathbf{R}^n$ *and for all* $\xi \in \mathbf{R}^n$, $\xi \neq 0$

$$\sum_{j,k=1}^{n} A_{jk}(x, u, p)\xi_j \xi_k > 0.$$

Let L denote the elliptic differential operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \tag{4.1}$$

and let $a_{ij} : \Omega \longrightarrow \mathbf{R}^1$ belong to $C^{\alpha}(\overline{\Omega})$, and let there exists $M > 0$ such that

$$M^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \leq M|\xi|^2$$

for every $\xi \in \mathbf{R}^n$, $x \in \overline{\Omega}$.

In this section we assume that Ω is a bounded domain with the boundary $\partial\Omega \in C^{2,\alpha}$.

Lemma 4.2. ([5]) *For every* $f \in C^{\alpha}(\overline{\Omega})$ *and* $\varphi \in C^{2,\alpha}(\overline{\Omega})$ *there exists a unique solution* u *of the Dirichlet problem*

$$\begin{cases} Lu(x) = f(x), & x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{4.2}$$

Furthermore, $u \in C^{2,\alpha}(\overline{\Omega})$ *and there exists a constant* C*(depending on* M, Ω, α *and* $\max \|a_{ij}\|_{\alpha}^{\overline{\Omega}}$*) such that*

$$\|u\|_{2,\alpha}^{\overline{\Omega}} \le C\left(\|\varphi\|_{2,\alpha}^{\partial\Omega} + \|f\|_{\alpha}^{\overline{\Omega}}\right).$$

The above estimate in fact holds for arbitrary $u \in C^{2,\alpha}(\overline{\Omega})$ *in the form*

$$\|u\|_{2,\alpha}^{\overline{\Omega}} \le C\left(\|\varphi\|_{2,\alpha}^{\partial\Omega} + \|Lu\|_{\alpha}^{\overline{\Omega}}\right)$$

if $u = \varphi$ *on* $\partial\Omega$.

Let $p, q \in C^{1,\alpha}(\partial\Omega)$ be nonnegative real valued functions which do not vanish simultaneously. Let $\nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ denote the unit outward normal vector field to $\partial\Omega$. For $u : \overline{\Omega} \longrightarrow \mathbf{R}^1$ define

$$(Bu)(x) = p(x)u(x) + q(x)\frac{du(x)}{d\nu} \tag{4.3}$$

where $\frac{du(x)}{d\nu}$ denotes the normal derivative of u on $\partial\Omega$.

Lemma 4.3. ([5]) *For every* $f \in C^{\alpha}(\overline{\Omega})$ *and* $\varphi \in C^{2,\alpha}(\overline{\Omega})$ *there exists a unique solution* u *of the boundary value problem*

$$\begin{cases} Lu(x) = f(x), & x \in \Omega \\ Bu(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{4.4}$$

Furthermore $u \in C^{2,\alpha}(\overline{\Omega})$ *and there exists a constant* C *(depending only on* $M,\ \Omega,\ \alpha,\ \max\|a_{ij}\|_{\alpha}^{\Omega},\ \|p\|_{\alpha}^{\partial\Omega},\ \|\nu\|_{C_0}^{\partial\Omega}$*) such that*

$$\|u\|_{2,\alpha}^{\overline{\Omega}} \leq C\left(\|Bu\|_{1,\alpha}^{\partial\Omega} + \|Lu\|_{\alpha}^{\overline{\Omega}}\right). \tag{4.5}$$

Remark. We call the boundary value problem (4.4) Dirichlet problem if q=0, Neumann problem if p=0, otherwise Robin(mixed boundary value) problem.

Lemma 4.4. ([5]) *For* $f \in C^{\alpha}(\overline{\Omega})$ *define* $T(f)$ *to be the unique solution of the problem*

$$\begin{cases} Lu(x) = f(x), & x \in \Omega \\ Bu(x) = 0, & x \in \partial\Omega \end{cases} \tag{4.4a}$$

For $\varphi \in C^{i,\alpha}(\overline{\Omega})$ *(if* $q = 0$, $i = 2$, *otherwise* $i = 1$*) we let* $S(\varphi)$ *to be unique solution of the problem*

$$\begin{cases} Lu(x) = 0, & x \in \Omega \\ Bu(x) = \varphi(x), & x \in \partial\Omega \end{cases} \tag{4.4b}$$

Then $u = T(f) + S(\varphi)$ *solves the problem*

$$\begin{cases} Lu(x) = f(x), & x \in \Omega \\ Bu(x) = \varphi(x), & x \in \partial\Omega \end{cases} \tag{4.4}$$

and $u = T(f) + S(\varphi) \in C^{2,\alpha}(\overline{\Omega})$. *Furthermore*

$$T : C^{\alpha}(\overline{\Omega}) \longrightarrow C^{2,\alpha}(\overline{\Omega})$$

is a continuous mapping.

Definition 4.5. *Let X and Y be Banach spaces, and let $f : X \longrightarrow Y$ be a map. We call f compact if for any open unit ball O in X the closure of $f(O)$ is compact in Y. X is called compactly embedded in Y if $X \subset Y$ and the inclusion map $i : X \longrightarrow Y$, given by $i(x) = x$, is compact*

Lemma 4.6. ([7]) *$C^{2,\alpha}(\overline{\Omega})$ is compactly embedded in $C^{\alpha}(\overline{\Omega})$.*

Proof. We note that $C^{2,\alpha}(\overline{\Omega})$ is compactly embedded in $C^{\alpha}(\overline{\Omega})$ if and only if the mapping $i : C^{2,\alpha}(\overline{\Omega}) \longrightarrow C^{\alpha}(\overline{\Omega})$ defined by $i(u) = u$ is compact if and only if for every bounded sequence $\{u_n\}$ in $C^{2,\alpha}(\overline{\Omega})$, the sequence $\{i(u_n)\}$ has a convergent subsequence in $C^{\alpha}(\overline{\Omega})$ if and only if for every ball B, $\overline{i(B)}$ is compact in C^{α}. Let $\{u_m\}$ be a bounded sequence in $C^{2,\alpha}(\overline{\Omega})$. Then by the Mean Value Theorem $\{u_m\}$ is equicontinuous and bounded, and so there is a convergent subsequence in $C^{\alpha}(\overline{\Omega})$ by Ascoli-Arzela Theorem. □

Remark. It follows from Lemma 4.6 that T may be thought of as a compact linear map of $C^{\alpha}(\overline{\Omega})$ into itself.

Lemma 4.7. ([8]) *For any $p \geq 1$, $C^{\alpha}(\overline{\Omega})$ is dense in $L^{p}(\Omega)$.*

Proof. We note that $C_0^{\infty}(\Omega) \subset C^{\alpha}(\overline{\Omega})$ and is dense in $L^{p}(\Omega)$. □

Remark. It follows from Lemma 4.7 that T has a unique bounded continuous extension (which we again denote by T) to $L^{p}(\Omega)$.

Lemma 4.8. [9] *There exists a constant γ such that for every $u \in C^{\alpha}(\overline{\Omega})$*

$$\|u\|_{W^{2,p}} \leq \gamma \|Lu\|_{L^p} \quad \text{if} \quad Bu = 0,$$

where γ depends only on Ω and p.

Remark. We note that $C^{\alpha}(\overline{\Omega}) \subset W^{2,p}(\overline{\Omega})$ We can show T is a continuous operator from $C^{\alpha}(\overline{\Omega})$ into $W^{2,p}(\Omega)$ for $p \geq 1$. To show that, let a sequence $\{u_m\}$ converge to u in $C^{\alpha}(\overline{\Omega})$. By Lemma 4.8

$$\begin{aligned}\|Tu_m - Tu\|_{W^{2,p}} &\leq \gamma \|L(Tu_m - Tu)\|_{L^p} \\ &= \gamma \|u_m - u\|_{L^p} \to 0.\end{aligned}$$

Therefore, Tu_m converges to Tu in $W^{2,p}(\Omega)$.

Remark. Since $C^{\alpha}(\overline{\Omega})$ is dense in $C(\overline{\Omega})$ and T is continuous on $C^{\alpha}(\overline{\Omega})$, we can extend T to $C(\overline{\Omega})$ continuously. For let $u \in C(\overline{\Omega})$, then there exists a sequence $\{u_m\}$ in $C^{\alpha}(\overline{\Omega})$ so that $u_m \to u$ in $C(\overline{\Omega})$. Then $\{u_m\}$ is a Cauchy sequence in $C(\overline{\Omega})$ with this inequality

$$\begin{aligned}\|Tu_m - Tu_l\|_{W^{2,p}} &\leq \gamma \|L(Tu_m - Tu_l)\|_{L^p} \\ &= \gamma \|u_m - u_l\|_{L^p} \\ &\leq \gamma |\Omega|^{\frac{1}{p}} \|u_m - u_l\|_{C^0}.\end{aligned}$$

We note that $\{Tu_m\}$ is a Cauchy sequence in $W^{2,p}(\Omega)$. Since $W^{2,p}(\Omega)$ is a Banach space, we can define Tu by

$$Tu = \lim_{m \to \infty} Tu_m.$$

To show that T is well-defined on $C(\overline{\Omega})$, we choose another Cauchy sequence $\{\hat{u}_m\}$ in $C(\overline{\Omega})$ so that $\hat{u}_m$ converges to u in $C(\overline{\Omega})$. By the inequality

$$\|T\hat{u}_m - Tu_m\|_{W^{2,p}} \leq \gamma |\Omega|^{\frac{1}{p}} \|\hat{u}_m - u_m\|_{C^0}$$

$T\hat{u}_m - Tu_m$ converges to 0 in $W^{2,p}(\Omega)$. We again denote the extension by T

$$T : C(\overline{\Omega}) \longrightarrow W^{2,p}(\Omega).$$

Lemma 4.9. ([7]) *If Ω be an open bounded domain with $\partial\Omega \in C^{2,\alpha}$, then $W^{2,p}(\Omega) \longrightarrow C^{1,\alpha}(\overline{\Omega})$ is a continuously embedded for sufficiently large p.*

Remark. We may view

$$T : (C(\overline{\Omega}), \| \cdot \|_{L^p}) \longrightarrow C^{1,\alpha}(\overline{\Omega})$$

as a bounded linear map.

Lemma 4.10. ([7]) *If Ω is a bounded domain with $\partial\Omega \in C^{2,\alpha}$, then $C^{1,\alpha}(\overline{\Omega}) \to C^1(\overline{\Omega})$ is compactly embedded.*

Remark. We may view

$$T : (C(\overline{\Omega}), \| \cdot \|_{L^p}) \longrightarrow C^1(\overline{\Omega}) \tag{4.5}$$

as a compact linear operator.

Definition 4.11. *Assume that $f : \overline{\Omega} \times \mathbf{R}^m \times \mathbf{R}^{nm} \longrightarrow \mathbf{R}^m$ belong to C^α and let $f = (f^1, f^2, \cdots, f^m)$. Let L^i denote the elliptic differential operator*

$$L^i = \sum_{j,k=1}^{n} a^i_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k}, \quad i = 1, 2, \cdots, m$$

where $a^i_{jk} : \overline{\Omega} \longrightarrow \mathbf{R}$ is α-Hölder continuous, and for each i let there exist $M_i > 0$ such that for every $\xi \in \mathbf{R}^n$, $x \in \overline{\Omega}$

$$M_i^{-1}|\xi|^2 \leq \sum_{j,k}^{n} a^i_{jk}(x)\xi_j\xi_k \leq M_i|\xi|^2.$$

Let B^i be the boundary operator defined in (4.3) by for $v : \overline{\Omega} \to \mathbf{R}$

$$(B^i v)(x) = p^i(x)v(x) + q^i(x)\frac{dv(x)}{d\nu}, \quad i = 1, 2, \cdots, m.$$

Then for $u = (u^1, u^2, \cdots, u^m)$ *and for boundary value* $\varphi = (\varphi^1, \cdots, \varphi^m)$

$$\begin{cases} L^i u^i = f^i(x, u, \partial u), & x \in \Omega \\ B^i u^i = \varphi, & x \in \partial\Omega \end{cases} \tag{4.6}$$

is called the coupled system of second order quasilinear elliptic boundary value problems. We write the system (4.6)

$$Lu(x) = f(x, u(x), \partial u(x)) \quad x \in \Omega$$

$$Bu(x) = \varphi(x) \quad x \in \partial\Omega$$

where $Lu = (L^1u^1, L^2u^2, \cdots, L^m u^m)$, $Bu = (B^1u^1, B^2u^2, \cdots, B^m u^m)$.

If f *is a real-valued function, we call* (4.6) *the boundary value problem for quasilinear second order elliptic equations.*

In the case if $q = 0$ *and* $p = 1$, *we call* (4.6) *Dirichlet Boundary Value Problem, if* $p = 0$ *and* $q = 1$, *we call* (4.6) *Neumann Boundary Value Problem, and the other case is Robin(Mixed) Boundary Value Problem.*

Definition 4.12. *Let* $f : \overline{\Omega} \times \mathbf{R}^m \times \mathbf{R}^{nm} \longrightarrow \mathbf{R}^m$ *belong to class* C^α. *For* $u \in C^{1,\tau}(\overline{\Omega})$, $0 < \tau < 1$, $u : \overline{\Omega} \to \mathbf{R}^m$, *define the Nemytskii operator* $F(u)$ *by* $F(u)(x) = f(x, u(x), \partial u(x))$.

Remark. $F : \left(C^{1,\tau}(\overline{\Omega}), \|\cdot\|^{\overline{\Omega}}_{1,\tau}\right) \longrightarrow \left(C^{\alpha\tau}(\overline{\Omega}), \|\cdot\|_{C^0}\right)$ is a continuous mapping. For $u \in C^{1,\tau}(\overline{\Omega})$,

$$\begin{aligned} |F(u)(x) - F(u)(y)| &= |f(x, u(x), \partial u(x)) - f(y, u(y), \partial u(y))| \\ &\le H_\alpha(f)\, |(x, u(x), \partial u(x)) - (y, u(y), \partial u(y))|^\alpha \\ &\le H_\alpha(f)\left(|x-y| + |u(x) - u(y)| + |\partial u(x) - \partial u(y)|\right)^\alpha \\ &\le H_\alpha(f)\left(|x-y| + H_\tau(u)|x-y|^\tau + H_\tau(\partial u)|x-y|^\tau\right)^\alpha \\ &= H_\alpha(f)\left(|x-y|^{1-\tau} + H_\tau(u) + H_\tau(\partial u)\right)|x-y|^{\alpha\tau}. \end{aligned}$$

Thus $F(u) \in C^{\alpha\tau}(\overline{\Omega})$, and the mapping F is well-defined.

If $u_m, u_n \in C^{1,\tau}(\overline{\Omega})$, then

$$|F(u_m)(x) - F(u_n)(x)| = |f(x, u_m(x), \partial u_m(x)) - f(x, u_n(x), \partial u_n(x))|$$
$$\leq H_\alpha(f)(|u_m(x) - u_n(x)| + |\partial u_m(x) - \partial u_n(x)|)^\alpha .$$

This implies that F is continuous.

Remark. If $u, v \in C(\overline{\Omega})$, $\|u-v\|_{L^p} \leq |\Omega|^{\frac{1}{p}} \|u-v\|_{C^0}$, where $|\Omega|$ denotes the Lebesgue measure of Ω. Thus $i : (C(\overline{\Omega}), \|\cdot\|_{C^0}) \longrightarrow (C(\overline{\Omega}), \|\cdot\|_{L^p})$ given by $i(u) = u$ is continuous. Hence it follows from the above Remark that

$$T \circ F : C^1(\overline{\Omega}) \longrightarrow C^1(\overline{\Omega}) \tag{4.7}$$

is thought as a continuous compact operator.

Theorem 4.13. *Let $f : \overline{\Omega} \times \mathbf{R}^m \times \mathbf{R}^{nm} \to \mathbf{R}^m$ belong to class C^α, $0 < \tau < 1$ and let $f = (f^1, f^2, \cdots, f^m)$. Let us assume we have the coupled system for $u = (u^1, u^2, \cdots, u^m)$. We denote by*

$$\mathcal{T} = (T^1, T^2, \cdots, T^m), \quad \mathcal{S} = (S^1, S^2, \cdots, S^m), \quad \varphi = (\varphi^1, \varphi^2, \cdots, \varphi^m)$$

where T^i, S^i are the operators defined in Lemma 4.2 with respect to L^i, B^i for each $i = 1, 2, \cdots, m$. Then the coupled system of boundary value problem (4.6) is equivalent to the system of operator equation

$$u = (\mathcal{T} \circ F)(u) + \mathcal{S}(\varphi)$$

where $\mathcal{T} \circ F : C^1(\overline{\Omega}) \longrightarrow C^1(\overline{\Omega})$ is a compact operator.

Proof. From (4.6), using what we have showed we can derive

$$u = (\mathcal{T} \circ F)(u) + \mathcal{S}(\varphi).$$

where $\mathcal{T} \circ F : C^1(\overline{\Omega}) \longrightarrow C^1(\overline{\Omega})$.

If $u = (\mathcal{T} \circ F)(u) + \mathcal{S}(\varphi)$, then $u \in C^1(\overline{\Omega})$, so $u \in C^\alpha(\overline{\Omega})$. Hence $F(u) \in C^{\alpha\tau}(\overline{\Omega})$. Thus $\mathcal{T} \circ F(u) \in C^{2,\alpha\tau}(\overline{\Omega})$, and therefore $u \in C^{2,\alpha\tau}(\overline{\Omega})$ and so $u \in C^{2,\alpha}(\overline{\Omega})$. Consequently we have that $u \in C^{2,\alpha}(\overline{\Omega})$ and is a solution of the problem (4.6). □

Let L be the scalar differential operator defined by (4.1) and let B be the boundary operator defined by (4.2).

Definition 4.14. *A function $f : \overline{\Omega} \times \mathbf{R}^m \times \mathbf{R}^{nm} \to \mathbf{R}^m$ is said to satisfy a Nagumo condition when:*

There exists a continuous function $g : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that

$$|f(x, u, p)| \leq g(|u|, |p|)(1 + |p|^2), \quad x \in \overline{\Omega} \tag{4.8}$$

where $\lim_{|p| \to \infty} \frac{g(|u|, |p|)}{|p|} = 0$ uniformly in $|u|$ if $m > 1$, and g is independent of p if $m = 1$.

In case $n = 1$ for every bounded u-set U there exists a continuous function $\psi : [0, \infty) \to [0, \infty)$ such that

$$|f(x, u, p)| \leq \psi(|p|), \quad x \in \overline{\Omega}, \quad u \in U \tag{4.9}$$

where ψ is nondecreasing and $\lim_{s \to \infty} \frac{s^2}{\psi(s)} = +\infty$ if $m > 1$, ψ is to satisfy $\int^\infty \frac{s}{\psi(s)} ds = +\infty$ if $m = 1$.

Lemma 4.15. ([5]) *Let f satisfy a Nagumo condition. Then for every constant $P > 0$ there exists a constant Q such that if $u : \overline{\Omega} \to \mathbf{R}^m$ is a solution of*

$$Lu = f(x, u, \partial u) \quad x \in \Omega \tag{4.11a}$$

which belong to $C^2(\overline{\Omega})$ *and* $|u(x)| < P$, $x \in \overline{\Omega}$, *then* $|\partial u(x)| \leq Q$, $x \in \Omega$. *The constant* Q *depends on* P *and the bounding functions* g, *respectively* ψ.

Lemma 4.16. ([1]) *Let* E *be a real Banach space and let* $\mathcal{O}$ *be a bounded open neighborhood of* $0 \in E$. *Let* $N : \overline{\mathcal{O}} \to E$ *be a compact continuous operator such that for all* $\lambda \in (0,1)$ *and* $u \in \partial\mathcal{O}$, $u \neq \lambda N u$. *Then the equation* $u = Nu$ *has a solution* $u \in \overline{\mathcal{O}}$.

Theorem 4.17. ([5]) *Let there exist a bounded open convex neighborhood* Σ *of* $0 \in \mathbf{R}^m$ *and let* $f : \overline{\Omega} \times \overline{\Sigma} \times \mathbf{R}^{nm} \to \mathbf{R}^m$ *belong to class* C^α. *For every* $u \in \partial\Sigma$ *let there exist an outer normal vector* $\mathbf{n}(u)$ *such that*

$$\mathbf{n}(u) \cdot f(x, u, p) > 0, \ x \in \overline{\Omega} \tag{4.10}$$

for all $p = (p^{ij})$ *such that* $\sum_{j=1}^{n} p^{ij} n^j(u) = 0$, $1 \leq i \leq m$. *Let* f *satisfy a Nagumo condition and assume that* $\varphi : \overline{\Omega} \to \overline{\Sigma}$ *belong to class* $C^{2,\alpha}$. *Then the boundary value problem*

$$\begin{cases} Lu(x) = f(x, u(x), \partial u(x)), & x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega \end{cases} \tag{4.11}$$

has a solution $u : \overline{\Omega} \to \overline{\Sigma}$, *and* $u \in C^{2,\tau}(\overline{\Omega})$ *for some* $\tau \in (0,1)$.

Proof. We note that the problem (4.11) is equivalent to the operator equation $u = (\mathcal{T} \circ F)(u) + \mathcal{S}(\varphi)$, where $\mathcal{S}(\varphi) \in C^{2,\alpha}(\overline{\Omega})$ and $\mathcal{T} \circ F : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ is compact continuous. We let E denote the real Banach space $C^1(\overline{\Omega})$ with the norm

$$\|u\| = \max_{\overline{\Omega}} |u(x)| + \max_{x \in \overline{\Omega}} |\partial u(x)|.$$

Let $P > 0$ be chosen such that if $u : \overline{\Omega} \longrightarrow \overline{\Sigma}$ is a solution $(4.11a)$, then $|u| \leq P$. Since f satisfies a Nagumo condition, we may determine the constant Q of Lemma 4.15 in terms of P determined above.

Define the bounded open set $\mathcal{O} \subset E$ as follows

$$\mathcal{O} = \{u \in E | u : \overline{\Omega} \to \Sigma, \ |\partial u(x)| < Q + 1, \ x \in \overline{\Omega}\}$$

and let $N : \overline{\mathcal{O}} \to E$ be defined by $Nu = (\mathcal{T} \circ F)(u) + \mathcal{S}(\varphi)$. The proof of the theorem will be complete by Lemma 4.16, once we show that for every $u \in \partial\mathcal{O}$ and $\lambda \in (0, 1)$, $u \neq \lambda N u$. To show this, we assume that there exist $\lambda \in (0, 1)$ and $u \in \partial\mathcal{O}$ such that

$$u = \lambda(\mathcal{T} \circ F(u) + \mathcal{S}(\varphi)).$$

Then u is a solution of the problem

$$\begin{cases} Lu(x) = \lambda f(x, u(x), \partial u(x)), & x \in \Omega \\ \ u(x) = \lambda\varphi(x), & x \in \partial\Omega. \end{cases}$$

Since $u : \overline{\Omega} \to \overline{\Sigma}$ it follows that $|u(x)| \leq P$, $x \in \Omega$ and hence by Lemma 4.15 we obtain $|\partial u(x)| \leq Q$, $x \in \partial\Omega$. Thus it must be the case that there exists $x_0 \in \overline{\Omega}$ such that $u(x_0) \in \partial\Sigma$. On the other hand since $\lambda \in (0, 1)$ and Σ is convex open $\lambda\varphi(x) \in \Sigma$, $x \in \partial\Omega$, and hence $x_0 \in \Omega$. Let $u(x_0) = u_0$ and define $r(x) = \mathbf{n}(u_0) \cdot (u(x) - u_0)$. Since $\mathbf{n}(u_0)$ is an outer normal to Σ at u_0 it follows that

$$r(x_0) = 0, \quad r(x) \leq 0 \text{ for } x \in \Omega, \quad r(x) < 0 \text{ for } x \in \partial\Omega.$$

Thus $r(x)$ assume its maximum at x_0 which is an interior point of Ω. Thus $\dfrac{\partial r(x_0)}{\partial x_i} = 0$, $1 \leq i \leq n$. Then $\mathbf{n}(u_0) \cdot \left(\dfrac{\partial u(x_0)}{\partial x_i}\right) = 0$, $1 \leq i \leq n$. It therefore follows from (4.10) that $\mathbf{n}(u_0)\cdot f(x, u(x), \partial u(x)) > 0$. Computing $(Lr)(x_0)$ we find

$$(Lr)(x_0) = \mathbf{n}(u_0)\cdot(Lu)(x_0) = \lambda\mathbf{n}(u_0)\cdot f(x_0, u_0, \partial u(x_0)) > 0,$$

hence there exists an open ball B centered at x_0, $\overline{B}\subset\Omega$, such that

$$(Lr)(x) = \lambda\mathbf{n}(u_0)\cdot f(x, u(x), \partial u(x)) \geq 0, \ \ x\in\overline{B},$$

which by Maximum principle implies that $r(x) = 0$, $x\in\overline{B}$. Hence $u(x)\in\partial\Sigma$, $x\in\overline{B}$. By means of a continuation argument we can obtain that $u(x)\in\partial\Sigma$, $x\in\overline{\Omega}$, contradiction to $u(x)\in\Sigma$ for $x\in\partial\Omega$. □

Corollary 4.18. ([5]) *Theorem 4.12 remains true if* $>$ *in* (4.10) *replaced by* $\geq$.

Proof. For $0 < \epsilon \leq 1$, let $f_\epsilon(x, u, p) = f(x, u, p) + \epsilon h(|p|)u$, where $h(|P|) = 1$ for $0 \leq |p| \leq 1$, $h(|p|) = \dfrac{1}{|p|}$ if $|p| \geq 1$. Then the modified problem

$$\begin{cases} Lu = f_\epsilon(x, u, p) & \text{in } \Omega \\ u = \varphi(x) & \text{on } \partial\Omega \end{cases}$$

has a solution $u_\epsilon : \overline{\Omega} \to \overline{\Sigma}$ by Theorem 4.17, and $|u_\epsilon(x)| \leq P$, $x\in\overline{\Omega}$ for some constant $P > 0$. Furthermore by Lemma 4.15 we obtain a constant $\tilde{Q}$(independent of ϵ) such that $|\partial u_\epsilon(x)| \leq \tilde{Q}$, $x\in\overline{\Omega}$. Let F_ϵ be the Nemytskii operators defined by f_ϵ. Then

$$\begin{aligned} |F_\epsilon(u_\epsilon)(x)| &\leq |f(x, u_\epsilon(x), \partial u_\epsilon(x))| + \epsilon h\big(\partial u_\epsilon(x)\big)|u_\epsilon(x)| \\ &\leq H_\alpha(f)\Big(|x| + |u_\epsilon(x)| + |\partial u_\epsilon(x)| + \epsilon|u_\epsilon(x)|\Big) \\ &\leq H_\alpha(f)\Big(\sup_{\overline{\Omega}}|x| + P + Q\Big) + \epsilon P. \end{aligned}$$

for all $x \in \overline{\Omega}$ and for $0 < \epsilon \leq 1$, and hence $\{F_\epsilon(u_\epsilon)\}$, $0 < \epsilon \leq 1$, is a bounded set in $C(\overline{\Omega})$. Consequently $\mathcal{T}\left(F_\epsilon(u_\epsilon)\right)$ is precompact in $C^1(\overline{\Omega})$. We hence obtain a subsequence $\{u_{\epsilon_n}\}$ such that $\lim_{\epsilon_n \to 0} u_{\epsilon_n} = u$ is a solution of $u = \mathcal{T} \circ F(u) + \mathcal{S}(\varphi)$. That is, u is a solution of (4.11). □

Corollary 4.19. ([5]) *Let Σ be a closed convex subset of $\mathbf{R}^m$ and assume that for every $u \in \partial\Sigma$ and every outer normal $\mathbf{n}(u)$ to Σ at u it is true that $\mathbf{n}(u) \cdot f(x, u, p) \geq 0$ for all $x \in \overline{\Omega}$ and all (p^{ij}) such that $\sum_{j=1}^{n} p^{ij} n^j(u) = 0$, $1 \leq i \leq m$, and let f satisfy the Nagumo condition. Let $\varphi : \overline{\Omega} \to \Sigma$. Then (4.13) has a solution $u : \overline{\Omega} \to \Sigma$.*

Proof. Let q denote a continuous retraction of $\mathbf{R}^m$ onto Σ and define $\tilde{f}$: $\overline{\Omega} \times \mathbf{R}^m \times \mathbf{R}^{nm} \longrightarrow \mathbf{R}^m$ by

$$\tilde{f}(x, u, p) = f(x, q(u), p).$$

Let $\Sigma_\epsilon = \{u \mid dist(\Sigma, u) \leq \epsilon\}$. Then Σ_ϵ is a bounded convex set with $int\Sigma_\epsilon \neq \phi$. Since $\tilde{f}$ satisfies the conditions in Corollary 4.18, the problem (4.11) with f replaced by $\tilde{f}$ and Σ by Σ_ϵ has a solution $u_\epsilon : \overline{\Omega} \to \Sigma_\epsilon$. Again using the Nagumo condition and a limiting argument as in the proof of Corollary 4.18 we obtain a solution of (4.11), $u : \overline{\Omega} \to \Sigma$, as a limit of the collection $\{u_\epsilon\}$, $0 < \epsilon \leq 1$. □

Theorem 4.20. ([5]) *Let $f : \overline{\Omega} \times \overline{\Sigma} \times \mathbf{R}^{nm} \to \mathbf{R}^m$ belong to class C^α and satisfy a Nagumo condition, let the condition (4.10) ($>$ replaced by $\geq$) be valid, and let the boundary operator B be given by (4.3) ;$Bu(x) = p(x)u(x) + q(x)\frac{du(x)}{d\nu}$, $x \in \partial\Omega$. Assume that*

$$\varphi \in C^{i,\alpha}(\overline{\Omega}) \ (i = 1, \text{or } 2) \quad \varphi(x) \in p(x)\overline{\Sigma},\ x \in \partial\Omega. \tag{4.12}$$

Then the boundary value problem

$$\begin{cases} Lu(x) = f(x, u(x), \partial u(x)), & x \in \Omega \\ Bu(x) = \varphi(x), & x \in \partial\Omega \end{cases} \tag{4.13}$$

has a solution.

Proof. We first assume Σ is a bounded open convex neighborhood of $0 \in \mathbf{R}^n$. Following the same method as in the proof of Theorem 4.17, we obtain that there exist $\lambda \in (0,1)$ and $u : \overline{\Omega} \to \overline{\Sigma}$ is a solution of the problem

$$\begin{cases} Lu(x) = \lambda f(x, u(x), \partial u(x)), & x \in \Omega \\ Bu(x) = \lambda\varphi(x), & x \in \partial\Omega \end{cases}$$

and $u(x_0) \in \partial\Sigma$ for some $x_0 \in \overline{\Omega}$.

Let $r(x) = \mathbf{n}(u_0) \cdot (u(x) - u_0)$ where $u_0 = u(x_0)$. Then $r(x_0) = 0$, $r(x) \le 0$, $x \in \overline{\Omega}$. Thus $r(x)$ assumes its maximum at $x_0 \in \overline{\Omega}$. If $x_0 \in \Omega$, then $\frac{\partial r(x_0)}{\partial x_i} = 0$, $1 \le i \le n$, a contradiction is obtained the same way as before. Thus $x_0 \in \partial\Omega$. Since $r(x)$ takes its maximum at a boundary point $x_0 \in \partial\Omega$, $\frac{dr(x_0)}{d\nu} \ge 0$, and hence $\mathbf{n}(u_0) \cdot \frac{du(x_0)}{d\nu} \ge 0$.

We now consider cases.

If $q(x_0) = 0$, then $p(x_0)u(x_0) = \lambda\varphi(x_0) \in \lambda p(x_0)\overline{\Sigma}$. Since $p(x_0) \ne 0$, we conclude that $u_0 = u(x_0) \in \lambda\overline{\Sigma} \subset \Sigma$, because $\lambda < 1$, contradicting that $u_0 \in \partial\Sigma$. Thus $q(x_0) \ne 0$ and hence

$$0 \le \mathbf{n}(u_0) \cdot \left(q(x_0)\frac{du(x_0)}{d\nu} \right) = \mathbf{n}(u_0) \cdot (\lambda\varphi(x_0) - p(x_0)u_0).$$

Since $\lambda\varphi(x_0) \in \lambda p(x_0)\overline{\Sigma} \subset p(x_0)\Sigma$ it follows that $\mathbf{n}(u_0) \cdot (\lambda\varphi(x_0) - p(x_0)u_0) < 0$ unless $p(x_0) = 0$. Thus if $p(x_0) > 0$, the proof is complete. Otherwise we replace $p(x)$ by $p(x) + \epsilon$, $0 < \epsilon \le 1$, apply what has just been proved to obtain a solution u_ϵ of the perturbed problem. A limiting argument then will yield a solution of the original problem.

In case of Σ is a bounded closed convex set we can obtain the same result by using similar manner with Corollary 4.19. $\square$

Theorem 4.21. ([5]) *Let $a_{ij} : \mathbf{R}^n \to \mathbf{R}$ and $f : \mathbf{R}^n \times \overline{\Sigma} \times \mathbf{R}^{nm} \longrightarrow \mathbf{R}^m$ be periodic with respect to x of period ω, where Σ is a bounded open convex neighborhood of $0 \in \mathbf{R}^m$ such that (4.10) holds ($>$ replaced by $\geq$). Let f satisfy a Nagumo condition. Then $(Lu)(x) = f(x, u(x), \partial u(x))$, $x \in \Omega$ has a periodic solution $u : \mathbf{R}^m \longrightarrow \overline{\Sigma}$ with period ω.*

Proof. We modify our problem as

$$(Lu)(x) - u = f(x, u(x), \partial u(x)) - u \tag{4.14}$$

and define the Nemytskii operator F by

$$(F(u))(x) = f(x, u(x), \partial u(x)) - u(x).$$

Then the problem (4.14) is equivalent to an operator equation $u = (\mathcal{T} \circ F)(u)$, where $\mathcal{T} : C^0_\omega(\mathbf{R}^n) \longrightarrow C^1_\omega(\mathbf{R}^n)$ is a compact linear operator(here $C^k_\omega(\mathbf{R}^n)$ denotes the set of functions $u : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ which are $k-$ times continuously differentiable and are periodic with period ω). Again we let

$$\mathcal{O} = \{u \in C^1_\omega(\mathbf{R}^n) | u : \mathbf{R}^n \longrightarrow \Sigma,\ |\partial u(x)| < Q + 1\}$$

where Q is a constant determined by the Nagumo condition on $f(x, u, p) - u$. In order to be able to employ Lemma 4.16, we now need to show that the equation

$$u = \lambda(\mathcal{T} \circ F)(u),\ \ 0 < \lambda < 1$$

has no solution $u \in \partial\mathcal{O}$. We argue indirectly and assume there exists $\lambda \in (0,1)$ and $u \in \partial\mathcal{O}$ such that $u = \lambda(\mathcal{T} \circ F)(u)$. As before it must be the case there exists $x_0 \in \mathbf{R}^n$ such that $u(x_0) = u_0 \in \partial\Sigma$ and consequently $\mathbf{n}(u_0) \cdot \left(\dfrac{\partial u(x_0)}{\partial x_i}\right) = 0$, $1 \leq i \leq n$ thus $\mathbf{n}(u_0) \cdot f(x_0, u_0, \partial u(x_0)) \geq 0$.

On the other hand u satisfies the equation

$$Lu(x) - u(x) = \lambda f(x, u(x), \partial u(x)) - \lambda u(x)$$

or

$$Lu(x) = \lambda f(x, u(x), \partial u(x)) + (1-\lambda)u(x).$$

Letting again $r(x) = \mathbf{n}(u_0) \cdot (u(x) - u_0)$, we find

$$\begin{aligned} Lr(x_0) &= \mathbf{n}(u_0) \cdot Lu(x_0) \\ &= \mathbf{n}(u_0) \cdot \lambda f(x_0, u_0, \partial u(x_0)) + (1-\lambda)\mathbf{n}(u_0) \cdot u_0 \\ &> 0. \end{aligned}$$

That is, $Lr(x_0) > 0$ and r assumes its maximum at x_0. Using a Maximum principle and continuation argument we conclude that $r(x)$ must be constant, that is, $r(x) = 0$, which is a contradiction. Thus the theorem follows from Lemma 4.16. □

Remark. ([5]) As in the case of Dirichlet and Mixed boundary conditions, we can obtain a similar result to Theorem 4.21 in case Σ is a closed bounded convex set with empty interior. Furthermore the requirement that $0 \in \Sigma$ may be removed by a change of variable argument.

Corollary 4.22. ([5]) *Let u and f be scalars and let there exist constant α and β such that $\alpha \leq 0 \leq \beta$ and*

$$f(x, \alpha, 0) \leq 0 \leq f(x, \beta, 0), \quad x \in \overline{\Omega}, \tag{4.15}$$

Let f satisfy a Nagumo condition. Then

(a) If $\varphi : \overline{\Omega} \longrightarrow [\alpha, \beta]$, the boundary value problem

$$\begin{cases} Lu(x) = f(x, u(x), \partial u(x)), & x \in \Omega \\ u(x) = \varphi(x), & x \in \partial\Omega \end{cases}$$

has a solution $u : \Omega \longrightarrow [\alpha, \beta]$.

(b) If $p(x)\alpha \leq \varphi(x) \leq p(x)\beta$, $x \in \partial\Omega$, the boundary value problem

$$\begin{cases} Lu(x) = f(x, u(x), \partial u(x)), & x \in \Omega \\ Bu(x) = \varphi(x), & x \in \partial\Omega \end{cases}$$

has a solution $u : \Omega \longrightarrow [\alpha, \beta]$.

(c) If a_{ij} and f, $1 \leq i, j \leq n$, are ω-periodic with respect to x and defined on all of $\mathbf{R}^n$ and if (4.15) holds for $x \in \mathbf{R}^n$,

$$Lu(x) = f(x, u(x), \partial u(x)), \quad x \in \Omega$$

has an ω-periodic solution $u : \mathbf{R}^n \longrightarrow [\alpha, \beta]$.

Proof. The condition (4.15) is the same as the condition (4.10) in the case of scalar equation. Hence (a), (b), (c) are valid. □

We now consider the scalar differential equation

$$Lu(x) = f(x, u(x), \nabla u(x)), \quad x \in \Omega \tag{4.16}$$

subject to the constraints

$$u(x) = \varphi(x), \quad x \in \partial\Omega \tag{4.17a}$$

or

$$p(x)u(x) + q(x)\frac{du(x)}{d\nu} = \varphi(x), \quad x \in \partial\Omega \tag{4.17b}$$

or in the case a_{ij} and f are periodic ω and defined on $\mathbf{R}^n$, the periodicity constraint

$$u(x + \omega) = u(x), \quad x \in \mathbf{R}^n. \tag{4.17c}$$

Definition 4.23. *A function $\alpha : \overline{\Omega} \longrightarrow \mathbf{R}$, $\alpha \in C(\overline{\Omega})$ is called a lower solution of* (4.16) *and* (4.17k), *k is either a, b, c, if for every $x \in \overline{\Omega}$ there exists a neighborhood U of x such that*

$$\alpha(y) = \max_{1 \le r \le s} \alpha_r(y) \tag{4.18}$$

(where s may depend on x) where $\alpha_r \in C^2(U \cap \overline{\Omega})$ and satisfies

$$(L\alpha_r)(y) \ge f(y, \alpha_r(y), \nabla \alpha_r(y)), \quad \in U \cap \overline{\Omega} \tag{4.19}$$

and further

$$\alpha_r(x) \le \varphi(x), \ x \in \partial\Omega \quad \text{in case } k = a \tag{4.20a}$$

$$p(x)\alpha_r(x) + q(x)\frac{d\alpha_r(x)}{d\nu} \le \varphi(x), \ x \in \partial\Omega, \ 1 \le r \le s, \quad \text{in case } k = b \tag{4.20b}$$

$$\alpha(x + \omega) = \alpha(x), \ x \in \mathbf{R}^n \quad \text{in case } k = c. \tag{4.20c}$$

An upper solution $\beta : \overline{\Omega} \longrightarrow \mathbf{R}$ is defined similarly replacing max *by* min *in (4.18) and reversely the inequalities in (4.19), (4.20a) and (4.20b).*

Remark. Lower and upper solutions are sometimes called quasi-sub and quasi-super solution, respectively.

Remark. If α and $\hat{\alpha}$ are two lower solutions of the same problem, then $\overline{\alpha}(y) = \max\{\alpha(y), \hat{\alpha}(y)\}$ is also a lower solution of this problem. And if β and $\hat{\beta}$ are two upper solutions, then $\overline{\beta}(y) = \min\{\beta(y), \hat{\beta}(y)\}$ is an upper solution also.

Theorem 4.24. *Assume there exist α, $\beta \in C(\overline{\Omega})$ (in the case of the periodic problem, $\Omega = \mathbf{R}^n$) which are respectively, lower and upper solutions of* (4.16), (4.17k), $k = a, b, c$. *Let $\alpha(x) \le \beta(x)$, $x \in \overline{\Omega}$. Assume that f satisfies a*

Nagumo condition. Then the problem (4.16), (4.17k) has a solution u such that $\alpha(x) \le u(x) \le \beta(x)$, $x \in \overline{\Omega}$.

Proof. We consider the problem (4.16), (4.17b). The other problems may be treated similarly. Then we define $\hat{f}(x,u,p)$ as follows

$$\hat{f}(x,u,p) = \begin{cases} f(x,\beta(x),p) + u - \beta(x), & \text{if} \quad u > \beta \\ f(x,u,p), & \text{if} \quad \alpha(x) \le u \le \beta(x) \\ f(x,\alpha(x),p) + u - \alpha(x), & \text{if} \quad u < \alpha(x) \end{cases} \tag{4.21}$$

and consider the problem

$$\begin{cases} Lu(x) = \hat{f}(x,u,\nabla u), \quad x \in \Omega \\ p(x)u(x) + q(x)\dfrac{du(x)}{d\nu} = \varphi(x), \quad x \in \partial\Omega. \end{cases} \tag{4.22}$$

Choose a constant $\bar{\beta} > 0$ so large so that

$$\hat{f}(x,-\bar{\beta},0) < \hat{f}(x,\bar{\beta},0), \quad x \in \overline{\Omega}$$

and that $|\varphi(x)| \le p(x)\bar{\beta}$. It follows from Corollary 4.22 (b) that (4.22) has a solution u such that $|u(x)| \le \bar{\beta}$. We show that $\alpha(x) \le u(x) \le \beta(x)$, $x \in \overline{\Omega}$, and thus conclude that u is a solution of the original problem. Let us show that $u(x) \le \beta(x)$ for $x \in \overline{\Omega}$, that $\alpha(x) \le u(x)$, $x \in \overline{\Omega}$ will follow similarly. Let $z(x) = u(x) - \beta(x)$ and assume that there exists $x \in \overline{\Omega}$ such that $z(x) > 0$. Then there exists a smallest positive constant ϵ such that $u(x) \le \beta(x) + \epsilon$, and hence a point $x_0 \in \overline{\Omega}$ such that $u(x_0) = \beta(x_0) + \epsilon$. If $x_0 \in \partial\Omega$, then there exists a neighborhood U of x_0 such that

$$\beta(x) = \min_{1 \le r \le s} \beta_r(x), \quad x \in U \cap \overline{\Omega}$$

where $\beta_r \in C^2(U \cap \overline{\Omega})$ for $r = 1, \cdots, s$. Let $\beta(x_0) = \beta_i(x_0)$ for some i. Then $u(x) \leq \beta_i(x) + \epsilon,\ x \in U \cap \overline{\Omega}$ and $u(x_0) = \beta_i(x_0) + \epsilon$. Thus

$$\frac{du(x_0)}{d\nu} \geq \frac{d\beta_i(x_0)}{d\nu}$$

and therefore

$$\begin{aligned}
\varphi(x_0) &= p(x_0)u(x_0) + q(x_0)\frac{du(x_0)}{d\nu} \\
&\geq p(x_0)u(x_0) + q(x_0)\frac{d\beta_i(x_0)}{d\nu} \\
&= p(x_0)\beta_i(x_0) + p(x_0)\epsilon + q(x_0)\frac{d\beta_i(x_0)}{d\nu} \\
&\geq \varphi(x_0) + p(x_0)\epsilon,
\end{aligned}$$

a contradiction if $p(x_0) \neq 0$. Thus $x_0 \in \Omega$, and as above we find a neighborhood U of x_0 and i such that $u(x) \leq \beta_i(x) + \epsilon,\ x \in U,\ u(x_0) = \beta_i(x_0) + \epsilon$. Thus it follows that

$$\frac{\partial u(x_0)}{\partial x_j} = \frac{\partial \beta_i(x_0)}{\partial x_j},\ 1 \leq j \leq n.$$

Computing $(Lu)(x_0)$, we obtain

$$\begin{aligned}
(Lu)(x_0) &= f(x_0, \beta_i(x_0), \nabla\beta_i(x_0)) + \epsilon \\
&\geq (L\beta_i)(x_0) + \epsilon.
\end{aligned}$$

Thus $L(u - \beta_i)(x_0) \geq \epsilon > 0$, and hence there exists a neighborhood V of x_0 such that $L(u - \beta_i) \geq 0$, and then by the maximum principle we obtain that $(u - \beta_i)(x)$ =constant in V, which is contradicting to $L(u - \beta_i)(x_0) \geq \epsilon$. $\square$

Remark. In case f is independent of ∇u the above argument can be extended to the case where $p(x)$ may vanish on $\partial\Omega$. If $p(x_0) = 0$, then $q(x_0) > 0$ and hence

$$\varphi(x_0) = q(x_0)\frac{du(x_0)}{d\nu} \geq q(x_0)\frac{d\beta_i(x_0)}{d\nu} \geq \varphi(x_0)$$

that is, $\frac{du(x_0)}{d\nu} = \frac{d\beta(x_0)}{d\nu}$. On the other hand

$$Lu(x_0) = f(x_0, \beta_i(x_0), \cdot) + \epsilon \geq L\beta_i(x_0) + \epsilon,$$

and hence there exists a neighborhood U of x_0 such that

$$L(u - \beta_i(x)) > 0, \quad x \in U \cap \overline{\Omega},$$

by the maximum principles we get a contradiction.

5. Existence of a Classical Solution on Some Elliptic Boundary Value Problems

In this section we shall deal with the differential equation of the form

$$\Delta u + b(x) \cdot \nabla u + f(u) = 0 \quad \text{in } \Omega$$

where u is a real-valued function and b is a vector field in $\mathbf{R}^n$.

Remark. We obtain an obvious result as a corollary of Theorem 4.24 as follows: Let $f(0) > 0$. If there exists $\beta \in [0, \infty)$ such that $f(\beta) \leq 0$, then the Dirichlet boundary value problem

$$\begin{aligned} &\Delta u + b(x) \cdot \nabla u + f(u) = 0 \quad \text{in } \Omega \\ &u = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{5.1}$$

has a solution $u : \overline{\Omega} \longrightarrow [0, \beta]$ such that $0 \leq u \leq \beta$, where $b(x) = (b_1(x), b_2(x), \cdots, b_n(x))$ is of class $C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$.

Remark. We denote for two real-valued function u, v defined in Ω $u \leq v$ when $u(x) \leq v(x)$ for all $x \in \Omega$, we write $u < v$ when $u \leq v$ and there exists at least one point $x \in \overline{\Omega}$ such that $u(x) < v(x)$. We say a function u is positive when $v > 0$, where 0 denotes a zero function.

Theorem 5.1. *Let $f : [0, \infty) \longrightarrow [0, \infty)$ belong to class C^α and be bounded, and let $b \in C^\alpha(\overline{\Omega})$. Then the problem (5.1) has a positive solution $u : \overline{\Omega} \longrightarrow [0, \infty)$.*

Proof. Since $\alpha = 0$ is a lower solution of (5.1) it suffices to find a upper solution β of the problem with $\beta \geq 0$. Let $M = \sup_{u \in [0, \infty)}\{f(u)\}$. Then

$0 \leq M < \infty$. We consider the linear boundary value problem

$$\begin{cases} \Delta u + b(x) \cdot \nabla u + M = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1a}$$

The problem $(5.1a)$ has a solution $\beta \in C^{2,\alpha}(\overline{\Omega})$ by Theorem 3.30. Then β is a upper solution of the problem (5.1), since

$$\Delta\beta + b(x) \cdot \nabla\beta + f(\beta) \leq \Delta\beta + b(x) \cdot \nabla\beta + M = 0 \quad \text{in } \Omega,$$

and $\beta = 0$ on $\partial\Omega$. Furthermore, $\beta(x) \geq 0$ for all $x \in \Omega$. For if there exists a point $x_0 \in \Omega$ such that $\beta(x_0) < 0$, then since $\Delta\beta(x) + b(x) \cdot \nabla\beta(x) = -M \leq 0$ for all $x \in \Omega$, β cannot have its minimum in Ω by Maximum principle. Hence since $\beta = 0$ on $\partial\Omega$, $\beta \geq 0$ throughout $\overline{\Omega}$. Consequently, we have a solution $u \in [0, \beta]$ of the problem (5.1) by Theorem 4.24. □

Now we give a existence theorem of one of our problems under suitable conditions in case that f is unbounded.

Theorem 5.2. *Let $f : [0, \infty) \longrightarrow [0, \infty)$ belong to class C^α with $f(0) > 0$. Assume that $b \in C^\alpha(\overline{\Omega})$ and $b(x) \cdot x \geq 0$ for $x \in \overline{\Omega}$. Let*

$$N^{\min} = \min\{|x|^2 : x \in \overline{\Omega}\}, \quad N^{\max} = \max\{|x|^2 \,|x \in \overline{\Omega}\},$$

and assume there exists $r > 0$ such that $r^2 - N^{\min} \geq 0$, $r^2 - N^{\max} \geq 0$ and $f(u) \leq 2n$ for all $u \in [0, r^2 - N^{\min}]$. Then the problem (5.1) has a positive solution.

Proof. Let $v(x) = v(x_1, x_2, \cdots, x_n) = r^2 - (x_1^2 + x_2^2 + \cdots + x_n^2)$. Then $v(x) \in [0, r^2 - N^{\min}]$ for all $x \in \overline{\Omega}$, and hence $f(v(x)) \leq 2n$. Thus $v(x) \geq 0$

for $x \in \partial\Omega$ and

$$\begin{aligned}\Delta v(x) + b(x) \cdot \nabla v(x) + f(v(x)) &= -2n - 2b(x) \cdot x + f(v(x)) \\ &\leq -2n + f(v(x)) \\ &\leq 0 \quad x \in \Omega.\end{aligned}$$

That is, v is a upper solution of the problem (5.1), and $u = 0$ is a lower solution of the problem. We note that $v = 0$ if and only if $|x|^2 = r^2$ for all $x \in \overline{\Omega}$, which is impossible since Ω is a domain, thus $v > 0$. Then by Theorem 4.24 we obtain a solution $u \in [0, v]$ of our problem. Since $f(0) > 0$, our solution u is positive. □

Theorem 5.3. *Let $b \in C^\alpha$. Assume that $f : [0, \infty) \longrightarrow [0, \infty)$ satisfies a Nagumo condition and is differentiable, $f(0) > 0$ and satisfies*

$$0 < \max_{u \in [0,\infty)} f'(u) \leq c, \quad 0 < c\gamma < 1$$

where γ is the constant in Lemma 4.8 for p sufficiently large so that Lemma 4.9 holds. Then the problem

$$\begin{cases} \Delta u + b(x) \cdot \nabla u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2}$$

has a positive solution.

Proof. We note that $u_0 = 0$ is a lower solution of (5.2). Consider a linear equation

$$\begin{cases} \Delta u + b(x) \cdot \nabla u + f(0) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3.0}$$

Then the problem (5.3.0) has a unique solution $u_1 \in C^{2,\alpha}(\overline{\Omega})$ and then u_1 is a lower solution of (5.2), since f is a strictly increasing function. Since

$\Delta u_1 + b(x)\cdot\nabla u_1 = -f(0) < 0$ in Ω and $u_1 = 0$ on $\partial\Omega$, by Maximum principle $u_1 \geq 0$ in $\overline{\Omega}$. Actually, $u > 0$, otherwise $f(0) = 0$.

For each $m = 0, 1, 2, \cdots$, define u_{m+1} as the solution of the problem

$$\begin{cases} \Delta u + b(x)\cdot\nabla u + f(u_m) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3.m}$$

Then u_{m+1} is a lower solution of (5.2).

To show $u_0 < u_1 < u_2 < \cdots\cdots$, assume $u_0 < u_1 < \cdots < u_{m-1} < u_m$. We shall show that $u_m < u_{m+1}$. Since $u_{m+1} - u_m = 0$ on $\partial\Omega$ and f is strictly increasing, we have

$$\begin{aligned} &\Delta(u_{m+1} - u_m) + b(x)\cdot\nabla(u_{m+1} - u_m) \\ &= f(u_{m-1}) - f(u_m) \\ &\leq 0, \quad x \in \Omega, \end{aligned}$$

and hence $u_{m+1} - u_m \geq 0$ throughout $\overline{\Omega}$, by Maximum principle. Moreover if $u_{m+1} = u_m$ in Ω, then

$$\begin{aligned} 0 &= \Delta u_{m+1} + b(x)\cdot\nabla u_{m+1} + f(u_m) \\ &= \Delta u_m + b(x)\cdot\nabla u_m + f(u_m) \\ &= f(u_m) - f(u_{m-1}) \quad \text{in } \Omega, \end{aligned}$$

contradiction to f is strictly increasing. Therefore, $u_{m+1}(x) > u_m(x)$ for some $x \in \Omega$, and consequently, we obtain a strictly increasing sequence

$$u_0 < u_1 < u_2 < \cdots\cdots.$$

We define $T(u_m)$ by the solution of the problem (5.3.m). Then we have seen that T is compact and continuous and $T(u_m) = u_{m+1}$. By Lemma 4.8,

$$\begin{aligned} \|u_{m+1}\|_{W^{2,p}} &\leq \|\Delta u_{m+1} + b(x)\cdot\nabla u_{m+1}\|_{L^p} \\ &\leq \gamma\|f(u_m)\|_{L^p}. \end{aligned}$$

Thus
$$\begin{aligned}\|u_{m+1}-u_m\|_{W^{2,p}} &\le \gamma\|f(u_m)-f(u_{m-1})\|_{L^p}\\ &\le c\gamma\|u_m-u_{m-1}\|_{L^p}.\end{aligned}$$
By the definition,
$$\begin{aligned}\|u_{m+1}-u_m\|_{L^p} &\le \|u_{m+1}-u_m\|_{W^{2,p}}\\ &\le c\gamma\|u_m-u_{m-1}\|_{L^p}.\end{aligned}$$
This implies
$$\begin{aligned}\|u_{m+1}-u_m\|_{L^p} &\le c\gamma\|u_m-u_{m-1}\|_{L^p}\\ &\le \cdots\\ &\le (c\gamma)^m\|u_1-u_0\|_{L^p}\\ &= (c\gamma)^m\|u_1\|_{L^p},\end{aligned}$$
and so
$$\begin{aligned}\|u_n-u_m\|_{L^p} &\le \|u_n-u_{n-1}\|_{L^p}+\|u_{n-1}-u_{n-2}\|_{L^p}+\cdots+\|u_{m+1}-u_m\|_{L^p}\\ &\le (c\gamma)^{n-1}\|u_1\|_{L^p}+(c\gamma)^{n-2}\|u_1\|_{L^p}+\cdots+\|u_1\|_{L^p}\\ &\le \frac{(c\gamma)^m-(c\gamma)^n}{1-c\gamma}\|u_1\|_{L^p}.\end{aligned}$$
Then $\|u_n-u_m\|_{L^p}\to 0$ as $n,m\to\infty$, since $0<c\gamma<1$. Thus $\{u_m\}$ forms a Cauchy sequence in $L^p(\overline{\Omega})$. Furthermore, since
$$\begin{aligned}\|u_n-u_m\|_{W^{2,p}} &\le \gamma\|f(u_{n-1})-f(u_{m-1})\|_{L^p}\\ &\le c\gamma\|u_{m-1}-u_{m-1}\|_{L^p},\end{aligned}$$
$\{u_m\}$ is also a Cauchy sequence in $W^{2,p}(\overline{\Omega})$. Since $W^{2,p}(\overline{\Omega})$ is continuously embedded in $C^{1,\alpha}(\overline{\Omega})$ by Lemma 4.9, $\{u_m\}$ is a Cauchy sequence in $C^{1,\alpha}(\overline{\Omega})$, and so there exists $u\in C^{1,\alpha}(\overline{\Omega})$ such that $\lim_{m\to\infty}u_m=u$ in $C^{1,\alpha}(\overline{\Omega})$. Since T is compact and continuous,
$$T=\lim_{m\to\infty}Tu_m=\lim_{m\to\infty}u_{m+1}=u.$$

and hence $u = Tu$ and $u \in C^{2,\alpha}(\overline{\Omega})$. Therefore u is a classical solution of

$$\begin{cases} \Delta u + b(x) \cdot \nabla u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

□

Corollary 5.4. *Assume the constant γ satisfies the hypotheses of Theorem 5.3. Let $f(u) = \alpha u + r$, where $r > 0$ and $0 < \alpha < c$ with $0 < c\gamma < 1$. Then for all functions $g : [0, \infty) \longrightarrow (0, \infty)$ such that*

$$0 < g(u) \leq \alpha u + r \qquad \textit{for all} \quad u,$$

the problem

$$\begin{cases} \Delta u + b(x) \cdot \nabla u + g(u) = 0 & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega \end{cases} \tag{5.4}$$

has a positive solution.

Proof. By Theorem 5.3 there exists a solution $\hat{u} > 0$ of the problem (5.2) with $f(u) = \alpha u + r$ and then $\hat{u}$ is a upper solution of the problem (5.4). Since $u_0 = 0$ is a lower sol·ˑtion of our problem, we have a solution $u \in [0, \hat{u}]$ of the problem (5.4). □

Remark. We denote $H_0^1(\Omega)$ the completion of $C_0^1(\overline{\Omega})$, that is, the class of Cauchy sequences of functions u in $C_0^1(\overline{\Omega})$ with norm defined by: if $u = \{u_1, u_2, \cdots\cdots\}$, $v = \{v_1, v_2, \cdots\cdots\}$ are Cauchy sequences in $H_0^1(\Omega)$,

$$\|u - v\| = \lim_{n \to \infty} \|u_n - v_n\|_{C^1}.$$

Then $H_0^1(\Omega)$ is a Hilbert space.

Corollary 5.5. *Let* $\lambda_0 = \inf_{u \in H_0^1(\Omega)} \left\{ \int_\Omega |\nabla u|^2 dx \,/ \int_\Omega u^2 dx \right\}$ *(λ_0 is well-defined and $\lambda_0 > 0$. See ref.[3], pp 125). Assume that the constants c, γ and α satisfy the hypotheses of Corollary 5.4, and that*

$$\nabla \cdot b(x) > m \quad \text{for all } x \in \Omega,$$

where $m > 0$ is a positive constant. Further we assume $0 < \alpha < \lambda_0 + \frac{m}{2}$. Let $0 \le f(u) \le \alpha u$ with $f(0) = 0$. Then the boundary value problem

$$\begin{cases} \Delta u + b(x) \cdot \nabla u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.5}$$

has no non-trivial positive solution.

Proof. Suppose that u is a positive solution of the problem (5.5). Then

$$\int_\Omega (\Delta u) u dx + \int_\Omega (b(x) \cdot \nabla u) u dx + \int_\Omega u f(u) dx = 0.$$

Since by Green's first identity

$$\int_\Omega (\Delta u) u dx = -\int_\Omega |\nabla u|^2 dx,$$

and by integration by parts

$$\begin{aligned} \int_\Omega (b(x) \cdot \nabla u) u dx &= -\frac{1}{2} \int_\Omega (\nabla \cdot b(x)) u^2 dx \\ &\le -\frac{m}{2} \int_\Omega u^2 dx. \end{aligned}$$

we obtain

$$0 \le -\int_\Omega |\nabla u|^2 dx - \frac{m}{2} \int_\Omega u^2 dx + \int_\Omega \alpha u^2 dx.$$

We note that

$$-\int_\Omega |\nabla u|^2 dx \leq -\lambda_0 \int_\Omega u^2 dx.$$

Thus we obtain

$$0 \leq -\lambda_0 \int_\Omega u^2 dx - \frac{m}{2} \int_\Omega u^2 dx + \int_\Omega \alpha u^2 dx$$

or

$$\left(\lambda_0 + \frac{m}{2} - \alpha\right) \int_\Omega u^2 dx \leq 0.$$

Since $\lambda_0 + \frac{m}{2} - \alpha > 0$, we obtain

$$\int_\Omega u^2 dx = 0.$$

Therefore $u(x) = 0$ for $x \in \Omega$, since u is continuous in $\overline{\Omega}$, which leads to a contradiction. □

Theorem 5.6. ([5]) *Let $\hat{u}$ be a solution of*

$$\begin{cases} Lu = f(x,u), & x \in \Omega \\ Bu = \varphi, & x \in \partial\Omega, \end{cases} \tag{5.6}$$

where f belongs to class C^α, and assume that $\frac{\partial f}{\partial u}$ is continuous. Further assume there exists an upper solution β of (5.6) with $\beta(x) \geq \hat{u}(x)$ for $x \in \overline{\Omega}$ and $B(\beta - \hat{u}) \neq 0$ on $\partial\Omega$. Let the linear variational equation $Lz = f_u(x, \hat{u}(x))z$ have no nontrivial solution z such that $z(x) \geq 0$ for $x \in \overline{\Omega}$ and that $Bz = \lambda(Bz - \varphi)$, $x \in \partial\Omega$, where $\lambda \geq 0$ is a constant. Then the boundary value problem (5.6) has another solution $u \neq \hat{u}$ such that $\hat{u}(x) \leq u(x) \leq \beta(x)$, $x \in \partial\Omega$.

Proof. Consider the sequence of problems

$$\begin{cases} Lu = f(x,u), & x \in \Omega \\ Bu = \dfrac{1}{m}(B\beta)(x) + \dfrac{m-1}{m}\varphi(x), & x \in \partial\Omega \end{cases} \tag{5.6.m}$$

for $m = 1, 2, \cdots$. Since β is an upper solution of (5.6),

$$
\begin{aligned}
B\beta &= \frac{1}{m}(B\beta)(x) + \frac{m-1}{m}B\beta(x) \\
&\geq \frac{1}{m}(B\beta)(x) + \frac{m-1}{m}\varphi(x),
\end{aligned}
$$

and also

$$
\begin{aligned}
B\hat{u} = \varphi(x) &\leq \varphi(x) + \frac{1}{m}(B\beta - \varphi) \\
&= \frac{1}{m}B\beta + \frac{m-1}{m}\varphi
\end{aligned}
$$

on $\partial\Omega$. Hence β is an upper solution and $\hat{u}$ is a lower solution of (5.6.m) for each $m = 1, 2, \cdots$. But for $m = 2, 3, \cdots$, neither β nor $\hat{u}$ is a solution of (5.6.m), since $B\beta - \varphi \neq 0$ on $\partial\Omega$. Applying Theorem 4.24 we obtain a solution u_1 of (5.6.1) such that $\hat{u}(x) \leq u_1(x) \leq \beta(x)$, $x \in \overline{\Omega}$. Since u_1 is an upper solution of (5.6.2) we obtain a solution u_2 of (5.6.2) such that $\hat{u}(x) \leq u_2(x) \leq u_1(x)$, $x \in \overline{\Omega}$, and $u_2 \neq u_1$. Thus by induction we obtain for every $m = 2, 3, \cdots$, a solution u_m of (5.6.m) such that $\hat{u} \neq u_m \neq u_{m-1}$, and $\hat{u}(x) \leq u_m(x) \leq u_{m-1}(x)$. The sequence $\{u_m\}$ is bounded in $C(\overline{\Omega})$ and hence by the Nagumo condition (which is trivially satisfied because f is gradient independent) we have that $\{u_m\}$ has a convergent subsequence. Since it is a decreasing sequence, it converges to a solution u of (5.6). In terms of the equivalent operator equations we obtain

$$u_m = (T \circ F)(u_m) + \frac{1}{m}S(B\beta) + \frac{m-1}{m}S(\varphi).$$

If it is the case that $u = \hat{u}$, we may use the Frechet differentiability of $T \circ F$ to write

$$u_m = (T\circ F)(u) + (T\circ F)'(u)(u_m - u) + \frac{1}{m}S(B\beta) + \frac{m-1}{m}S(\varphi) + o(\|u_m - u\|_{C^1})$$

where $(T \circ F)'(u)$ is the Frechet derivative of $T \circ F$ at u and

$$\lim_{m \to \infty} \frac{o(\|u_m - u\|_{C^1})}{\|u_m - u\|_{C^1}} = 0.$$

But $(T \circ F)(u) = u - S(\varphi)$ and hence

$$u_m - u = (T \circ F)'(u)(u_m - u) + \frac{1}{m}\big(S(B\beta - \varphi)\big) + o(\|u_m - u\|_{C^1}).$$

Since $u_m \to u$ we may let $y_m = \dfrac{u_m - u}{\|u_m - u\|_{C^1}}$ and obtain

$$y_m = (T \circ F)'(u) y_m + (m\|u_m - u\|_{C^1})^{-1} S(B\beta - \varphi) + \frac{o(\|u_m - u\|_{C^1})}{\|u_m - u\|_{C^1}}.$$

We note that if $K \in C^1$ is a compact operator, then K' is also compact. Thus $(T \circ F)'(u)$ is a compact linear operator and $\|y_m\|_{C^1} = 1,\ m = 1, 2, \cdots$, it follows that we may find a sequence of integers m_k such that

$$\lim_{k \to \infty} y_{m_k} = y \in C^1(\overline{\Omega}), \qquad \lim_{k \to \infty} (m_k \|u_{m_k} - u\|_{C^1})^{-1} = \lambda \geq 0,$$

exist, and

$$y = (T \circ F)'(u) y + \lambda S(B\beta - \varphi)$$

therefore, y is a solution of the boundary value problem

$$\begin{cases} Lz = f_u(x, \hat{u}(x))z, & x \in \Omega \\ Bz = \lambda(B\beta - \varphi), & x \in \partial\Omega \end{cases}$$

and $\|y\|_{C^1} = 1,\ y(x) \geq 0,\ x \in \overline{\Omega}$, which is a contradiction to our assumption. Therefore $\hat{u} \neq u$ and $\hat{u}(x) \leq u(x) \leq \beta(x),\ x \in \overline{\Omega}$. $\square$

Lemma 5.7. ([10]) *There exists a positive constant μ such that the boundary value problem*

$$\begin{cases} \Delta\varphi - x\cdot\nabla\varphi + \mu\varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{5.7}$$

has a positive solution.

Proof. Let T be the inverse operator of $-\Delta + x\cdot\nabla$ subject to zero Dirichlet boundary condition. The maximum principle implies that T is a positive operator. T is also compact. Hence by the Krein-Rutman Theorem ([1]) there exists a positive eigenfunction φ and a positive eigenvalue μ satisfying

$$\begin{cases} \Delta\varphi - x\cdot\nabla\varphi + \mu\varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

□

Theorem 5.8. *Let μ be the constant defined in Lemma 5.7 and let c be a constant satisfying the hypotheses of Theorem 5.3. Let $f : [0,\infty) \longrightarrow [0,\infty)$, $f(0) = 0$. Assume that $f'(0) > \mu + n$, $\limsup_{u\to\infty} \frac{f(u)}{u} < c$. Then the boundary value problem*

$$\begin{cases} \Delta u + x\cdot\nabla u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.8}$$

has a positive solution.

Proof. We note that $u_0 = 0$ is a solution of (5.8). Since $\limsup_{u\to\infty} \frac{f(u)}{u} < c$, there exists a real number r sufficiently large so that $f(u) \le cu + r$ for all u. Let $\Omega_\epsilon = \{x \in \mathbf{R}^n : dist(x,\Omega) \le \epsilon\}$, then $\overline{\Omega} \subset \Omega_\epsilon$. Taking ϵ sufficiently small so that $f(u(x)) \leq cu(x) + r$ for $x \in \Omega_\epsilon$ and applying Theorem 5.3 the problem

$$\begin{cases} \Delta u + x \cdot \nabla u + cu + r = 0, & x \in \Omega_\epsilon \\ u = 0, & x \in \partial\Omega_\epsilon \end{cases}$$

has a positive solution β, and then β is an upper solution of (5.8) such that $B\beta \neq 0$ on Ω, since $\beta > 0$ in Ω_ϵ by the maximum principle. We consider a boundary value problem for the linear variational equation

$$\begin{cases} \Delta u + x \cdot \nabla u + f'(0)u = 0 & \text{in } \Omega \\ u = \lambda & \text{on } \partial\Omega \end{cases} \tag{5.8.a}$$

where λ is a nonnegative constant. Suppose that (5.8.a) has a nontrivial positive solution u and let φ be the positive solution (by Lemma 5.7) of the problem

$$\begin{cases} \Delta\varphi - x \cdot \nabla\varphi + \mu\varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.8.b}$$

Compute

$$\int_\Omega (\varphi\Delta u + \varphi\, x \cdot \nabla u + f'(0)\varphi u)\, dx = 0 \tag{5.8.c}$$

and

$$\int_\Omega (u\Delta\varphi - u\, x \cdot \nabla\varphi + \mu u\varphi)\, dx = 0. \tag{5.8.d}$$

Since $\varphi = 0$ on $\partial\Omega$ by Green's identity,

$$\int_\Omega (\varphi\Delta u - u\Delta\varphi)\, dx = -\int_{\partial\Omega} u \frac{d\varphi}{d\nu} dx.$$

Integration by parts yields

$$\int_\Omega \varphi\, x \cdot \nabla u dx = -\int_\Omega u\, x \cdot \nabla\varphi dx - n\int_\Omega \varphi u dx.$$

Subtracting (5.8.d) from (5.8.c) we obtain

$$\int_{\partial\Omega} u \frac{d\varphi}{d\nu} dx = (f'(0) - \mu - n) \int_{\Omega} \varphi u dx.$$

Since $\frac{d\varphi}{d\nu} \leq 0$ on $\partial\Omega$,

$$(f'(0) - \mu - n) \int_{\Omega} \varphi u dx \leq 0.$$

But $\varphi(x) > 0$, $u(x) > 0$ for $x \in \Omega$ by the maximum principle and $f'(0) - \mu - n > 0$ and hence

$$(f'(0) - \mu - n) \int_{\Omega} \varphi u dx > 0,$$

which leads to a contradiction. Therefore the problem (5.8.a) does not have any nontrivial positive solution, and hence by Theorem 5.6 our problem (5.8) has a solution u such that $u \not\equiv 0$, $0 \leq u(x) \leq \beta(x)$, $x \in \overline{\Omega}$. □

Lemma 5.9. *Let p be an even natural number, $p > 2$. Then there is a constant $\lambda > 0$ which depends only on Ω and p such that*

$$\int_{\Omega} u^{p-2} |\nabla u|^2 dx \geq \lambda \int_{\Omega} u^p dx,$$

for all $u \in C_0^1(\overline{\Omega})$.

Proof. Let $p = 2q$. Since Ω is a bounded domain it can be enclosed in a cube $\Gamma = \{x \in \mathbf{R}^n : |x_i| \leq a \quad \text{for } i = 1, 2, \cdots, n\}$. We continue u as identically zero outside Ω. Then for any $x = (x_1, x_2, \cdots, x_n) \in \Gamma$

$$\begin{aligned} u^q(x) &= \int_{-a}^{x_1} \frac{\partial}{\partial x_1} (u^q(t, x_2, \cdots, x_n))\, dt \\ &= q \int_{-a}^{x_1} u^{q-1} \frac{\partial u}{\partial x_1} dt. \end{aligned}$$

and by Schwartz inequality

$$q\int_{-a}^{x_1} u^{q-1}\frac{\partial u}{\partial x_1}dt \le q\left[\int_{-a}^{x_1} u^{2(q-1)}\Big(\frac{\partial u}{\partial x_1}\Big)^2 dt\right]^{\frac{1}{2}}(x_1+a)^{\frac{1}{2}}.$$

Then

$$\begin{aligned} u^p = u^{2q} &\le q^2(x_1+a)\int_{-a}^{x_1} u^{2(q-1)}\Big(\frac{\partial u}{\partial x_1}\Big)^2 dt \\ &\le 2a(\frac{p}{2})^2\int_{-a}^{a} u^{p-2}\Big(\frac{\partial u}{\partial x_1}\Big)^2 dt. \end{aligned}$$

Thus

$$\int_{-a}^{a} u^p dx_1 \le 4a^2(\frac{p}{2})^2\int_{-a}^{a} u^{p-2}\Big(\frac{\partial u}{\partial x_1}\Big)^2 dx_1,$$

and then integrating over $x_2,\cdots,x_n$ from $-a$ to a we find

$$\begin{aligned} \int_\Omega u^p dx &= \int_\Gamma u^p dx \\ &= \int_{-a}^{a}\cdots\int_{-a}^{a} u^p dx_1\cdots dx_n \\ &\le 4a^2(\frac{p}{2})^2\int_{-a}^{a}\cdots\int_{-a}^{a} u^{p-2}\Big(\frac{\partial u}{\partial x_1}\Big)^2 dx_1\cdots dx_n \\ &= a^2p^2\int_\Omega u^{p-2}\Big(\frac{\partial u}{\partial x_1}\Big)^2 dx \\ &\le a^2p^2\int_\Omega u^{p-2}|\nabla u|^2 dx. \end{aligned}$$

Therefore

$$\int_\Omega u^{p-2}|\nabla u|^2 dx \ge \lambda\int_\Omega u^p dx,$$

where $\lambda = \frac{1}{a^2p^2} > 0$. □

Lemma 5.10. *Let p be an even natural number sufficiently large so that Lemma 4.9 holds. Let γ be the constant defined in Lemma 4.8, and assume $\nabla\cdot b(x) > m > 0$ for all $x\in\Omega$. Assume that $0 < c\gamma < 1$ and $0 < c <$*

$\lambda(p-1)+\frac{m}{p}$, where λ is the constant defined in Lemma 5.9. For $r>0$ let u be a positive solution of the problem

$$\begin{cases} \Delta u + b(x)\cdot\nabla u + cu + r = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.10}$$

Then there exists a constant δ depending only on Ω and p such that

$$\|u\|_{C^0} \le r\delta.$$

Proof. Let u be a positive solution of (5.10). Then

$$\int_\Omega \left(u^{p-1}\Delta u + u^{p-1}b(x)\cdot\nabla u + cu^p + ru^{p-1}\right)dx = 0.$$

By Green's identity and direct computation,

$$\begin{aligned}\int_\Omega u^{p-1}\Delta u dx &= \int_\Omega u\Delta(u^{p-1})dx \\ &= \int_\Omega u\left[(p-1)(p-2)u^{p-3}|\nabla u|^2 + (p-1)u^{p-2}\Delta u\right]dx \\ &= (p-1)(p-2)\int_\Omega u^{p-2}|\nabla u|^2 dx + (p-1)\int_\Omega u^{p-1}\Delta u dx,\end{aligned}$$

thus

$$\int_\Omega u^{p-1}\Delta u dx = -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx.$$

By integration by parts,

$$\begin{aligned}\int_\Omega u^{p-1}b(x)\cdot\nabla u dx &= -(p-1)\int_\Omega u^{p-1}b(x)\cdot\nabla u dx - \int_\Omega u^p\big(\nabla\cdot b(x)\big)dx \\ &\le -(p-1)\int_\Omega u^{p-1}b(x)\cdot\nabla u dx - m\int_\Omega u^p dx,\end{aligned}$$

and hence

$$\int_\Omega u^{p-1}b(x)\cdot\nabla u dx \le -\frac{m}{p}\int_\Omega u^p dx.$$

Then by Lemma 5.9 we obtain

$$0 \le -\lambda(p-1)\int_\Omega u^p dx - \frac{m}{p}\int_\Omega u^p dx + c\int_\Omega u^p dx + r\int_\Omega u^{p-1} dx,$$

and then by Hölder inequality

$$\begin{aligned}(\lambda(p-1) + \frac{m}{p} - c)\int_\Omega u^p dx &\le r\int_\Omega u^{p-1} dx \\ &\le r\left[\int_\Omega (u^{p-1})^{\frac{p}{p-1}} dx\right]^{\frac{p-1}{p}} \left[\int_\Omega 1 dx\right]^{\frac{1}{p}} \\ &= r\left[\int_\Omega u^p dx\right]^{1-\frac{1}{p}} |\Omega|^{\frac{1}{p}},\end{aligned}$$

or

$$(\lambda(p-1) + \frac{m}{p} - c)\left[\int_\Omega u^p dx\right]^{\frac{1}{p}} \le r|\Omega|^{\frac{1}{p}},$$

or $\|u\|_{L^p} \le rC_1$, where $C_1 = \dfrac{|\Omega|^{\frac{1}{p}}}{\lambda(p-1) + \frac{m}{p} - c}$.

By Lemma 4.9 there exists a constant C_2 such that

$$\|u\|_{C^1} \le C_2\|u\|_{W^{2,p}},$$

and by Lemma 4.8

$$\begin{aligned}\|u\|_{W^{2,p}} &\le \gamma\|\Delta u + x\cdot\nabla u\|_{L^p} \\ &\le \gamma\|cu + r\|_{L^p}.\end{aligned}$$

Therefore

$$\begin{aligned}\|u\|_{C^0} &\le C_2\gamma\|cu + r\|_{L^p} \\ &\le C_2\gamma c\|u\|_{L^p} + C_2\gamma|\Omega|^{\frac{1}{p}} r \\ &\le (cC_1C_2\gamma)r + (C_2\gamma|\Omega|^{\frac{1}{p}})r \\ &\le \delta r,\end{aligned}$$

where $\delta = 2\max\{cC_1C_2\gamma, C_2\gamma|\Omega|^{\frac{1}{p}}\}$. □

Theorem 5.11. *Assume that the hypotheses of Lemma 5.10 are satisfied and let $f : [0,\infty) \longrightarrow [0,\infty)$ with $f(0) > 0$. If $0 \leq f(u) \leq cu + r$ for $0 \leq u \leq \delta r$, where δ is the constant defined in Lemma 5.10, then the boundary value problem*

$$\begin{cases} \Delta u + b(x) \cdot \nabla u + f(u) = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{5.11}$$

has a solution.

Proof. Let $\hat{u}$ be a positive solution of (5.10). Then $0 \leq \hat{u}(x) \leq \delta r$, and $\hat{u}$ is an upper solution of (5.11), since

$$\Delta \hat{u} + b(x) \cdot \nabla \hat{u} + f(\hat{u}) \leq \Delta \hat{u} + b(x) \cdot \nabla \hat{u} + c\hat{u} + r = 0.$$

We note that $u_0 = 0$ is a lower solution of (5.11). Hence there exists a solution u of (5.11) such that $0 \leq u(x) \leq \hat{u}(x)$, $x \in \overline{\Omega}$. $\square$

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<**Abstract** >

SECOND ORDER SEMI-LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS AND DIFFERENTIAL INEQUALITIES

In this paper we introduce established theorems about existence of solutions of second order linear and semi-linear elliptic boundary value problems and then use them to derive the existence of positive solutions of the following second order semi-linear elliptic boundary value problems defined in a bounded domain Ω in which the boundary is differentiable;

$$\begin{cases} \Delta u(x) + b(x) \cdot \nabla u(x) + f(u(x)) = 0, & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

To discuss the existence of the solutions, we use the upper solution- lower solution method and integral inequality.

감사의 글

2년간 제가 중도에 포기하지 않고 학위과정을 마칠 수 있도록 조언과 채찍질을 아끼지 않으시고 본 논문이 나올 수 있도록 세세하게 지도해 주신 고봉수교수님께 진심으로 감사드립니다.

제게 늘 관심 가져 주시고, 학업에 열중할 수 있도록 격려와 배려를 해 주신 수학과, 수학교육과 모든 교수님께도 깊은 감사를 드립니다.

그리고 4학기 동안 지속적으로 공부할 수 있도록 도와준 대학원생 이지순과 친구 오성철에게도 고마움을 전합니다.

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