ON THE SPACE OF COMPLEX MEASURES

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감 사 의 글

이 논문이 완성될 수 있도록 세심한 배려를 하여 주신 한철순 교수님께 충심으로 감사를 드립니다.
또한 끝까지 지도와 편달을 하여주신 현진오 교수님, 홍성관 교수님 그리고 수학교육과 여러 교수님께 깊은 감사를 드립니다.
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김 희 순

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국 문 초 록

복소축도들의 공간에 관한 연구

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이 논문은 복소측도들의 집합에다 하나의 거리함수를 도입하여, 그것이 완비성을 갖추고 있는지 조사한다. 아울러 특정한 형태인 복소정칙 BOREL 측도들의 공간에 대한 완비성도 조사한다.

1. Introduction

In this paper we are going to study the following facts:

The preliminary section contains definitions, properties, and the Riesz representation theorem.

In section 2, It will be proved that the space of all complex measures on a σ − algebra $\mathbf{M}$ can be a Banach space under the operation defined by

$$(\mu + \lambda)(E) = \mu(E) + \lambda(E)$$

$$(\alpha\mu)(E) = \alpha\mu(E), \text{ where for any scalar } \alpha,$$

$E \in \mathbf{M}$ and μ, λ are complex measures.

and the norm defined by

$$\|\mu\| = |\mu|(X) \quad \text{and} \quad |\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|,$$

where the supremum is taken over all partitions $\{E_i\}$ of E.

It will be shown, in a special case, that the space of all complex regular Borel measures on a locally compact Hausdorff space is complete under the norm defined as above by using the Riesz Representation theorem.

2. Preliminaries

Definition 2-1 Let X be an arbitrary set. A collection of subsets of X is said to be a σ-algebra in X if **M** has the following properties: (i) $X \in \mathbf{M}$. (ii) If $A \in \mathbf{M}$, then $A^c \in \mathbf{M}$, wher A^c is the complement of A relative to X. (iii) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \mathbf{M}$ for $n = 1, 2, \cdots\cdots$, then $A \in \mathbf{M}$.

Definition 2-2 Let X be any set and let **M** be a σ-algebra in X. Call a countable collection $\{E_i\}$ of members of **M** a partition of E if $E_i \cap E_j = \phi$ whenever $i \neq j$, and if $E = \bigcup_{i=1}^{\infty} E_i$.

A complex measure μ on **M** is then a complex function on **M** such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \quad (E \in \mathbf{M}) \text{ for every partition } \{E_i\} \text{ of } E.$$

Definition 2-3 A set function $|\mu|$ on σ-algebra defined by $|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|$ $(E \in \mathbf{M})$, where the supremum is taken over all partitions $\{E_i\}$ of E, is called the total variation measure and the term "total variation of μ" is also frequently used to denote the number $|\mu|(X)$.

Proposition 2-1 The total variation $|\mu|$ of a complex measure μ on **M** is a positive measure on **M**.

Proof Let $\{E_i\}$ be a partition of E in **M** and let t_i be real numbers such that $t_i < |\mu|(E_i)$. Then each E_i has a partition $\{A_{ij}\}$ such that

$$|\mu|\ (E_i) \geqq \sum |\mu(A_{ij})| > t_i \quad (i=1,2,3,\cdots\cdots) \qquad (1)$$

Since $\{A_{ij}\}$ $(i,j=1,2,3\cdots\cdots)$ is a partition of E.

it follow that

$$\sum_i t_i \leqq \sum_{i,j} |\mu(A_{ij})| \leqq |\mu|\ (E) \qquad (2)$$

Taking the supremum of the left side of (2), over all admissible choice of $\{t_i\}$, we see that

$$\sum_i |\mu|\ (E_i) \leqq |\mu|\ (E) \qquad (3)$$

since $\sum_i |\mu|(E_i) = \sup \sum_{i,j} |\mu(A_{ij})|$.

To prove the opposite inequality, let $\{A_j\}$ be any partition of E. Then for any fixed j, $\{A_j \cap E_i\}$ is a partition of A_j, and for any fixed i, $\{A_j \cap E_i\}$ is a partition of E_i. Hence

$$\begin{aligned}\sum_j |\mu(A_j)| &= \sum_j \left|\sum_i \mu(A_j \cap E_i)\right| \\ &\leqq \sum_j \sum_i |\mu(A_j \cap E_i)| \qquad (4) \\ &= \sum_i \sum_j |\mu(A_j \cap E_i)| \\ &\leqq \sum_i |\mu|\ (E_i).\end{aligned}$$

Since (4) holds for every partition $\{A_j\}$ of E, we have

$$|\mu|\ (E) \leqq \sum_i |\mu|\ (E_i) \qquad (5)$$

By (3) and (5), $|\mu|$ is countably additive. that is

$$|\mu|\ (E) = \sum_i |\mu|\ (E_i)$$

Thus this proof is complete.

Lemma 2-2 If $z_1, z_2, \cdots\cdots, z_n$ are complex numbers, there is a subset S of $\{1,2,\cdots\cdots,n\}$ such that

$$\left|\sum_{j\in S} z_j\right| \geqq \frac{1}{6} \sum_{j=1}^{n} |z_j|$$

Proof Put $\omega = |z_1| + \cdots\cdots + |z_n|$. The complex plane is the union of four closed quadrants, bounded by the lines $y = \pm x$, and at least one of these quadrants Q has the property that the sum of the $|z_j|$ for which $z_j \in Q$ is at least $\omega/4$. For $z \in Q$, we have

$$\mathrm{Re}\, z \geqq |z|/\sqrt{2};$$

if S is the set of all j such that $z_j \in Q$, it follows that

$$\left|\sum_{j \in S} z_j\right| \geqq \sum_{j \in S} \mathrm{Re}\, z_j \geqq \frac{1}{\sqrt{2}} \sum_{j \in S} |z_j| \geqq \frac{\omega}{4\sqrt{2}} \geqq \frac{\omega}{6}$$

Hence $\left|\sum_{j \in S} z_j\right| \geqq \frac{1}{6} \sum_{j=1}^{n} |z_j|$

Theorem 2-3 If μ is a complex measure on X, then the total variation of μ is a finite measure.

Proof We first show: If $E \in \mathbf{M}$ and $|\mu|(E) = \infty$, then $E = A \cup B$, where A and $B \in \mathbf{M}$, $A \cap B = \emptyset$, and

$$|\mu(A)| > 1, \quad |\mu|(B) = \infty \qquad (1)$$

Indeed, the definition of $|\mu|$ shows that to every $t < \infty$ there corresponds a partition $\{E_i\}$ of E such that

$$\Sigma\, |\mu(E_j)| > t.$$

Let us take $t = 6(1 + |\mu(E)|)$. Then

$$\Sigma \quad |\mu(E_j)| > t \qquad (2)$$

for some n; and if we apply Lemma 2-2 with $z_j = \mu(E_j)$ and put

$$A = \bigcup_{j \in S} E_j \qquad (3)$$

it follows that $A \in E$ and $|\mu(A)| > t/6 \geqq 1$. If $B = E - A$, then

$|\mu(B)| = |\mu(E) - \mu(A)| \geqq |\mu(A)| - |\mu(E)| > t/6 - |\mu(E)| = 1$.

Since $|\mu|(E) = |\mu|(A) + |\mu|(B)$, by proposition 2 − 1, we have $|\mu|(A) = \infty$ or $|\mu|(B) = \infty$, and we obtain 1) by interchanging A and B.

Now assume that $|\mu|(X) = \infty$. put $B_0 = X$. Suppose $n \geqq 0$, and B_n is chosen so that $|\mu|(B_n) = \infty$. Then, applying (1) with B_n in place of E, we see that B_n is the union of two disjoint sets A_{n+1} and B_{n+1}, such that $|\mu(A_{n+1})| > 1$ and $|\mu|(B_{n+1}) = \infty$.

Thus we inductively obtain disjoint sets $A_1, A_2, A_3, \cdots$, with $|\mu(A_n)| > 1$. If $C = \cup A_n$, the countable additivity of μ show that

$$\mu(C) = \sum_{n=1}^{\infty} \mu(A_n)$$

But this seris can not converge, since $\mu(A_n)$ does not tend to 0 as $n \to \infty$. This contradiction shows $|\mu(X)| < \infty$.

Propostion 2−4. (Riesz Representation Theorem)

Let X be a locally Hausdorff space and let $C_0(X)$ be the class of all continuous complex functions on X which vanish at infinity. Then to each bounded linear functional Φ on $C_0(X)$, there corresponds a unique complex reqular Borel measure μ such that

$$\Phi(f) = \int_X f\, du \quad (f \in C_0(X)) \qquad (1)$$

Moreover, if Φ and μ are related as in (1), then

$$\|\Phi\| = |\mu|(X)$$

Proof. See Ref [1] Page 139.

3. Main Theorems

Definition 3 - 1 The operation on the space of complex measures on the σ-algebra $\mathbf{M}$ of any set X can be defined as

$$(\lambda + u)(E) = \lambda(E) + u(E)$$

$$(\alpha\lambda)(E) = \alpha\lambda(E), \text{ for any } E \text{ in } \mathbf{M}, \text{any scalar.}$$

and the norm on the space of complex measures can be defined as $\|u\| = |u|(X) \quad u(E) = \sup \sum_{i=1}^{\infty} |u(E_i)|$,

where the supremum is taken over all partitons $\{E_i\}$ of E.

Theorem 3 - 1 The collection of all complex measures of a σ-algebra $\mathbf{M}$ forms a normed vector space under the operation and norm defined as above.

Proof Let $\mathcal{J}$ be the collection of all complex measures of a σ-algebra $\mathbf{M}$. then this $\mathcal{J}$ is a vector space over $\mathbf{C}$.

i) $(\lambda + u)(\bigcup_{i=1}^{\infty} E_i) = \lambda(\bigcup_{i=1}^{\infty} E_i) + u(\bigcup_{i=1}^{\infty} E_i)$

$$= \sum_{i=1}^{\infty} \lambda(E_i) + \sum_{i=1}^{\infty} u(E_i)$$

$$= \sum_{i=1}^{\infty} (\lambda(E_i) + u(E_i))$$

$$= \sum_{i=1}^{\infty} (\lambda+u)(E_i), \text{ for each } E_i \in \mathbf{M}, E = \bigcup_{i=1}^{\infty} E_i$$

and $E_i \cap E_j = \phi$ if $i \neq j$.

ii) $(c\lambda)(\bigcup_{i=1}^{\infty} E_i) = c\lambda(\bigcup_{i=1}^{\infty} E_i)$

$$= c\left(\sum_{i=1}^{\infty} \lambda(E_i)\right)$$

$$= \sum_{i=1}^{\infty} c\lambda(E_i)$$

$$= \sum_{i=1}^{\infty} (c\lambda)(E_i), \text{ for } E \in \mathbf{M},\ c \in \mathbf{C}.$$

ii) Take z by the rule

$$z(E) = 0 \quad \text{for all } E \in \mathbf{M}$$

Then $(\lambda + z)(E) = \lambda(E) + z(E)$

$= \lambda(E)$

$= z(E) + \lambda(E)$

$= (z + \lambda)(E).$

This means z is a identity with respect to the addition.

For $1 \in \mathbf{C}$

$(1 \cdot \lambda)(E) = 1(\lambda(E)) = \lambda(E)$ that is $1 \cdot \lambda = \lambda$.

iii) $-\lambda$ is the inverse of λ in the sense that

$-\lambda(E) + \lambda(E) = z(E)$, that is $-\lambda + \lambda = z$

iv) Another requirements for vector space are also satisfied, one by one autonomically.

Second, we must prove $\mathcal{J}$ is the normed vector space

i) $\|u\| = |u|(X) \geqq 0$ for each $u \in \mathcal{J}$

ii) $\|u\| = |u|(X) = 0$ iff for any $E \in \mathbf{M}$ $|u|(E) = 0$

iff $|u|(E) = \sup \sum_{i=1}^{\infty} |u(E_i)| = 0$ for all partitions $\{E_i\}$ of E

iff $|u(E)| = 0$

iff $u(E) = 0$.

iii) $||\alpha u|| = |\alpha u|(X)$

for any $E \in \mathbf{M}$

$$|\alpha u|(E) = \sup \sum_{i=1}^{\infty} |\alpha u(E_i)|$$

$$= \sup \sum_{i=1}^{\infty} |\alpha|\ |u(E_i)|$$

$$= |\alpha| \sup \sum_{i=1}^{\infty} |u(E_i)|$$

$$= |\alpha|\,|u|(E), \text{ for all partitions of } E.$$

iv) $||u+v|| = |u+v|(X)$

for any $E \in \mathbf{M}$

$$|u+v|(E) = \sup \sum_{i=1}^{\infty} |(u+v)(E_i)|$$

$$= \sup \sum_{i=1}^{\infty} |u(E_i) + v(E_i)|$$

$$\leqq \sup \sum_{i=1}^{\infty} (|u(E_i)| + |v(E_i)|)$$

$$\leqq \sup \sum_{i=1}^{\infty} |u(E_i)| + \sup \sum_{i=1}^{\infty} |v(E_i)|$$

$$\leqq |u|(E) + |v|(E), \text{ for all partition } \{E_i\} \text{ of } E.$$

Hence we get

$$||u+v|| \leqq ||u|| + ||v||.$$

Definition 3 − 2 The collection $L = L(X, \mathbf{M}, u)$ of integrable functions consists of all complex-valued $\mathbf{M}$-measurable functions f defined on X, such that both the positive and negative parts f^+, f^- of f have finite integrals with respect to u.

Lemma 3 - 2 If $f_n \in L(X, \mathbf{M}, u)$ and if $\sum_{i=1}^{\infty} \int |f_n| \, du < +\infty$, Then the series $\sum f_n(X)$ converges almost evergwhere to a function f in $L(X, \mathbf{M}, u)$. Moreover

$$\int f \, du = \sum_{i=1}^{\infty} \int f_n \, du$$

Proof Let $A = \{x \in X : \sum_{n=1}^{\infty} |f_n(x)| < +\infty\}$

$$B = \{x \in X : \sum_{n=1}^{\infty} |f_n(x)| = \infty\}$$

Define $f(x) = \sum_{n=1}^{\infty} (f_n \chi_A)(x)$

we must show $u(B) = 0$ and $f \in L(X, \mathbf{M}, u)$

$$\sum_{n=1}^{\infty} \int |f_n| \, du = \lim_{n \to \infty} \sum_{k=1}^{n} \int |f_k| \, du$$

$$= \lim_{n \to \infty} \int \sum_{k=1}^{n} |f_k| \, du$$

By Monotone Canvergence Theorem,

$$\lim_{n \to \infty} \int \sum_{k=1}^{n} |f_k| \, du = \int \lim_{n \to \infty} \sum_{k=1}^{n} |f_k| \, du$$

$$= \int \sum_{n=1}^{\infty} |f_n| \, du$$

$$= \int_A \sum_{n=1}^{\infty} |f_n(x)| \, du + \int_B \sum_{n=1}^{\infty} |f_n| \, du < \infty$$

Hence $u(B) = 0$ and $\int |f| \, du \leqq \int \sum_{n=1}^{\infty} | f_n \chi_A | \, du < +\infty$

that is,

$f \in L(X, \mathbf{M}, u)$ and

$|\sum_{k=1}^{\infty} (f_k \chi_A(x)|$ is an integrable function.

By Lebesque Dominated Convergence Theorem.

$$\int f \, du = \int \lim_{n \to \infty} \sum_{k=1}^{n} f_k \chi_A \, du$$

$$= \lim_{n\to\infty} \int \sum_{k=1}^{n} f_k \chi_A \, du$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k \chi_A \, du$$

$$= \sum_{n=1}^{\infty} \int f_n \, du$$

Corollary 3-3 If $\alpha_{nk} \in \mathbf{C}$ and $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_{nk}| < \infty$, then $\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{nk} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{nk}$.

Proof Let X=N, $\mathbf{M}$ to be the σ-algebra of all subset of N and μ to be the counting measure on X.

Then $\sum_{n=1}^{\infty} \int |f_n| \, du = \sum_{n=1}^{\infty} \int_N |\alpha_{nk}| \, du = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_{nk}| < \infty$.

By Lemma 3-2

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{nk} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{nk} .$$

Lemma 3-4 Let λ_n's be complex measures on X such that $\sum_{n=1}^{\infty} \lambda_n$ is complex measure. Then,

$$\| \sum_{n=1}^{\infty} \lambda_n \| \leqq \sum_{n=1}^{\infty} \|\lambda_n\|$$

Proof Let λ_n's be complex measures on X. Then we have the following:

$$\| \sum_{n=1}^{\infty} \lambda_n \| = | \sum_{n=1}^{\infty} \lambda_n | (X).$$

For any $E \in \mathbf{M}$

$$| \sum_{n=1}^{\infty} \lambda_n | (E) = \sup \sum_{i=1}^{\infty} | \sum_{n=1}^{\infty} \lambda_n (E_i) |$$

$$\leq \sup \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |\lambda_n(E_i)|$$

$$\leq \sum_{n=1}^{\infty} \sup \sum_{i=1}^{\infty} |\lambda_n(E_i)|$$

$$= \sum_{n=1}^{\infty} |\lambda_n|(E), \quad \text{for all partitions } \{E_i\} \text{ of } E.$$

Hence we get

$$\| \sum_{n=1}^{\infty} \lambda_n \| \leq \sum_{n=1}^{\infty} \|\lambda_n\| .$$

Theorem 3－5 Let X be a set and let **M** be a σ-algebra on X. Then the normed vector space of complex measures is a complete metric space with the metric $d(\lambda, \mu) = \|\lambda - \mu\|$

Proof Let $\mathcal{J}$ be the normed vector space of complex measures. Choose a Cauchy sequence (λ_n) in $\mathcal{J}$.

We can always select a subsequence $(\lambda_{nk}) \subset (\lambda_n)$ so that $\|\lambda_{nk} - \lambda_{n\,k+1}\| < \frac{1}{2^{k+1}}$. This can be assured by the fact that (λ_n) is a Cauchy sequence in $\mathcal{J}$.

Define $\lambda(E)$ by the rule

$$\lambda(E) = \sum_{k=1}^{\infty} (\lambda_{nk} - \lambda_{n\,k+1})(E).$$

We must prove that λ is a complex measure on **M**.

Let $E = \bigcup_{i=1}^{\infty} E_i$ and $E_i \cap E_j = \phi$ whenever $i \neq j$.

Then
$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |(\lambda_{nk} - \lambda_{n\,k+1})(E_i)| \leq \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |\lambda_{nk} - \lambda_{n\,k+1}|(E_i)$$

$$= \sum_{k=1}^{\infty} |\lambda_{nk} - \lambda_{n\,k+1}|(E)$$

$$\leq \sum_{k=1}^{\infty} \|\lambda_{nk} - \lambda_{n\,k+1}\| < \infty .$$

Hence by corollary 3-3,

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}(\lambda_{nk}-\lambda_{n\,k+1})(E_i)=\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}(\lambda_{nk}-\lambda_{n\,k+1})(E_i)$$
$$=\sum_{k=1}^{\infty}(\lambda_{nk}-\lambda_{n\,k+1})(E).$$

that is,
$$\sum_{i=1}^{\infty}\lambda(E_i)=\lambda(E).$$

These complete the proof of $\lambda\in\mathcal{J}$.

Since $\sum_{k=1}^{j-1}(\lambda_{nk}-\lambda_{n\,k+1})=\lambda_{n_1}-\lambda_{n_j}$, by Lemma 3-4,

$$\|\lambda-(\lambda_{n_1}-\lambda_{n_j})\|=\|\sum_{k=j}^{\infty}(\lambda_{nk}-\lambda_{n\,k+1})\|$$
$$\leqq\sum_{k=j}^{\infty}\|\lambda_{nk}-\lambda_{n\,k+1}\|$$

and $\sum_{k=1}^{\infty}\|\lambda_{nk}-\lambda_{n\,k+1}\|\leqq 1$.

We can conclude that $\lim_{j\to\infty}\sum_{k=j}^{\infty}\|\lambda_{nk}-\lambda_{n\,k+1}\|=0$

and so

$$\lim_{j\to\infty}\|\lambda_{nj}-(\lambda_{n_1}-\lambda)\|=0$$

Denote $\lambda_{n_1}-\lambda$ by μ.

For $\varepsilon>0$, there exist N such that $m,\ n\geqq N\Rightarrow\|\lambda_m-\lambda_n\|<\frac{\varepsilon}{2}$

choose k so large that

$$\|\lambda_{nk}-\lambda_n\|<\frac{\varepsilon}{2}\quad \text{for } n\geqq N$$
$$\|\lambda_{nk}-\mu\|<\frac{\varepsilon}{2}$$

Then, for every $n\geqq N$

$$\|\lambda_n-\mu\|\leqq\|\lambda_n-\lambda_{nk}\|+\|\lambda_{nk}-\mu\|<\varepsilon$$

Hence there exist $\mu\in\mathcal{J}$ such that

$$\lim_{n\to\infty}\|\lambda_n-\mu\|=0$$

that is, every cauchy sequence in $\mathcal{J}$ has a limit in $\mathcal{J}$.

4. A Special Case

Definition 4-1 X is a Hausdorff space if the following holds : If $p \in X$, $q \in X$, and $P \neq q$, then p has a neighborhood μ and q has a neighborhood V such that $U \cap V = \phi$.

Definition 4-2 Hausdorff space X is locally compact if every point of X has a neighborhood whose closure is compact.

Definition 4-3 A complex function f on a locally compact Hausdorff space X is said to vanish at infinity if to every $\varepsilon > o$ there exists a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ for all x not in K.

The class of all continuous f on X which vanish at infinity is called $C_0(X)$.

Definition 4-4 Let $C_0(X)$ be a vector space over $\mathbf{C}$. A linear functional on $C_0(X)$ is a linear map whose values lie in $\mathbf{C}$.

Definition 4-5 Let X be a Hausdorff space and let $\mathfrak{B}(x)$ be the σ-algebra generated by the open subsets of X. A Borel measure on X is a measure whose domain is $\mathfrak{B}(x)$. Suppose that $\mathbf{M}$ is a σ-algebra on X such that $\mathfrak{B}(X) \subset \mathbf{M}$.

A positive measure μ on $\mathbf{M}$ is regular if

(a) each compact subset K of X satisfies $\mu(K) < +\infty$,

(b) each set E in $\mathbf{M}$ satisfies,

$\mu(E) = \inf \{\mu(F) : E \subset F$ and F is open$\}$, and

(C) each open subset F of X satisfies

$\mu(F) = \sup \{\mu(K) : K \subset F$ and K is compact$\}$.

A regular Borel measure on X is a regular measure whose domain is $\mathcal{B}(X)$.

Lemma 4-1 The space of all bounded linear functional on $C_0(X)$ forms a Banach space with respect to the norm defined by $\|\Phi\| = \sup |\Phi(f)|$, $\|f\| \leq 1$, for any bounded linear functional Φ and $f \in C_0(X)$, where X is a locally compact Hausdorff space.

Proof Let C_0^* be the space of all bounded linear functional on $C_0(X)$, that is,

$$C_0^*(X) = \{\Phi : C_0(X) \to \mathbf{C} \text{ such that } \|\Phi\| < \infty\}.$$

Clearly $C_0^*(X)$ is a vector space with a usual operation.
Therefore it is sufficient to prove $C_0^*(X)$ is complete with respect to the norm defined as above.

Let (Φ_n) be a Cauchy sequence in $C_0^*(X)$.
For any f in $C_0(X)$

$$|\Phi_n(f) - \Phi_m(f)| \leq \|\Phi_n - \Phi_m\| \; \|f\|$$

and so $(\Phi_n(f))$ is a Cauchy sequence in $\mathbf{C}$.

Thus we can have a function on $C_0(X)$ defined by

$$\Phi : C_0(X) \to \mathbf{C} \text{ with } \Phi(f) = \lim_{n\to\infty}(\Phi_n(f)).$$

we must prove the following conditions ;

1) Φ is a linear functional

2) $\|\Phi\| < \infty$

3) $\lim_{n\to\infty} \Phi_n = \Phi$

Let's , in the first place, prove 1) holds.

Let $f, g \in C_0(X)$, $\alpha \in \mathbf{C}$

$$\begin{aligned} \Phi(f+g) &= \lim_{n\to\infty} \Phi_n(f+g) \\ &= \lim_{n\to\infty} \Phi_n(f) + \lim_{n\to\infty} \Phi_n(g) \\ &= \Phi(f) + \Phi(g) \end{aligned}$$

and

$$\begin{aligned} \Phi(\alpha f) &= \lim_{n\to\infty} \Phi_n(\alpha f) \\ &= \lim_{n\to\infty} \alpha\, \Phi_n(f) \\ &= \alpha \lim_{n\to\infty} \Phi_n(f) \\ &= \alpha\, \Phi(f). \end{aligned}$$

Thus 1) has proved.

Next, let's prove 2) holds

For every ε there is a N such that

$$n \geqq N \Rightarrow \|\Phi_n - \Phi_N\| < \varepsilon .$$

Thus $\|\Phi_n\| \leqq \|\Phi_N - \Phi_n\| + \|\Phi_N\|$ if $n \geq N$.

So, $\|\Phi_n\| \leqq M$ for all n .

Fos every f in $C_0(X)$ with $\|f\| \leq 1$,

$|\Phi_n(f)| \leqq \|\Phi_n\| \leqq M$, and so

$|\Phi_n(f)| \leqq M$, that is,

$\|\Phi_n\| \leqq M < +\infty$.

Now, we have the issue if 3) holds or not.

Let $\varepsilon > o$ be give, since (Φ_n) is a Cauchy sequence there exist N

such that $m, n > N$ $\|\Phi_m - \Phi_n\| < \frac{\varepsilon}{2}$.

For every $f \in C_0(X)$ with $\|f\| \leqq 1$, there exist $n_f > N$ such that

$|\Phi_{n_f}(f) - \Phi(f)| < \frac{\varepsilon}{2}$.

Thus, if $n \geqq N$,

$|\Phi_n(f) - \Phi(f)| \leqq |\Phi_n(f) - \Phi_{n_f}(f)| + |\Phi_{n_f}(f) - \Phi(f)| \leqq \|\Phi_n - \Phi_{n_f}\|$
$+ \frac{\varepsilon}{2} < \varepsilon$.

This implies $|\Phi_n(f) - \Phi(f)| < \varepsilon$ for every $n \geqq N$, for $f \in C_0(X)$ with $\|f\| \leqq 1$.

Consequently

$\|\Phi_n - \Phi\| < \varepsilon$ if $n \geqq N$.

This proves the lemma completely.

Theorem 4-2. Let X be a locally compact Hausdorff space. The vector space M(X) of all complex reqular Borel measures on X is a Banach space if $\|\mu\| = |\mu|(X)$.

Proot We will prove the given problem. By the Riesz Represent-ation Theorem, with each bounded linear functional Φ on $C_0(X)$, there corresponds a unique regular Borel measure μ such that

$$\Phi(f) = \int_X f du \quad \text{for} \quad f \in C_0(X),$$

$$\|\Phi\| = \|\mu\|$$

Thus we can define $\Psi = C_0(X) \to M(X)$ by $\Psi(\Phi) = \mu$,

where $\Phi(f) = \int f du$, $f \in C_0(X)$, $\|\Phi\| = \|\mu\|$.

Can Ψ be a norm which preserves the vector space isomophism?

The answer is affirmative. Let's prove it. Cleary Ψ is 1 - 1

by Riesz Represemtation Theorem.

Let μ be a complex regular Borel measure on X,

that is, $\mu \in M(X)$. When we use Random - Nikodym Theorem,

we get a measurable function h such that $|h(X)| = 1$ for all

$x \in X$ and such that

$$d\mu = hd|\mu|$$

So, for every $f \in C_0(X)$ with $\|f\| \leqq 1$,

$$\begin{aligned} |\Phi(f)| &= |\int fd\mu| \\ &\leqq \int |f| d|\mu| \\ &\leqq |\mu|(X) < +\infty. \end{aligned}$$

Therefore, for every $\mu \in M(X)$,

$$\Phi(f) = \int fd\mu \text{ for } f \in C_0(X)$$

is a bounded linear functional on $C_0(X)$,

that is,

$$\Phi \in C_0^*(X).$$

Let $\Phi, \Phi' \in C_0^*(X)$ and $\Psi(\Phi) = \mu$, $\Psi(\Phi') = \mu'$.

Substituting $\lambda = \mu + \mu'$, we get

$$\int fd\lambda = \int fdu + \int fdu'.$$

since
$$\begin{aligned} \int \chi_E d\lambda &= \lambda(E) \\ &= \mu(E) + \mu'(E) \\ &= \int \chi_E du + \int \chi_E d\mu'. \end{aligned}$$

So, $(\Phi+\Phi')(f)=\Phi(f)+\Phi'(f)$

$$=\int f d\lambda, \quad \lambda\in M(X).$$

Therefore,

$$\Psi(\Phi+\Phi')=\lambda \quad \text{by uniqueness.}$$

So, $(\Phi+\Phi')(f)=\int fd\lambda$

$$=\int fdu + \int fdu'$$

$$=\Phi(f)+\Phi'(f).$$

Sinc $\lambda=\mu+\mu'$,

$$\Psi(\Phi+\Phi')=\Psi(\Phi)+\Psi(\Phi').$$

Let $\alpha\in \mathbf{C}$, $\Phi\in C_0^*(X)$, $\Psi(\Phi)=\mu$.

Then $(\alpha\Phi)(f)=\alpha(\Phi(f))$

$$=\alpha\int fdu.$$

Substituting $\lambda=\alpha\mu$, we get

$$\int fd\lambda=\alpha\int fdu, \quad \text{since}$$

$$\int \chi_E d\lambda = \lambda(E)$$

$$=\alpha\mu(E)$$

$$=\alpha\int \chi_E d\mu.$$

Therefore

$$(\alpha\Phi)(f)=\int fd\lambda, \quad \text{that is,}$$

$$\Psi(\alpha\Phi)=\lambda$$

$$=\alpha\mu$$

$$=\alpha\Psi(\Phi).$$

We have proved Ψ is a norm which preserves the vector space isomorphism.

By Lemma 4-1, $C_0^*(X)$ is a Banach space, and hence M(X) is a Banach space since Ψ is a norm which preserves the vector space isomorphism from a Banach space $C_0^*(X)$ onto a nomed linear space.

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