博士學位論文

On the Class Q^*, 2-isometries, Quasi-isometries and Posiquasi-isometries

濟州大學校 大學院

數 學 科

金 哲 準

2009年 2月

On the Class Q^*, 2-isometries, Quasi-isometries and Posiquasi-isometries

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A thesis submitted in partial fulfillment of the requirement for the degree of Doctor of Science

2008. 12.

This thesis has been examined and approved.

Date Approved :

Department of Mathematics

GRADUATE SCHOOL

CHEJU NATIONAL UNIVERSITY

$Q*$-작용소, 2-등거리변환 작용소, 유사-등거리변환 작용소 그리고 양유사-등거리변환 작용소에 관한 연구

指導敎授 梁 永 五

金 哲 準

이 論文을 理學 博士學位 論文으로 提出함

2008年 12月

金哲準의 理學 博士學位 論文을 認准함

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濟州大學校 大學院

2008年 12月

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Abstract(Korean)

Acknowledgements(Korean)

<Abstract>

On the Class Q^*, 2-isometries, Quasi-isometries and Posiquasi-isometries

In this thesis we shall study some algebraic and spectral properties of several classes of operators: Q-operators, 2-isometries, quasi-isometries, and two new operators that are defined below as Q^*-operators and posiquasi-isometries; The class of posiquasi-isometries is an extension of the class of quasi-isometries and includes all invertible operators. And we investigate the relationship between these and other operators, i.e., hyponormal, paranormal operators, and so on.

Moreover, we give necessary and sufficient conditions for a unilateral weighted shift to be a Q-operator, Q^*-operator, 2-isometry, quasi-isometry, and posiquasi-isometry respectively. In particular we show that if an operator $T \in L(H)$ on a Hilbert space H is either 2-isometry or quasi-isometriy, then the Weyl's theorem holds for T and for every $f \in H(\sigma(T))$, its Weyl spectrum satisfies the spectral mapping theorem for $f(T)$, where $H(\sigma(T))$ denotes the set of analytic functions on an open neighborhood of $\sigma(T)$. Furthermore, we show that the Weyl's theorem holds for $f(T)$.

Also we give necessary and sufficient conditions for an operator to be a posiquasi-isometry and show that every quasinilpotent posiquasi-isometry is zero, any power of a posiquasi-isometry is also a posiquasi-isometry, and the set of all posiquasi-isometries is not closed in the operator norm topology on $L(H)$.

1. Introduction

Recently paranormal operators have been much investigated ([39],[11],[25]). and S. Prasanna ([34]) showed that the Weyl's theorem holds for every paranormal operator. Let H be a complex Hilbert space and let $L(H)$ be the set of all bounded linear operators on H. In particular, it is well known ([3]) that an operator $T \in L(H)$ on a complex Hilbert space is paranormal if and only if

$$0 \leq T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I$$

for all $\lambda > 0$. Also $*$-paranormal operators have been studied ([5],[6],[24]). It is well known ([5]) that T is $*$-paranormal if and only if

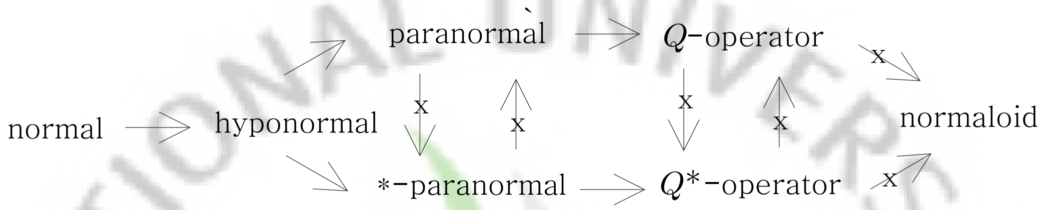
$$0 \leq T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I$$

for all $\lambda > 0$. Evidently, hyponormal operators are both paranormal and $*$-paranormal, but paranormality is independent of $*$-paranormality ([6]).

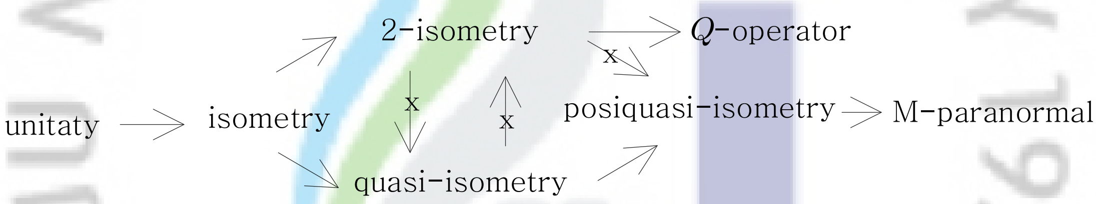
Put $Q = T^{*2}T^2 - 2T^*T + I$. If Q is positive, i.e., $0 \leq Q$, T is called an operator of class Q introduced by B. P. Duggal, et al. ([15]). Clearly every paranormal operator is of class Q.

In particular if Q is zero, i.e., $T^{*2}T^2 - 2T^*T + I = 0$, then T is said to be a $2-isometry$, and a $quasi-isometry$ if $T^*T = T^{*2}T^2$. These concepts are introduced by S. M. Patel ([31],[32]). The two classes of 2-isometries and quasi-isometries are extensions of the class of isometries but they are independent.

In this thesis we shall study some algebraic properties of operators of class Q , 2-isometries and quasi-isometries. Also we introduce two new classes of operators defined as follows: T is called an operator of class Q^* if $0 \leq T^{*2}T^2 - 2TT^* + I$ and $posiquasi-isometry$ if there exists a positive operator $P \in L(H)$ called the interrupter, such that $T^*T = T^{*2}PT^2$. Clearly every $*$-paranormal operators is of class Q^*. And the class of posiquasi-isometries is an extension of the class of quasi-isometries. The diagram below summarizes the proper inclusion relationship among these classes that will be required later in this thesis.



[Fig. 1-1]



[Fig. 1-2]

This thesis is organized as follows:

In Chapter 2, we shall give the preliminary definitions and basic properties of a bounded linear operator needed throughout the thesis.

In Chapter 3, we shall study several properties about the class Q and explore a new class Q^*. Its new concept is motivated by class Q. Also we give examples and counterexamples in order to put this class Q^* in its due place and show that classes of Q and Q^* are independent as giving an example. If T_x is the weighted shift with non-zero weights (see Example 3.23), then we give necessary and sufficient conditions for T_x to be Q-operator, Q^*-operator, paranormal, and $*$-paranormal respectively.

In Chapter 4, we investigate some algebraic and spectral properties of 2-isometries. In particular, we show that the Weyl's theorem holds for 2-isometries and also show that for every $f \in H(\sigma(T))$, the Weyl spectrum, $w(T)$, satisfies spectral mapping theorem for $f(T)$, where $H(\sigma(T))$ denotes the set of analytic functions on an open neighborhood of $\sigma(T)$. Furthermore, we show that the Weyl's theorem holds for $f(T)$. And we prove that if T is a 2-isometry, then $\ker(T^*T-I)$ is a unique maximal invariant subspace such that $T \mid \ker(T^*T-I)$ is an isometry. Also we give an example that a non isometric unilateral weighted shift is a 2-isometry.

In Chapter 5, we shall study some properties of quasi-isometries. In particular we show that if $T \in L(H)$ is a quasi-isometry and λ is an isolated point of $\sigma(T)$, then $EH = \ker(T-\lambda)$, where E is the Riesz spectral projection E with respect to λ (see (2.2)) and $\mathrm{ran}(T-\lambda)$ is closed. Also we prove that the Weyl's theorem holds for quasi-isometries and the Weyl spectrum, $w(T)$, satisfies spectral mapping theorem for $f(T)$. Furthermore, we show that the Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

In Chapter 6, we define a new class of posiquasi-isometries which is an extension of the class of quasi-isometries and includes all invertible operators. Its concept is motivated by posinormal operators which are introduced by Rhaly, Jr. ([36]). Here we investigate many algebraic and spectral properties of posiquasi-isometries and also we give necessary and sufficient conditions for an operator to be a posiquasi-isometry. The main results are as follows:

(a) T is a posiquasi-isometriy if and only if $T^*T \le \lambda^2 T^{*2}T^2$ for some $\lambda \ge 0$ if and only if $\mathrm{ran}\, T^* = \mathrm{ran}\, T^{*2}$.

(b) If T and S are commuting posiquasi-isometries, then the product TS is a posiquasi-isometry. Thus any power of a posiquasi-isometry is a posiquasi-isometry.

(c) Every invertible operator is a posiquasi-isometry with the unique interrupter. And if T is invertible with interrupter P, then P is invertible and P^{-1} is a positive operator.

(d) Let T be a unilateral weighted shift T with non-zero weights $\{\alpha_n\}$. Then T is a posiquasi-isometriy if and only if $\sup_{n \geq 1}(1/|\alpha_n|) < \infty$.

(e) Every quasinilplotent posiquasi-isometriy T is zero.

(f) Let $P(H)$ be the set of all posiquasi-isometries on H. Then $P(H)$ is not closed in the operator norm topology on $L(H)$.

(g) Let T is a posiquasi-isometriy with interrupter P. Then

$$0 \in \sigma(T) \setminus \omega(T) \text{ if and only if } 0 \in \pi_{00}(T).$$

2. Preliminaries and Basic Results

Let H be a complex Hilbert space and let $L(H)$ be the set of all bounded linear operators on H. An operator $T \in L(H)$ is said to be *self-adjoint* if $T = T^*$; *unitary* if $T^*T = TT^* = I$; *isometry* if $\| Tx \| = \| x \|$ for all $x \in H$; *contraction* if $\| T \| \leq 1$ (i.e., $\| Tx \| \leq \| x \|$ for all $x \in H$; equivalently, $T^*T \leq I$). We denote the kernel of T and the range of T by $\ker T$ and $\operatorname{ran} T$ respectively.

Theorem 2.1. ([17, p80]) *For any $T \in L(H)$, the following properties hold.*

(a) $\ker T = (\operatorname{ran} T^*)^{\perp}$.

(b) $\ker T^* = (\operatorname{ran} T)^{\perp}$.

(c) $\overline{\operatorname{ran} T} = (\ker T^*)^{\perp}$.

(d) $\overline{\operatorname{ran} T^*} = (\ker T)^{\perp}$.

Theorem 2.2. ([9, p36]) *For any $T \in L(H)$, the following statements are equivalent.*

(a) *T is left invertible.*

(b) $\operatorname{ran} T$ *is closed and* $\ker T = \{0\}$.

(c) $\inf\{ \| Tx \| : \| x \| = 1 \} > 0$.

(d) *T is bounded below, i.e., $\| Tx \| \geq c \| x \|$ for some $c > 0$ and all $x \in H$.*

We write $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\}$ for the spectrum of T ; $\partial\sigma(T)$ for the boundary of $\sigma(T)$; $\rho(T) = \sigma(T)^c$ for the resolvent of T ; $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda) \neq \{0\}\}$ for the set of eigenvalues of T ; $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity.

A complex number $\lambda \in \mathbb{C}$ is said to be an *approximate eigenvalue* of T if there exists a sequence (x_n) with $\| x_n \| = 1$ such that $(T-\lambda)x_n \to 0$. Let $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \lambda$ is an approximate eigenvalue of $T\}$. Then $\sigma_{ap}(T)$ is called the *approximate point spectrum* of T. Also we denote $\sigma_{ap}(T)$ by $\pi(T)$.

A point $\lambda \in \mathbb{C}$ is called a *normal eigenvalue* of T if eigenspace corresponding to λ reduces T. Equivalently,
$\lambda \in \mathbb{C}$ is a normal eigenvalue if and only if $\ker(T-\lambda) \subseteq \ker(T-\lambda)^*$. (2.1).
Also if $\lambda \in \mathbb{C}$ is a normal eigenvalue, then $T_1 = T \mid \ker(T-\lambda)$ is normal.

Theorem 2.3. ([10, p353]) *For any $T \in L(H)$, the following statements are equivalent.*

(a) $\lambda \notin \sigma_{ap}(T)$.

(b) $\operatorname{ran}(T-\lambda)$ *is closed and* $\ker(T-\lambda) = \{0\}$.

(c) $T-\lambda$ *is bounded below, i.e.,* $\| (T-\lambda)x \| \geq c \| x \|$ *for some* $c > 0$ and all $x \in H$.

(d) $\operatorname{ran}(T^* - \overline{\lambda}) = H$.

A closed linear subspace M of H is invariant under the operator T if $T(M) \subseteq M$. A closed linear subspace M reduces the operator T if both M and $M^\perp$ are invariant under the operator T where $M^\perp$ is orthogonal complement of M. We write $\operatorname{Lat}(T)$ for the collection of all invariant subspace for T. $T \mid M$ denotes the restriction of T to M, which is invariant subspace for T. If M reduces the operator T, then T can decomposed into the direct sum : $T = T \mid M \oplus T \mid M^\perp$.

An operator $P \in L(H)$ is called a *projection operator* if $P^2 = P$. If P is any projection on H, then $\operatorname{ran} P$ and $\ker P$ are complementary subspaces of H, i.e., $H = \operatorname{ran} P + \ker P$ and $\operatorname{ran} P \cap \ker P = \{0\}$. Also $I - P$ is a projection and furthermore, $\ker P = \operatorname{ran}(I - P)$, $\operatorname{ran} P = \ker(I - P)$. An operator $P \in L(H)$ is called an *orthogonal projection* if $P^2 = P$ and in addition $P^* = P$. If P is an orthogonal projection on H, then $\operatorname{ran} P$ and $\ker P$ are orthogonal complements in H ([12]).

Theorem 2.4. ([12, p164]) *Let $M \in Lat(T)$ and P be an orthogonal projection of H onto M. Then*

(a) *M is invariant under the operator T if and only if $TP = PTP$.*

(b) *M reduces the operator T if and only if $PT = TP$.*

Theorem 2.5. ([18, p10]) *Let $T \in L(H)$ and λ be an isolated point in $\sigma(T)$. Consider the Riesz spectral projection E with respect to λ, given by*

$$E = \frac{1}{2\pi i} \int_{\partial D} (\lambda - T)^{-1} d\lambda, \tag{2.2}$$

where D is an open disk of center λ which contains no other points of $\sigma(T)$. Then

(a) *The operator E is a projection, i.e., $E^2 = E$ and $ET = TE$.*

(b) *Put $M = \operatorname{ran} E$, and $L = \ker E$. Then $H = M \oplus L$, the space M and L are invariant under the operator T and*

$$\sigma(T|M) = \{\lambda\}, \ \sigma(T|L) = \sigma(T) \setminus \{\lambda\}.$$

The *ascent* (resp., *descent*) of T, denoted by $a(T)$, (resp., $d(T)$) is the smallest non-negative integer n such that $\ker T^n = \ker T^{n+1}$ (resp., $\operatorname{ran} T^n = \operatorname{ran} T^{n+1}$). If no such n exists, then $a(T) = \infty$ (resp., $d(T) = \infty$). If $a(T) < \infty$ and $d(T) < \infty$, then $a(T) = d(T)$ ([13]). This notion encompasses injectivity: an operator T is injective if and only if $a(T)=0$.

An operator $T\in L(H)$ is said to be *semi−Fredholm* if $\operatorname{ran}T$ is closed and either $\ker T$ or $\ker T^*$ are finite dimensional. If T is semi-Fredholm, The *index* of T, denoted by $\operatorname{ind}(T)$, is defined by

$$\operatorname{ind}(T)=\dim\ker T-\dim\ker T^*.$$

If T is semi-Fredholm and $\operatorname{ind}(T)$ is finite, then T is called *Fredholm*. It is well known ([20, Theorem 2.6]) that

$$\text{if } T\in L(H) \text{ is Fredholm of finite accent then } \operatorname{ind}(T)\leq 0 : \tag{2.3}$$

indeed, either if T has finite decent, then $\operatorname{ind}(T)=0$, or if T does not have finite decent, then $\operatorname{ind}(T)<0$.

An operator $T\in L(H)$ is *left−Fredholm* if $\operatorname{ran}T$ is closed and $\ker T$ is finite dimensional and *right−Fredholm* if $\operatorname{ran}T$ is closed and $\ker T^*$ is finite dimensional. The essential spectrum of T, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T)=\{\lambda\in\mathbb{C} : T-\lambda I \text{ is not Fredholm}\}$$

and the left essential spectrum of T, denoted by $\sigma_{le}(T)$, is defined by

$$\sigma_{le}(T)=\{\lambda\in\mathbb{C} : \dim\ker(T-\lambda)=\infty \text{ or } \operatorname{ran}(T-\lambda) \text{ is not closed}\}$$

and the right essential spectrum of T, denoted by $\sigma_{re}(T)$, is defined by

$$\sigma_{re}(T)=\{\lambda\in\mathbb{C} : \dim\ker(T-\lambda)^*=\infty \text{ or } \operatorname{ran}(T-\lambda) \text{ is not closed}\}$$

Clearly

$$T-\lambda\in L(H) \text{ is semi-Fredholm if and only if } \lambda\notin\sigma_{le}(T)\cap\sigma_{re}(T). \tag{2.4}$$

An operator $T\in L(H)$ is said to be *Weyl* if it is Fredholm of index zero and *Browder* if it is Fredholm of finite ascent and descent. The Weyl spectrum, $w(T)$, and Browder spectrum, $\sigma_b(T)$, are defined by

$$w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \ \text{ is not Wely}\},$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \ \text{is not Browder}\}.$$

Then by [21]

$$\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup acc\,\sigma(T) \tag{2.5}$$

where we write $acc\,\sigma(T)$ for the accumulation point of $\sigma(T)$. We say that the Weyl's theorem hold for T if $\sigma(T) \setminus w(T) = \pi_{00}(T)$ or equivalently, $\sigma(T) \setminus \pi_{00}(T) = w(T)$.

It is well known ([30]) that the mapping $T \rightarrow w(T)$ is upper semi-continuous, but not continuous at T. However if $T_n \rightarrow T$ with $T_n T = T T_n$ for all $n \in N$, then

$$\lim w(T_n) = w(T).$$

It is known that $w(T)$ satisfies the one-way spectral mapping theorem for analytic function: If f is analytic on an open neighborhood of $\sigma(T)$, denoted by $f \in H(\sigma(T))$, then

$$w(f(T)) \subseteq f(w(T)). \tag{2.6}$$

Theorem 2.6. ([10], p362) *For any* $T \in L(H)$, $\mathrm{ind}(T - \lambda)$ *is constant on the components of* $\mathbb{C} \setminus \sigma_{le}(T) \cap \sigma_{re}(T)$. *If* λ *is an isolated point of* $\sigma(T)$ *and* $\lambda \notin \sigma_{le}(T) \cap \sigma_{re}(T)$, *then* $\mathrm{ind}(T - \lambda) = 0$.

Theorem 2.7. ([10]) *For any* $T \in L(H)$, *the following properties hold.*

(a) $\sigma_{le}(T) \cup \sigma_{re}(T) = \sigma_e(T)$.

(b) $\sigma_{le}(T) \cap \sigma_{re}(T) \subseteq \sigma_{ap}(T)$.

(c) $\partial\sigma(T) \subseteq \sigma_{ap}(T)$.

(d) $\partial w(T) \subseteq \sigma_e(T) \subseteq w(T)$.

Theorem 2.8. *For any $T \in L(H)$, the following properties hold.*

(a) *If* $\operatorname{ran} T$ *is closed, then* $\operatorname{ran} T^*$ *is closed* ([10, p173]).

(b) *If* $\lambda \in \partial\sigma(T)$ *and* λ *is not an isolated point of* $\sigma(T)$, *then* $\operatorname{ran}(T-\lambda)$ *is not closed* ([35]).

An operator $T \in L(H)$ is called *positive*, denoted by $T \geq 0$, if $< Tx, x > \geq 0$ for all $x \in H$. T is *normal* if $T^*T - TT^* = 0$. T is *hyponormal* if $T^*T - TT^* \geq 0$ or equivalently, $\| Tx \| \geq \| T^*x \|$ for all $x \in H$. T is *paranormal* if $\| Tx \|^2 \leq \| T^2x \| \| x \|$ for all $x \in H$ or equivalently ([3]),

$$0 \leq T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I \quad \text{for all } \lambda > 0.$$

T is *M−paranormal* if $\| Tx \|^2 \leq M \| T^2x \| \| x \|$ for all $x \in H$. Also an operator T is *$*$-paranormal* if $\| T^*x \|^2 \leq \| T^2x \| \| x \|$ for all $x \in H$ or equivalently ([5]),

$$0 \leq T^{*2}T^2 - 2\lambda T T^* + \lambda^2 I \quad \text{for all } \lambda > 0.$$

An operator $T \in L(H)$ is called *normaloid* if its norm $\| T \|$ and its spectral radius of $r(T) = \sup\{|z| : z \in \sigma(T)\}$ are equal. It is well known that $r(T) \leq \| T \|$ and $r(T) = \lim_{n\to\infty} \| T^n \|^{1/n}$. Clearly if $\| T^n \| = \| T \|^n$, then T is normaloid. $T \in L(H)$ is said to be *nilpotent* if $T^n = 0$ for some $n \in N$ and *quasinilpotent* if $\| T^n \|^{1/n} \to 0$ as $n \to \infty$. Evidently, if T is quasnilpotent, then $\sigma(T) = 0$.

These operators are related by proper inclusion as follows:

Normal $\subsetneq$ Hyponormal

$\subsetneq$ Paranormal (or $*$-Paranormal) $\subsetneq$ Normaloid.

An operator $T \in L(H)$ is called *isoloid* if isolated points of $\sigma(T)$ are eigenvalues of T, i.e., $\operatorname{iso}\sigma(T) \subseteq \sigma_p(T)$ where we write $\operatorname{iso}\sigma(T)$ for the isolated points of $\sigma(T)$ and *reguloid* if $T-\lambda I$ has closed range for each $\lambda \in \text{iso}\sigma(T)$. Clearly if T is reguloid, then T is isoloid. It is well known ([39]) that

$$\text{if } T \text{ is paranormal, then } T \text{ is isoloid and reguloid.} \tag{2.7}$$

Theorem 2.9. ([7]) *Let $T \in L(H)$ be positive, i.e., $T \geq 0$. Then*

(a) *T is self-adjoint.*

(b) *$S^*TS \geq 0$ for any operator S.*

(c) $|< Tx, y >|^2 \leq < Tx, x > < Ty, y >$ *for all $x, y \in H$.*

(d) *$Tx = 0$ if and only if $< Tx, x > = 0$.*

Theorem 2.10. *(The Spectral Mapping Theorem) If $T \in L(H)$ and f is analytic on a neighborhood of $\sigma(T)$, then $\sigma(f(T)) = f(\sigma(T))$.*

3. Class Q and Class Q^* of operators

3.1 Class Q of operators

Definition 3.1. An operator T is of class Q, shortened to $Q-$*operator* if $0 \le T^{*2}T^2 - 2T^*T + I$. Equivalently, T is an operator of class Q if

$$\| Tx \|^2 \le \frac{1}{2}(\| T^2x \|^2 + \| x \|^2) \quad \text{for every } x \in H.$$

Remark 3.2. Every paranormal operator is clearly of class Q. Since $0 \le T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I$ if and only if $\lambda^{-1/2}T \in Q$ for any $\lambda > 0$,

$$T \text{ is paranormal if and only if } \lambda T \in Q \text{ for all } \lambda > 0.$$

Theorem 3.3. ([16]) *Let T be an operator of class Q.*

(a) *The restriction of T to an invariant subspace is again of class Q.*

(b) *If T is invertible, then T^{-1} is of class Q.*

Theorem 3.4. ([16]) *For any $T \in L(H)$, the following properties hold.*

(a) *If $\| T \| \le 1/\sqrt{2}$, then $T \in Q$.*

(b) *If $T^2 = 0$, then $T \in Q$ if and only if $\| T \| \le 1/\sqrt{2}$.*

Theorem 3.5. *Let T be an operator of class Q.*

(a) *If S is unitarily equivalent to T, then S is of class Q.*

(b) *If T commutes with an isometry S, then the product TS is of class Q.*

(c) *$T \otimes I$ and $I \otimes T$ are both of class Q.*

Proof. (a) Let $S = U^*TU$ where U is unitary. Then

$$S^{*2}S^2 - 2S^*S + I$$
$$= U^*T^{*2}T^2U - 2U^*T^*TU + U^*U$$
$$= U^*(T^{*2}T^2 - 2T^*T + I)U \geq 0.$$

Hence S is of class Q.

(b) Let $A = TS$. We must show that $A^{*2}A^2 - 2A^*A + I \geq 0$. By hypothesis, we have $S^*S = I$, $ST = TS$, $S^*T^* = T^*S^*$. Thus

$$A^{*2}A^2 - 2A^*A + I$$
$$= S^*T^*S^*T^*TSTS - 2S^*T^*TS + I$$
$$= T^{*2}T^2 - 2T^*T + I \geq 0.$$

Hence $A = TS$ is of class Q.

(c) Since T is of class Q, $(T^{*2}T^2 - 2T^*T + I) \otimes I \geq 0$ and we have

$$[(T\otimes I)^*]^2(T\otimes I)^2 - 2(T\otimes I)^*(T\otimes I) + (I\otimes I)$$
$$= (T^{*2}\otimes I)(T^2\otimes I) - 2(T^*\otimes I)(T\otimes I) + (I\otimes I)$$
$$= (T^{*2}T^2\otimes I) - 2(T^*T\otimes I) + (I\otimes I)$$
$$= (T^{*2}T^2 - 2T^*T + I)\otimes I \geq 0.$$

Hence $T\otimes I$ is of class Q and similarly $I\otimes T$ is of class Q.

Example 3.6. Let $S = \lambda\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be an operator on a two-dimensional Hilbert space $\mathbb{C}^2$. Then $\|S\| = |\lambda|$, $S^2 = 0$ and $\sigma(S) = \{0\}$. So by Theorem 3.4(b),

$$S = \lambda\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Q \text{ if and only if } |\lambda| \leq 1/\sqrt{2}. \tag{3.1}$$

Thus S is not normaloid for all $\lambda \neq 0$ since $\|S\| \neq r(S)$, so that S is not paranormal for all $\lambda \neq 0$.

The above example shows that an operator of class Q need not to be normaloid and hence paranormal. Thus the following classes are related by proper inclusion :

$$\text{Unitary} \subsetneq \text{Hyponormal} \subsetneq \text{Paranormal} \subsetneq \text{Class } Q.$$

Theorem 3.7. *Let T be a unilateral weighted shift with weights $\{\alpha_n\}_{n=0}^{\infty}$. Then T is of class Q if and only if for all $n \geq 0$,*

$$|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 \geq 0.$$

Proof. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for H. Then $Te_n = \alpha_n e_{n+1}$ for all $n \geq 0$ and $T^*e_0 = 0$, $T^*e_n = \overline{\alpha}_{n-1} e_{n-1}$ for all $n \geq 1$. Thus

$$(T^{*2}T^2 - 2T^*T + I)e_n = (|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1)e_n$$

for all $n \geq 0$, so that this implies the result.

Isolated points of the spectrum of a paranormal operator are eigenvalues, but an operator of class Q need not to be isoloid.

Example 3.8. Let T be a weighted shift with weights $\{1/(n+1)\}_{n=1}^{\infty}$. Then T is a compact operator, $\sigma(T) = \{0\}$, $\sigma_p(T) = \varnothing$ and $\|T\| = 1/2$ ([12, p170]). Thus T is an operator of class Q since

$$|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 \geq 0$$

for all $n \geq 1$, as easily checked. But T is not isoloid.

Remark 3.9. In the Example 3.6 if $\lambda = 1/2$, then $S \in Q$, but if $\lambda = 2$, then $2S$ is not an operator of class Q from (3.1). Hence a multiple of a Q-operator may not be of class Q.

Theorem 3.10. ([16]) *Let T be an operator of class Q.*

(a) *If T^2 is a contraction, then so is T.*

(b) *If T^2 is an isometry, then T is paranormal.*

Proof. (a) Observe that T is of class Q if and only if $2(T^*T-I) \leq T^{*2}T^2-I$. Thus $T^{*2}T^2 \leq I$ implies $T^*T \leq I$. So T is a contraction whenever T^2 is.

(b) Take any x in H and note that T is of class Q if and only if

$$2\|Tx\|^2 \leq (\|T^2x\| - \|x\|)^2 + 2\|T^2x\|\,\|x\|.$$

Hence $\|T^2x\| = \|x\|$ implies $\|Tx\|^2 \leq \|T^2x\|\,\|x\|$ for every $x \in H$.

3.2 Class Q^* of operators

Definition 3.11. An operator T is of class Q^*, shortened to Q^*-*operator* if $0 \leq T^{*2}T^2 - 2TT^* + I$. Equivalently, T is an operator of class Q^* if

$$\|T^*x\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2) \quad \text{for every } x \in H.$$

Remark 3.12. Clearly every $*$-paranormal operator is an operator of class Q^*. Since $0 \leq T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I$ if and only if $\lambda^{-1/2}T \in Q^*$ for any $\lambda > 0$,

$$T \text{ is } *\text{-paranormal if and only if } \lambda T \in Q^* \text{ for all } \lambda > 0.$$

Theorem 3.13. *For any $T \in L(H)$, the following properties hold.*

(a) *If* $\|T\| \leq 1/\sqrt{2}$, *then* $T \in Q^*$.

(b) *If* $T^2 = 0$, *then* $T \in Q^*$ *if and only if* $\|T\| \leq 1/\sqrt{2}$.

(c) *If* $T \in Q^*$, $T^2 \neq 0$ *and* $|\alpha| \leq \min\{1, \|T^2\|^{-1}\}$, *then* $\alpha T \in Q^*$. *In particular, if* $T \in Q^*$ *is a contraction, then* $\alpha T \in Q^*$ *whenever* $|\alpha| \leq 1$.

(d) *A contraction* $T \in Q^*$ *is* $*$-*paranormal if and only if*

$$0 \leq T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \quad \text{for all } \lambda \in (0,1).$$

Proof. (a) $\| T \| \le 1/\sqrt{2}$ if and only if $\| \sqrt{2}\,T \| \le 1$ if and only if $2TT^* \le I$. Hence $\| T \| \le 1/\sqrt{2}$ implies $0 \le T^{*2}T^2 - 2TT^* + I$.

(b) If $T^2 = 0$, then $0 \le T^{*2}T^2 - 2TT^* + I$ if and only if $2TT^* \le I$. Hence $T \in Q^*$ if and only if $\| T \| \le 1/\sqrt{2}$.

(c) If $T \in Q^*$, then $2TT^* \le T^{*2}T^2 + I$, so for each scalar α,

$$2|\alpha|^2 TT^* \le |\alpha|^2 T^{*2}T^2 + |\alpha|^2 I$$

and hence for each scalar α,

$$\begin{aligned} &2|\alpha|^2 TT^* - |\alpha|^4 T^{*2}T^2 - I \\ &\le |\alpha|^2 T^{*2}T^2 + |\alpha|^2 I - |\alpha|^4 T^{*2}T^2 - I \\ &= (1 - |\alpha|^2)(|\alpha|^2 T^{*2}T^2 - I). \end{aligned}$$

Suppose $T^2 \ne 0$. If $|\alpha| \le \min\{1, \| T^2 \|^{-1}\}$, then $1 - |\alpha|^2 \ge 0$ since $|\alpha| \le 1$. Also $|\alpha|^2 T^{*2}T^2 - I \le 0$ since $|\alpha| \le \| T^2 \|^{-1}$ if and only if $\| \alpha T^2 \| \le 1$ if and only if αT^2 is a contraction. Thus $(1 - |\alpha|^2)(|\alpha|^2 T^{*2}T^2 - I) \le 0$. Hence $2|\alpha|^2 TT^* \le |\alpha|^4 T^{*2}T^2 + I$, so that $\alpha T \in Q^*$.

In particular, let $T \in Q^*$ be a contraction. then in the case of $T^2 \ne 0$, $\alpha T \in Q^*$ whenever $|\alpha| \le 1$ since $\min\{1, \| T^2 \|^{-1}\} = 1$. And in the case of $T^2 = 0$, we have $\| T \| \le 1/\sqrt{2}$ by (b). Also $|\alpha| \| T \| \le 1/\sqrt{2}$ and $\| \alpha T \| \le 1/\sqrt{2}$ for $|\alpha| \le 1$. Therefore $\alpha T \in Q^*$ for $|\alpha| \le 1$ by (a).

(d) If $T \in Q^*$ is a contraction, then $\alpha T \in Q^*$ for $\alpha \in (0, 1]$ or equivalently, $0 \le \alpha^4 T^{*2}T^2 - 2\alpha^2 TT^* + I$ for $\alpha \in (0, 1]$, i.e.,

$$0 \le T^{*2}T^2 - 2\frac{1}{\alpha^2} TT^* + \frac{1}{\alpha^4} I.$$

Let $\lambda = 1/\alpha^2$. Then $0 \le T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I$ for all $\lambda \ge 1$. Hence a contraction $T \in Q^*$ is $*$-paranormal if and only if

$$0 \le T^{*2}T^2 - 2\lambda TT^* + \lambda^2 I \quad \text{for all } \lambda \in (0, 1).$$

Corollary 3.14. *If* $T^2=0$*, then* $T\in Q$ *if and only if* $T\in Q^*$.

Proof. It follows from Theorem 3.4(b) and Theorem 3.13(b).

Remark 3.15. An operator of class Q^* need not to be normaloid and hence not to be $*$-paranormal. For example, by Corollary 3.14,

$$S=\lambda\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\in Q \text{ if and only if } S\in Q^* \text{ if and only if } |\lambda|\leq 1/\sqrt{2}$$

since $S^2=0$. And also S is not normaloid for all $\lambda\neq 0$ and hence not $*$-paranormal (see Example 3.6).

The above remark shows that the following classes are related by proper inclusion :

$$\text{Unitary} \subsetneq \text{Hyponormal} \subsetneq *\text{-Paranormal} \subsetneq \text{Class } Q^*.$$

And a multiple of a Q^*-operator may not be of class Q^* (see Remark 3.9).

Theorem 3.16. *Let* T *be an operator of class* Q^*.

(a) *If* $M\subseteq H$ *is an invariant subspace for* T*, then* $T|M$ *is of class* Q^*.

(b) *If* S *is unitarily equivalent to* T*, then* S *is of class* Q^*.

(c) *If* T *commutes with a unitary operator* S*, then the product* TS *is of class* Q^*.

(d) $T\otimes I$ *and* $I\otimes T$ *are both of class* Q^*.

Proof. (a) Let P be the orthogonal projection of H onto M and let $A=T|M$ denote the restriction of T to M. Then for every $x\in M$,

$$\begin{aligned}\|A^*x\|^2 &= \|PT^*x\|^2 \leq \|T^*x\|^2 \\ &\leq 1/2(\|T^2x\|^2+\|x\|^2) = 1/2(\|A^2x\|^2+\|x\|^2).\end{aligned}$$

Hence $A=T|M$ is of class Q^*.

(b) Let $S=U^*TU$ where U is unitary. Then

$$\begin{aligned}&S^{*2}S^2-2SS^*+I \\ &=U^*T^{*2}T^2U-2U^*TT^*U+U^*U \\ &=U^*(T^{*2}T^2-2TT^*+I)U \geq 0.\end{aligned}$$

Hence S is of class Q^*.

(c) Let $A = TS$. We must show that $A^{*2}A^2 - 2AA^* + I \geq 0$. By hypothesis, we have $S^*S = SS^* = I$, $ST = TS$, $S^*T^* = T^*S^*$. Thus

$$\begin{aligned}
&A^{*2}A^2 - 2AA^* + I \\
&= S^*T^*S^*T^*TSTS - 2TSS^*T^* + I \\
&= T^{*2}(S^*S^*SS)T^2 - 2T(SS^*)T^* + I \\
&= T^2T^2 - 2TT^* + I \geq 0.
\end{aligned}$$

Hence $A = TS$ is of class Q^*.

(d) Since T is of class Q^*, $(T^{*2}T^2 - 2TT^* + I)\otimes I \geq 0$ and we have

$$\begin{aligned}
&[(T\otimes I)^*]^2(T\otimes I)^2 - 2(T\otimes I)(T\otimes I)^* + (I\otimes I) \\
&= (T^{*2}\otimes I)(T^2\otimes I) - 2(T\otimes I)(T^*\otimes I) + (I\otimes I) \\
&= (T^{*2}T^2\otimes I) - 2(TT^*\otimes I) + (I\otimes I) \\
&= (T^{*2}T^2 - 2TT^* + I)\otimes I \geq 0.
\end{aligned}$$

Hence $T\otimes I$ is of class Q^* and similarly $I\otimes T$ is of class Q^*.

Theorem 3.17. *Let T be a unilateral weighted shift with weights $\{\alpha_n\}_{n=0}^{\infty}$. Then T is of class Q^* if and only if for all $n \geq 1$,*

$$|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_{n-1}|^2 + 1 \geq 0.$$

Proof. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for H. Then $Te_n = \alpha_n e_{n+1}$ for all $n \geq 0$ and $T^*e_0 = 0$, $T^*e_n = \overline{\alpha}_{n-1}e_{n-1}$ for all $n \geq 1$. Thus

$$(T^{*2}T^2 - 2TT^* + I)e_n = (|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_{n-1}|^2 + 1)e_n$$

for all $n \geq 1$ and $(T^{*2}T^2 - 2TT^* + I)e_0 = (|\alpha_0|^2|\alpha_1|^2 + 1)e_0$. This implies the result.

Example 3.18. Let T be a weighted shift with weights $\{1/(n+1)\}_{n=1}^{\infty}$. Then since $|\alpha_{n-1}|^2 \leq 1/2$, $|\alpha_{n-1}|^2 \leq 1/2\,(|\alpha_n|^2|\alpha_{n+1}|^2+1)$ for $n \geq 2$. Thus T is a Q^*-operator by Theorem 3.17, but T is not isoloid (see Example 3.8). This means that an operator of class Q^* need not to be isoloid.

The following results are well known ([24],[25]): Let T be a unilateral weighted shift with non-zero weights $\{\alpha_n\}_{n=0}^{\infty}$. Then

(a) T is paranormal if and only if $|\alpha_n| \leq |\alpha_{n+1}|$ for all $n \geq 0$.

(b) T is $*$-paranormal if and only if $|\alpha_{n-1}|^2 \leq |\alpha_n||\alpha_{n+1}|$ for all $n \geq 1$.

The following example shows that classes of Q-operators and Q^*-operators are independent.

Example 3.19. Let T be a unilateral weighted shift with weights $\{\alpha_n\}_{n=0}^{\infty}$ $=(1, 1/2, 2, 2, 2, \cdots)$. Then

(a) T is a Q^*-operator since $|\alpha_{n-1}|^2 \leq 1/2\,(|\alpha_n|^2|\alpha_{n+1}|^2+1)$ for all $n \geq 1$, as easily checked. In fact T is $*$-paranormal since $|\alpha_{n-1}|^2 \leq |\alpha_n||\alpha_{n+1}|$ for all $n \geq 1$.

(b) By Theorem 3.7, T is not a Q-operator since $|\alpha_0|^2|\alpha_1|^2 - 2|\alpha_0|^2 + 1 < 0$, so that T is not paranormal.

(c) $\|T^2\| = 4$ since $T^2(x_1, x_2, x_3 \cdots) = (0, 0, 1/2x_1, x_2, 4x_3, 4x_4, \cdots)$. So $\alpha T \in Q^*$ whenever $|\alpha| \leq 1/4$ by Theorem 3.13(c).

Theorem 3.20. *Let T be an operator of class Q^*.*

(a) *If T^2 is a contraction, then so is T.*

(b) *If T^2 is an isometry, then T is $*$-paranormal.*

Proof. (a) T is of class Q^* if and only if $2TT^* \leq T^{*2}T^2 + I$ if and only if $2TT^* \leq 2I$ since T^2 is a contraction. Thus $TT^* \leq I$, which means that T is a contraction.

(b) Take any x in H and note that T is of class Q^* if and only if

$$2\|T^*x\|^2 \leq \|T^2x\|^2 + \|x\|^2$$
$$= (\|T^2x\| - \|x\|)^2 + 2\|T^2x\|\,\|x\|.$$

Hence $\|T^2x\| = \|x\|$ implies $\|T^*x\|^2 \leq \|T^2x\|\,\|x\|$ for all $x \in H$. Therefore T is $*$-paranormal.

Corollary 3.21. *Let T^2 be an isometry. Then $T \in Q^*$ if and only if T is a contraction.*

Proof. Since $T^{*2}T^2 = I$, $T \in Q^*$ if and only if $2TT^* \leq 2I$ if and only if T is a contraction.

Note that there exists a non-zero operator $T \notin Q^*$ that T^2 is an isometry.

Example 3.22. Let T be a unilateral weighted shift with weights $(\alpha_n) = (2, 1/2, 2, 1/2, \cdots)$. Then

(a) $T^2(x_1, x_2, x_3, \cdots) = (0, 0, x_1, x_2, x_3, \cdots)$, i.e., T^2 is an isometry, but T is not of class Q^* since T is not a contraction ($\|T\| = 2$).

(b) $r(T) = \lim \|T^n\|^{1/n} = 1$. In fact, $\|T^n\| = 1$ if n is even and $\|T^n\| = 2$ if n is odd. So T is not normaloid since $2 = \|T\| \neq r(T) = 1$ and hence T is not $*$-paranormal.

Example 3.23. Let T_x be a unilateral weighted shift with non-zero weights

$$\alpha_0 = x\,,\ \alpha_1 = \sqrt{\frac{2}{3}}\,,\ \alpha_2 = \sqrt{\frac{3}{4}}\,, \cdots,\ \alpha_n = \sqrt{\frac{n+1}{n+2}}\,, \cdots.$$

(a) $T_x \in Q$ if and only if $0 < x \le \frac{\sqrt{3}}{2}$.

(b) $T_x \in Q^*$ if and only if $0 < x \le \frac{\sqrt{3}}{2}$.

(c) T_x is $*$-paranormal if and only if $0 < x \le \frac{1}{\sqrt[4]{2}}$.

(d) T_x is paranormal if and only if $0 < x \le \sqrt{\frac{2}{3}}$.

(e) If $\frac{1}{\sqrt[4]{2}} < x \le \frac{\sqrt{3}}{2}$, then T_x is of class $Q \cap Q^*$, but not $*$-paranormal.

Proof. (a) For $n \ge 1$, $2|\alpha_n|^2 \le |\alpha_n|^2|\alpha_{n+1}|^2 + 1$ since

$$2|\alpha_n|^2 = \frac{2(n+1)}{n+2} < \frac{2(n+2)}{n+3} = |\alpha_n|^2|\alpha_{n+1}|^2 + 1.$$

When $n=0$, $2|\alpha_0|^2 \le |\alpha_0|^2|\alpha_1|^2 + 1$ for $0 < x \le \frac{\sqrt{3}}{2}$.

(b) For $n \ge 2$, $2|\alpha_{n-1}|^2 \le |\alpha_n|^2|\alpha_{n+1}|^2 + 1$ since

$$2|\alpha_{n-1}|^2 = \frac{2n}{n+1} < \frac{2(n+2)}{n+3} = |\alpha_n|^2|\alpha_{n+1}|^2 + 1.$$

When $n=1$, $2|\alpha_0|^2 \le |\alpha_1|^2|\alpha_2|^2 + 1$ for $0 < x \le \frac{\sqrt{3}}{2}$.

(c) T_x is $*$-paranormal if and only if $|\alpha_{n-1}|^2 \le |\alpha_n||\alpha_{n+1}|$ for all $n \ge 1$. Now for $n \ge 2$, $|\alpha_{n-1}|^2 \le |\alpha_n||\alpha_{n+1}|$ since

$$|\alpha_{n-1}|^2 = \frac{n}{n+1} < \sqrt{\frac{n+1}{n+3}} = |\alpha_n||\alpha_{n+1}|.$$

When $n=1$, $|\alpha_0|^2 \le |\alpha_1||\alpha_2|$ for $0 < x \le \frac{1}{\sqrt[4]{2}}$.

(d) This is clear from the following the fact that T_x is paranormal if and only if $|\alpha_n| \le |\alpha_{n+1}|$ for all $n \ge 0$, i.e., (α_n) is increasing.

(e) It follows from the above part (a), (b), and (c).

4. 2-isometric operators

Definition 4.1. An operator $T \in L(H)$ is defined to be a $2-isometry$ if

$$T^{*2}T^2 - 2T^*T + I = 0. \qquad (4.1)$$

Equivalently, T is a 2-isometry if

$$2\|Tx\|^2 = \|T^2x\|^2 + \|x\|^2 \text{ for every } x \in H.$$

Clearly every isometry is a 2-isometry since $T^*T = I$. And every 2-isometry is a Q-operator.

Remark 4.2. *For any* 2-isometry T, *the following properties hold.*

(a) T is left invertible since $(2T^* - T^{*2}T)T = I$. And hence $\operatorname{ran} T$ is closed and $\ker T = \{0\}$ (see Theorem 2.2).

(b) $\|T\| \geq 1$ since $T^*T - I \geq 0$ ([2, Proposition 1.5]).

(c) T is invertible if and only if T is unitary ([2]).

Theorem 4.3. *For any* 2-isometry T, *the following properties hold.*

(a) *T is not compact if H is infinite dimensional.*

(b) *If T is invertible, then T^{-1} is also a 2-isometry.*

(c) *If T is normal, then T^* is also a 2-isometry.*

(d) *If T^2 is an isometry, then T is also an isometry.*

Proof. (a) If T is a 2-isometry, then $T^*T \geq I$. In general I is not compact on an infinite dimensional Hilbert space. Thus T^*T is not compact. Hence T is not compact.

(b) The hypothesis that T is an invertible 2-isometry yield T is unitary by Remark 4.2(c). So T^{-1} is unitary and hence T^{-1} is a 2-isometry.

(c) If T is normal, then $T^*T = TT^*$, so $T^{*2}T^2 = T^2T^{*2}$. From (4.1), we obtain $T^2T^{*2} - 2TT^* + I = 0$, which implies T^* is a 2-isometry.

(d) If T^2 is an isometry, then $T^{*2}T^2 = I$ and hence $2(I - T^*T) = 0$ from (4.1), so $T^*T = I$. This implies T is an isometry.

We denote D to be an open unit disk, i.e., $D = \{z \in \mathbb{C} : |z| < 1\}$ and also write ∂D for the topological boundary of D.

Theorem 4.4. *If both T and T^* are 2-isometries, then $\sigma(T) \subseteq \partial D$.*

Proof. By Remark 4.2(a), $\operatorname{ran} T$ is closed and both T and T^* are injective, so T is invertible and hence T is unitary by Remark 4.2.(c). This implies $\sigma(T) \subseteq \partial D$.

Corollary 4.5. *Let T be a 2-isometry. Then the following statements are equivalent.*

(a) *T is invertible.*

(b) *T is unitary.*

(c) *T is normal.*

(d) *T has it's spectrum on the unit circle.*

Proof. (a) implies (b) by Remark 4.2(c). Clearly (b) implies (c). Using Theorem 4.3(c) and Theorem 4.4, (c) implies (d). (d) implies (a) since $0 \notin \sigma(T)$.

Theorem 4.6. *For any 2-isometry T, the following properties hold.*

(a) *If S is unitarily equivalent to T, then S is a 2-isometry.*

(b) *If $M \subseteq H$ is an invariant subspace for T, then $T|M$ is a 2-isometry.*

Proof. (a) Let $S = U^*TU$ where U is unitary. Then

$$S^{*2}S^2 - 2S^*S + I = U^*T^{*2}T^2U - 2U^*T^*TU + U^*U.$$

$$= U^*(T^{*2}T^2 - 2T^*T + I)U = 0$$

Hence S is a 2-isometry.

(b) If $u \in M$, then

$$2\|T|Mu\|^2 = 2\|Tu\|^2 = \|T^2u\|^2 + \|u\|^2 = \|(T|M)^2u\|^2 + \|u\|^2.$$

So $T|M$ is a 2-isometry.

Theorem 4.7. *Let T be a 2-isometry. If T commutes with an isometry S, then the product TS is a 2-isometry.*

Proof. Let $A = TS$. We must show that $A^{*2}A^2 - 2A^*A + I = 0$.
By hypothesis, we have $S^*S = I$, $ST = TS$, $S^*T^* = T^*S^*$. Thus

$$\begin{aligned} &A^{*2}A^2 - 2A^*A + I \\ &= S^*T^*S^*T^*TSTS - 2S^*T^*TS + I \\ &= T^{*2}T^2 - 2T^*T + I = 0. \end{aligned}$$

Hence TS is a 2-isometry.

Theorem 4.8. *Let T be a 2-isometry. Then*

αT is a 2-isometry if and only if $|\alpha| = 1$ or αT^2 is an isometry.

Proof. If T is a 2-isometry, then $2|\alpha|^2 T^*T = |\alpha|^2 T^{*2}T^2 + |\alpha|^2 I$ for any $\alpha \in \mathbb{C}$. So we have for any $\alpha \in \mathbb{C}$,

$$|\alpha|^4 T^{*2}T^2 - 2|\alpha|^2 T^*T + I = (|\alpha|^2 - 1)(|\alpha|^2 T^{*2}T^2 - I),$$

which implies the result.

Corollary 4.9. *If T and αT are 2-isometries. Then $|\alpha| \leq 1$.*

Proof. Note T^2 is a 2-isometry ([32, Theorem 2.1]), and so $1 \leq \| T^2 \|$. Let αT be a 2-isometry. If $|\alpha| \neq 1$, then $\| \alpha T^2 \| = 1$ by Theorem 4.8, which implies $|\alpha| < 1$.

Remark 4.10. According to [2], If T is a 2-isometry, then $\sigma_{ap}(T) \subseteq \partial D$. And either $\sigma(T) \subseteq \partial D$ if T is invertible or $\sigma(T) = \overline{D}$ if T is not invertible. Thus if T is an isometry, then either $\sigma(T) \subseteq \partial D$ or $\sigma(T) = \overline{D}$. In particular if T is unitary, then $\sigma(T) \subseteq \partial D$.

Recall that an operator $T \in L(H)$ is isoloid if isolated points of $\sigma(T)$ are eigenvalues of T and reguloid if $T-\lambda I$ has closed range for each isolated points of $\sigma(T)$.

Theorem 4.11. *If T is a 2-isometry, then T is isoloid and reguloid.*

Proof. If T has isolated points of $\sigma(T)$, then it is clear from the above remark that T is unitary since $\sigma(T) \subseteq \partial D$. Thus T is paranormal, so that the result follows (see (2.7)).

In the next theorems we explore several properties of the spectrum of a 2-isometric non-unitary operator and also we prove that the Weyl's theorem holds for 2-isometries.

Theorem 4.12. *If T is a 2-isometry and non-unitary, then*

(a) $\sigma(T) = \overline{D}$.

(b) $\sigma_{ap}(T) = \partial D$.

(c) $\sigma_{le}(T) \cap \sigma_{re}(T) = \partial D$.

(d) $\sigma(T) = w(T)$.

Proof. (a) Since T is not invertible, $\sigma(T) = \overline{D}$ by preceding Remark 4.10.

(b) For any operator $T \in L(H)$, $\partial\sigma(T) \subseteq \sigma_{ap}(T)$ (see Theorem 2.7), so $\partial D \subseteq \sigma_{ap}(T)$ by part (a) and $\sigma_{ap}(T) \subseteq \partial D$ by preceding Remark 4.10. Thus the result follows.

(c) For any operator $T \in L(H)$, $\sigma_{le}(T) \cap \sigma_{re}(T) \subseteq \sigma_{ap}(T)$ (see Theorem 2.7) and using part (b), $\sigma_{le}(T) \cap \sigma_{re}(T) \subseteq \partial D$.

Conversely if $\lambda \in \partial D$, then λ is not isolated point of $\sigma(T)$ by part (a). Thus $\mathrm{ran}(T-\lambda)$ is not closed (see Theorem 2.8). Hence $\lambda \in \sigma_{le}(T) \cap \sigma_{re}(T)$.

(d) Let $\lambda \in \sigma(T)$. If $\lambda \in \partial D$, then $\text{ran}(T-\lambda)$ is not closed, so $\lambda \in w(T)$. If $\lambda \in D$, then $T-\lambda$ is closed and $\ker(T-\lambda)=\{0\}$ since $\lambda \notin \sigma_{ap}(T)$ (see Theorem 2.3). And since $\lambda \in \sigma(T)$, we must have $\dim\ker(T-\lambda)^* \neq 0$. Thus $\text{ind}(T-\lambda) < 0$, so that $\lambda \in w(T)$. Therefore $\sigma(T) \subseteq w(T)$ by part (a). This proved (d).

Remark 4.13. The above Theorem 4.12 shows that if T is a 2-isometric non-unitary operator, then

(a) T is not a Weyl operator since $\overline{D} = w(T)$.

(b) $\partial D \subseteq \sigma_e(T) \subseteq \overline{D}$ since $\partial w(T) \subseteq \sigma_e(T) \subseteq \sigma(T)$ for any operator $T \in L(H)$ (see Theorem 2.7).

(c) $T-\lambda$ is semi-Fredholm for $\lambda \in D$ (see (2.4)).

(d) $\text{ind}(T-\lambda) \leq 0$ for $|\lambda| \neq 1$. In fact, if $|\lambda| < 1$, then $\text{ind}(T-\lambda) < 0$ in the proof of Theorem 4.12(d) and if $|\lambda| > 1$, then $T-\lambda$ is invertible, so that $\text{ind}(T-\lambda) = 0$.

(e) the function from D into $Z \cup \{\pm\infty\}$ given by $\lambda \to \text{ind}(T-\lambda)$ is constant (see Theorem 2.6).

Corollary 4.14. *Let T be a 2-isometric non-unitary operator. If $\lambda \notin \sigma_e(T)$ i.e, $T-\lambda$ is a Fredholm, then* $\text{ind}(T-\lambda) \leq 0$.

Proof. This proof is immediate by part (b) and (d) of the Remark 4.13.

The following theorem appeared in [32, Corollary 2.13]. Here we will prove this with alternate argument using the Theorem 4,12.

Theorem 4.15. *The Weyl's theorem holds for 2-isometries.*

Proof. Let T be a 2-isometry. If T is unitary, the result is obvious. If T is non-unitary, then since $\sigma(T) = \overline{D}$ by Theorem 4.12(a), $\pi_{00}(T) = \varnothing$. Thus $\sigma(T) - \pi_{00}(T) = w(T)$ by Theorem 4.12(d), as desired.

Theorem 4.16. *Let T be a unilateral weighted shift with weights $\{\alpha_n\}_{n=0}^{\infty}$. Then T is a 2-isometry if and only if for all $n \geq 0$,*

$$|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 = 0.$$

Proof. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for H. Then $Te_n = \alpha_n e_{n+1}$ for all $n \geq 0$ and $T^*e_0 = 0$, $T^*e_n = \overline{\alpha}_{n-1} e_{n-1}$ for all $n \geq 1$. Thus

$$(T^{*2}T^2 - 2T^*T + I)e_n = (|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1)e_n$$

for all $n \geq 0$, so that this implies the result.

Next we shall give an example that a non isometric unilateral weighted shift is a 2-isometry.

Remark 4.17. In [32, Theorem 2.2], S. M. Patel proved that a non isometric unilateral weighted shift T with weights $\{\alpha_n\}$ is a 2-isometry if and only if (i) $|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 = 0$ for each n; (ii) $|\alpha_n| \neq 1$ for each n.

Example 4.18. Define $T: l_2 \to l_2$ by $T(x_1, x_2, \cdots) = (0, \alpha_1 x_1, \alpha_2 x_2, \cdots)$ where $\alpha_n = \sqrt{1 + \frac{1}{n}}$. Then T is a non isometric unilateral weighted shift and a 2-isometry since $|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 = 0$ and $|\alpha_n| \neq 1$ for each n, easily checked. And $\| T \| = \sqrt{2}$ since $\sqrt{2} \geq |\alpha_n| > 1$ for each n.

Theorem 4.19. *Define $T: l_2 \to l_2$ by $T(x_1, x_2, \cdots) = (0, \alpha_1 x_1, \alpha_2 x_2, \cdots)$ where $\{\alpha_n\}$ is non-zero weights for each n. If T is a 2-isometry, then*

(a) $\sigma(T) = w(T) = \overline{D}$.

(b) $\sigma_{ap}(T) = \partial D$.

(c) $\sigma_p(T) = \varnothing$.

(d) *for* $|\lambda| < 1$, $\mathrm{ran}(T - \lambda)$ *is closed and* $\mathrm{ind}(T - \lambda) = -1$.

(e) $\sigma_e(T)=\partial D$ *and* $\sigma_{le}(T)=\sigma_{re}(T)=\partial D$.

Proof. Since T is a 2-isometric non-unitary operator, part (a) and (b) are obvious by Theorem 4.12.

(c) Since $\sigma_p(T)\subseteq\sigma_{ap}(T)=\partial D$, $0\notin\sigma_p(T)$. Suppose $x=(x_1,x_2,\cdots)\in l_2$ and $\lambda\neq 0$. If $Tx=\lambda x$, then $0=\lambda x_1$, $\alpha_1 x_1=\lambda x_2,\cdots$. Thus $0=x_1=x_2=\cdots$. Hence $\sigma_p(T)=\varnothing$.

(d) If $|\lambda|<1$, then since $\lambda\notin a_{ap}(T)=\partial D$, so that $\mathrm{ran}(T-\lambda)$ is closed and $\dim\ker(T-\lambda)=0$. To prove $\mathrm{ind}(T-\lambda)=-1$, it suffices to show that $\dim\ker(T-\lambda)^*=1$ for $|\lambda|<1$. If $x=(x_1,x_2,\cdots)\in l_2$ and $T^*x=\overline{\lambda}x$, then $(\alpha_1x_2,\alpha_2x_3,\cdots)=\overline{\lambda}(x_1,x_2,\cdots)$. So $x_{n+1}=\dfrac{\overline{\lambda^n}}{\alpha_1\alpha_2\cdots\alpha_n}x_1$ for all n. That is, if $x_{\overline{\lambda}}=\left(1,\dfrac{\overline{\lambda}}{\alpha_1},\dfrac{\overline{\lambda^2}}{\alpha_1\alpha_2},\cdots,\dfrac{\overline{\lambda^n}}{\alpha_1\alpha_2\cdots\alpha_n},\cdots\right)$, then $x=(x_1,x_2,\cdots)=x_1x_{\overline{\lambda}}$. Clearly $x_{\overline{\lambda}}\in\ker(T-\lambda)^*$. This implies that $\ker(T-\lambda)^*$ is the one dimensional space spanned by $x_{\overline{\lambda}}$, as desired.

(e) Using $\partial D\subseteq\sigma_e(T)\subseteq\overline{D}$ (see Remark 4.13(b)) and part (d), we obtain $\sigma_e(T)=\partial D$. Since $\sigma_{le}(T)\cap\sigma_{re}(T)=\partial D$ by Theorem 4.12(c) and $\sigma_{le}(T)\cup\sigma_{re}(T)=\sigma_e(T)$ by Theorem 2.7(a), we have $\sigma_{le}(T)=\sigma_{re}(T)=\partial D$.

Since a unilateral shift T is a 2-isometric non-unitary operator, the following corollary is obvious by the above Theorem 4.19.

Corollary 4.20. *Let* T *be a unilateral shift defined* $T:l_2\to l_2$ *by* $T(x_1,x_2,\cdots)=(0,x_1,x_2,\cdots)$. *Then*

(a) $\sigma(T)=w(T)=\overline{D}$.

(b) $\sigma_p(T)=\varnothing$.

(c) *For* $|\lambda|<1$, $\operatorname{ran}(T-\lambda)$ *is closed with* $\dim\ker(T-\lambda)^*=1$.

(d) $\sigma_{le}(T)=\sigma_{re}(T)=\sigma_e(T)$.

Remark 4.21. Every isometry is normaloid since $r(T)=\|T\|=1$ (see Remark 4.10). But the Example 4.18 shows that a 2-isometry T need not to be normaloid since $r(T)\neq\|T\|$. Also Example 3.8(Chapter 3) and Example 4.18 show that the following classes are related by proper inclusion :

$$\text{Unitary} \subsetneq \text{Isometry} \subsetneq \text{2-isometry} \subsetneq Q\text{-operator}.$$

Next we show that the $w(T)$ satisfies spectral mapping theorem for $f(T)$ and furthermore, the Weyl's theorem holds for $f(T)$ where T is a 2-isometry and f is analytic on a neighborhood of $\sigma(T)$.

Theorem 4.22. *If T is a 2-isometry and f is analytic on a neighborhood of $\sigma(T)$, then $w(f(T))=f(w(T))$.*

Proof. Let T be a 2-isometry. If T is a unitary, the result is obvious. Assume that T is a non-unitary. Suppose $p(z)$ is any polynomial. Let $p(T)-\lambda=a_0(T-\mu_1)\cdots(T-\mu_n)$ where $p(u_i)-\lambda=0$ $i=1,2,\cdots,n$. We first show that $p(w(T))\subseteq w(p(T))$. If $\lambda\notin w(p(T))$, then

$$p(T)-\lambda=a_0(T-\mu_1)\cdots(T-\mu_n)$$

is Weyl. Since $T-\mu_i$ commutes each other, every $T-\mu_i$ is Fredlhom. Thus by Corollary 4.14, $\operatorname{ind}(T-\mu_i)\leq 0$ for each $i=1,2,\cdots,n$, so that $\operatorname{ind}(T-\mu_i)=0$ since

$$\operatorname{ind}(p(T)-\lambda)=\operatorname{ind}((T-\mu_1))+\cdots+\operatorname{ind}((T-\mu_n))=0.$$

Thus $\mu_i\notin w(T)$ for each $i=1,2,\cdots,n$ and $\lambda\notin p(w(T))$ since $p(\mu_i)=\lambda$ $i=1,2,\cdots,n$. Hence this implies $p(w(T))\subseteq w(p(T))$.

The converse assertion $p(w(T))\supseteq w(p(T))$ is trivial (see (2.6)). Hence we have $p(w(T))=w(p(T))$ for any polynomial $p(z)$.

If f is analytic on a neighborhood of $\sigma(T)$, Then by the Runge's theorem, there is a sequence $(p_n(z))$ of polynomials converging uniformly on a neighborhood of $\sigma(T)$ to $f(z)$ so that $p_n(T) \to f(T)$. Note that the mapping $T \to w(T)$ is upper semi-continuous. Since each $p_n(T)$ commutes with $f(T)$, it follows from [30] that

$$f(w(T)) = \lim_{n\to\infty} p_n(w(T)) = \lim_{n\to\infty} w(p_n(T)) = w(f(T)).$$

Hence $w(f(T)) = f(w(T))$.

Oberai showed that if T is isoloid and the Weyl's theorem holds for T, then the Weyl's theorem holds for $p(T)$ if and only if $w(p(T)) = p(w(T))$ for any polynomial $p(z)$ ([30]). Thus the following statement is true.

Corollary 4.23. *If T is a 2-isometry and f is analytic on a neighborhood of $\sigma(T)$, then the Weyl's theorem holds for $f(T)$.*

Proof. Recall that if $T \in L(H)$ is isoloid, then

$$f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T))$$

for every $f \in H(\sigma(T))$ ([29], [30]). Since T is a 2-isometry, T is isoloid by Theorem 4.11. Also the Weyl's theorem holds for T by Theorem 4.15. Thus we have

$$\begin{aligned}\sigma(f(T)) \setminus \pi_{00}(f(T)) &= f(\sigma(T) \setminus \pi_{00}(T)) \\ &= f(w(T)) = w(f(T)) \text{ (Theorem 4.22).}\end{aligned}$$

Therefore $\sigma(f(T)) \setminus \pi_{00}(f(T)) = w(f(T))$, so that the Weyl's theorem holds for $f(T)$.

The following results were discussed in [2]. Here we prove them in detail in the case of a 2-isometry.

Theorem 4.24. *If* T *is a 2-isometry, then* $\ker(\triangle_T)$ *is an invariant subspace for* T *where* $\triangle_T = T^*T - I$.

Proof. Since T is a 2-isometry, $T^*\triangle_T T - \triangle_T = 0$ and $\triangle_T \geq 0$ by Remark 4.2(b). Now if $x \in \ker(\triangle_T)$, then

$$\begin{aligned} \langle \triangle_T T x, \ T x \rangle &= \langle T^* \triangle_T T x, \ x \rangle \\ &= \langle \triangle_T x, \ x \rangle = 0. \end{aligned}$$

Thus $\triangle_T T x = 0$ since $\triangle_T \geq 0$ (see Theorem 2.9(d)). So $T x \in \ker(\triangle_T)$.

Theorem 4.25. *Let* T *be a 2-isometry. Then*

(a) $T \mid \ker \triangle_T$ *is an isometry.*

(b) If $M \subseteq H$ *is an invariant subspace for* T *and* $T \mid M$ *is an isometry, then* $M \subseteq \ker \triangle_T$.

Proof. (a) Let P be the orthogonal projection of H onto $\ker\triangle_T$ and let $A = T \mid \ker\triangle_T$. We shall show that $A^*A = I$. Let $x \in \ker\triangle_T$. Then $T^*Tx = x$ and $A^*A = (PT^*T) \mid \ker\triangle_T$. So $A^*Ax = PT^*Tx = x$, as desired.

(b) Let $B = T \mid M$ and P_M be the orthogonal projection of H onto M. Given $x \in M$, we have $B^*Bx = P_M T^*T x = x$ since $T \mid M$ is an isometry by hypothesis. Thus we see that

$$\begin{aligned} 0 &= \langle B^*Bx - x, x \rangle = \langle (P_M T^*T - I) x, x \rangle \\ &= \langle (T^*T - I) x, P_M x \rangle = \langle \triangle_T x, x \rangle. \end{aligned}$$

So $\langle \triangle_T x, x \rangle = 0$ and $\triangle_T x = 0$. Hence $x \in \ker\triangle_T$ and so $M \subseteq \ker\triangle_T$.

Remark 4.26. For a 2-isometry, $\ker\triangle_T$ is a maximal invariant subspace such that $T \mid \ker\triangle_T$ is an isometry. Also $\ker\triangle_T$ is unique by the above Theorem 4.25.

5. Quasi-isometric operators

Definition 5.1. An operator $T \in L(H)$ is said to be a $quasi-isometry$ if

$$T^* T = T^{*2} T^2.$$

Equivalently, T is a quasi-isometry if

$$\| Tx \| = \| T^2 x \| \text{ for every } x \in H. \tag{5.1}$$

Remark 5.2. Every isometry is a quasi-isometry, whereas an idempotent is a quasi-isometry, but need not be an isometry. For example, every orthogonal projection operator is an idempotent, but is not an isometry. On the other hand, a quasi-isometry which is an 2-isometry is an isometry. Thus the classes of 2-isometries and quasi-isometries are extentions of isometries and they are independent.

The above (5.1) immediately gives us the following facts:

(a) $\| T \| = \| T^2 \| \leq \| T \|^2$.

(b) For $n \geq 2$, $\| T^n x \| = \| T^2 T^{n-1} x \| = \| T^{n+1} x \|$ for every x in H.

(c) If $T^2 x = 0$, then $Tx = 0$.

(d) For any unit vector x in H,

$$\| Tx \|^2 \leq \| T^2 x \| \| T^2 x \| \leq \| T \|^2 \| T^2 x \| .$$

From the above facts, we can obtain the following properties.

Theorem 5.3. *For any quasi-isometry T, the following properties hold.*

(a) $\| T \| = \| T^2 \|$. *Furthermore,* $\| T^n \| = \| T^{n+1} \|$ *for every* $n \geq 1$.

(b) *If T is non-zero, then* $1 \leq \| T \|$.

(c) $\ker T = \ker T^2$.

(d) *T is M-paranormal where* $M = \| T \|^2$.

Recall that Lat(T) is the collection of all invariant subspace for T.

Theorem 5.4. *For any quasi-isometry T, the following properties hold.*

(a) *If S is unitarily equivalent to T, then S is a quasi-isometry.*

(b) *If $M \in Lat(T)$, then $T|M$ is a quasi-isometry.*

(c) *If T is invertible, then T is unitary.*

(d) *If T is invertible, then T^{-1} is also a quasi-isometry.*

Proof. (a) Let $S = UTU^*$ where U is unitary. Then

$$\begin{aligned}(UTU^*)^*(UTU^*) &= U(T^*T)U^* = U(T^{*2}T^2)U^* \\ &= UT^{*2}(U^*U)T^2U^* \\ &= (UTU^*)^{*2}(UTU^*)^2.\end{aligned}$$

Hence $S^*S = S^{*2}S^2$.

(b) Let P be the orthogonal projection of H onto M. Since $M \in Lat(T)$, $TP = PTP$ or taking adjoint, $PT^* = PT^*P$. Thus

$$PT^*T = PT^{*2}T^2 = (PT^*P)T^*T^2 = (PT^*)(PT^*)T^2.$$

Hence $(T|M)^*(T|M) = (T|M)^{*2}(T|M)^2$, as desired.

(c) If T is invertible, then by hypothesis $T^*T = I$. So T is invertible isometry and hence T is unitary.

(d) By part (c), T is unitary. So T^{-1} is unitary and hence T^{-1} *is a* quasi-isometry.

Remark 5.5. Let U be a unilateral shift on l_2 defined in Corollary 4.20. Then U is a quasi-isometry since U is an isometry. But U^* is not a quasi-isometry since $\ker U^* \neq \ker U^{*2}$. So a quasi-isometry need not have a quasi-isometry adjoint.

Theorem 5.6. *Let T be a unilateral weighted shift with non-zero weights $\{\alpha_n\}_{n=0}^{\infty}$. Then T is a quasi-isometry if and only if*

$$|\alpha_{n+1}| = 1 \quad \textit{for } n = 0,1,2,3 \cdots .$$

Proof. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for H. Then $Te_n = \alpha_n e_{n+1}$ for all $n \geq 0$ and $T^*e_0 = 0$, $T^*e_n = \overline{\alpha}_{n-1} e_{n-1}$ for all $n \geq 1$. Thus

$$(T^{*2}T^2 - T^*T)e_n = (|\alpha_n|^2 |\alpha_{n+1}|^2 - |\alpha_n|^2)e_n \quad \text{for all } n \geq 0.$$

So this implies the result.

Example 5.7. Let T be a unilateral weighted shift T with weights

$$(\alpha_n) = (1, \omega, \omega^2, \omega^2, 1, \omega^2, \omega, \omega : 1, \omega, \omega^2, \omega^2, 1, \omega^2, \omega, \omega : \cdots)$$

where $\omega^3 = 1$, i.e., $\omega = \frac{-1+\sqrt{3}i}{2}$. Then $|\alpha_{n+1}| = 1$ for all $n \geq 0$ and $|\alpha_n||\alpha_{n+1}| - |\alpha_{n-1}| = 0$ for $n = 1,2,3 \cdots$. Hence T is a quasi-isometry.

Theorem 5.8. *If T is a non-zero quasi-isometry and if T is hyponormal, then $\| T \| = 1$.*

Proof. If T is a non-zero quasi-isometry, then $1 \leq \| T \|$ by Theorem 5.3(b). And by hypothesis, $TT^* \leq T^*T$ and $T^*(TT^*)T \leq T^*T$. So $\| T^*Tx \| \leq \| Tx \|$ and hence $\| TT^* \| \leq \| T \|$ and $\| T \|^2 \leq \| T \|$. This means $\| T \| \leq 1$, as desired.

Remark 5.9. S. M. Patel proved the fact: If T is a quasi-isometry and if $\| T \| = 1$, then T is hyponormal ([31, Theorem 2.2]). Thus Theorem 5.8 implies that for a non-zero quasi-isometry T,

$$T \text{ is hyponormal if and only if } \| T \| = 1.$$

Example 5.10. Let $T=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ be defined on $\mathbb{C}^2$. Then

(a) T is a quasi-isometry since T is an idempotent operator.

(b) $\ker T = \ker T^2 = \{(0, y) : y \in \mathbb{C}\}$, but $\ker T \subset \ker T^*$ is failed since $\ker T^* = \{(x, -x) : x \in \mathbb{C}\}$.

(c) $\sigma_p(T) = \sigma(T) = \{0, 1\}$ and $\| T \| = \sqrt{2}$. Hence a quasi-isometry is not necessarily normaloid.

Recall that $\sigma_{ap}(T)$ is the approximate point spectrum of T. Also we denote $\sigma_{ap}(T)$ by $\pi(T)$.

The following theorem will be appeared to be true in the next section : Posiquasi-isometric operators (Corollary 6.24 and Corollary 6.26).

Theorem 5.11. *Let T be a quasi-isometry. Then*

(a) *If T is quasinilplotent, then $T=0$.*

(b) *$\sigma_{ap}(T) \setminus \{0\}$ is a subset of the unit circle.*

Remark 5.12. Let T be a quasi-isometry. If T is invertible, then $\sigma(T) \subseteq \partial D$ where $D = \{z \in \mathbb{C} : |z| < 1\}$ since T is unitary. If T is not invertible, then using Theorem 5.11(b) and $\partial\sigma(T) \subseteq \sigma_{ap}(T)$, we have either $\sigma(T) \subseteq \{0\} \cup \partial D$ if 0 is an isolated point of $\sigma(T)$ or $\sigma(T) = \overline{D}$ if 0 is not an isolated point of $\sigma(T)$.

Theorem 5.13. ([31]) *Let T be a quasi-isometry. Then isolated points of $\sigma(T)$ are eigenvalues of T.*

Proof. Let λ be an isolated point of $\sigma(T)$. Then we consider the Riesz spectral projection E with respect to λ,

$$E = \frac{1}{2\pi i} \int_{\partial D} (T - z)^{-1} dz, \tag{5.2}$$

where D is an open disk of center λ which contains no other points of $\sigma(T)$. Then E is a non-zero idempotent operator commuting with T and EH is invariant under the operator T. And also $\sigma(T|EH) = \{\lambda\}$ (see Theorem 2.5) and $T|EH$ a quasi-isometry by Theorem 5.4(b). If $\lambda = 0$, then $T|EH = 0$ by Theorem 5.11(a). If $\lambda \neq 0$, then $T|EH$ is invertible and so must be unitary. Thus $T|EH = \lambda I|EH$. In either case, $\lambda \in \sigma_p(T)$, which completes the proof.

It is well known([37, p.424]) that if E be Riesz spectral projection with respect to λ where λ is an isolated point in $\sigma(T)$ defined by (5.2), then

$$EH = \{ x \in H : \| (T-\lambda)^n x \|^{1/n} \to 0 \} .$$

Evidently, for any positive interger n,

$$\ker(T-\lambda)^n \subseteq EH. \tag{5.3}$$

Corollary 5.14. *Let T be a quasi-isometry and λ be an isolated point of $\sigma(T)$. Then the Riesz spectral projection E with respect to λ defined by (5.2) satisfies $EH = \ker(T-\lambda)$.*

Proof. In general, $\ker(T-\lambda) \subseteq EH$ from (5.3) and in the proof of Theorem 5.13, $(T-\lambda)|EH = 0$, so that $EH \subseteq \ker(T-\lambda)$. Hence $EH = \ker(T-\lambda)$.

Recall that an operator T is reguloid if $\text{ran}(T-\lambda)$ is closed for the isolated points of $\sigma(T)$.

Theorem 5.15. *If T is a quasi-isometry, then T is reguloid.*

Proof. Let λ be an isolated point of $\sigma(T)$. and let E be the Riesz spectral projection with respect to λ defined by (5.2). Then

$$H = EH + (1-E)H ,$$

both EH and $(1-E)H$ are closed subspace, and they both are invariant under the operator T. Note that $\sigma(T\,|\,EH)=\{\lambda\}$ and $\sigma(T\,|(1-E)H)=\sigma(T)\setminus\{\lambda\}$. If we use the decomposition $H=EH+(1-E)H$, we have

$$(T-\lambda)H=(T-\lambda)EH+(T-\lambda)(1-E)H=(1-E)H$$

since $EH=\ker(T-\lambda)$ and $(T-\lambda)|(1-E)H$ is invertible. Hence $\operatorname{ran}(T-\lambda)$ is closed, as desired.

S. M. Patel proved that the Weyl's theorem holds for quasi-isometries ([33, Theorem 3.19]). Here we will prove this with alternate argument using the following Lemma.

Lemma 5.16. *Let* $T=\begin{pmatrix}\lambda & S\\ 0 & T_1\end{pmatrix}$ *on* $H=\ker(T-\lambda)\oplus\overline{\operatorname{ran}(T-\lambda)^*}$ *be a quasi-isometry, where* $\lambda\in\sigma_p(T)$. *Then*

(a) *If* $\lambda\neq 0$, *then* $ST_1=0$ *and* T_1 *is a quasi-isometry.*

(b) $\ker(T_1-\lambda)=\{0\}$.

Proof. (a) Suppose $\lambda\neq 0$. Then $|\lambda|=1$ by Theorem 5.11(b). Thus $T^*T=\begin{pmatrix}1 & \bar{\lambda}S\\ \lambda S^* & S^*S+T_1^*T_1\end{pmatrix}$ and $T^{*2}T^2=$

$$\begin{pmatrix}1 & \bar{\lambda}S+ST_1\\ \lambda S^*+\lambda^2T_1^*S^* & S^*S+\lambda T_1^*S^*S+\bar{\lambda}S^*ST_1+T_1^*S^*ST_1+T_1^{*2}T_1^2\end{pmatrix}.$$

Since $T^*T=T^{*2}T^2$, we have $ST_1=0$ and $T_1^*T_1=T_1^{*2}T_1^2$.

(b) Suppose $x\in\ker(T_1-\lambda)$.

Case 1. $\lambda=0$. In this case,

$Tx=Sx\oplus T_1x=Sx$ and $T^2x=TSx=0$ since $Sx\in\ker T$. And since T is a quasi-isometry, $\|Tx\|=\|T^2x\|=0$ and hence $Tx=0$. So $x\in\ker T$ and $x\in\ker T\cap\overline{\operatorname{ran}T^*}=\{0\}$. Therefore $x=0$.

Case 2. $\lambda \neq 0$. In this case, since $T_1 x = \lambda x$,

$$Tx = Sx \oplus \lambda x \text{ and } T^2 x = 2\lambda Sx \oplus \lambda^2 x. \tag{5.4}$$

Since T is a quasi-isometry, $\| Tx \|^2 = \| T^2 x \|^2$ and from (5.4),

$$\| Sx \|^2 + |\lambda|^2 \| x \|^2 = 4|\lambda|^2 \| Sx \|^2 + |\lambda|^4 \| x \|^2. \tag{5.5}$$

Since $\| \lambda \| = 1$ by Theorem 5.11(b), we have $\| Sx \|^2 = 4 \| Sx \|^2$ from (5.5) and $Sx = 0$. So $Tx = 0 \oplus \lambda x$ from (5.4) and hence $x \in \ker(T-\lambda)$ and $x \in \ker(T-\lambda) \cap \overline{\operatorname{ran}(T-\lambda)^*} = \{0\}$. Thus $x = 0$. The proof is completed.

Theorem 5.17. *Let λ and μ be non-zero distinct eigenvalues of a quasi-isometry T. Then $\ker(T-\lambda) \perp \ker(T-\mu)$.*

Proof. Let T have the matrix representation corresponding λ as in Lemma 5.16. Let $x = x_1 \oplus x_2 \in \ker(T-\mu)$. Then

$$0 = (T-\mu)x = [(\lambda-\mu)x_1 + Sx_2] \oplus (T_1-\mu)x_2.$$

Since $(T_1-\mu)x_2 = 0$, $0 = S(T_1-\mu)x_2 = \mu S x_2$ by Lemma 5.16(a), so $Sx_2 = 0$ and hence $(\lambda-\mu)x_1 = 0$ and $x_1 = 0$ since $\lambda \neq \mu$. Therefore

$$x = [0 \oplus x_2] \perp \ker(T-\lambda) \text{ since } x_2 \in \overline{\operatorname{ran}(T-\lambda)^*}.$$

Theorem 5.18. *The Weyl's theorem holds for quasi-isometries.*

Proof. First we show that $\sigma(T) \setminus w(T) \subset \pi_{00}(T)$.

Let $\lambda \in \sigma(T) \setminus w(T)$. Then $T-\lambda$ is a Fredholm operator with index 0. Hence $\ker(T-\lambda)$ is a non-zero finite demensional subspace and $\lambda \in \sigma_p(T)$.

Let $T = \begin{pmatrix} \lambda & S \\ 0 & T_1 \end{pmatrix}$ on $H = \ker(T-\lambda) \oplus \operatorname{ran}(T-\lambda)^*$. Then $\ker(T_1 - \lambda) = \{0\}$ by Lemma 5.16(b) and

$$\begin{aligned}\mathrm{ind}(T-\lambda) &= \mathrm{ind}\left[\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & T_1-\lambda \end{pmatrix}\right] \\ &= \mathrm{ind}\begin{pmatrix} 0 & 0 \\ 0 & T_1-\lambda \end{pmatrix} \\ &= \mathrm{ind}(T_1-\lambda)\end{aligned}$$

since S is a finite rank operator. So $\ker(T_1-\lambda)^*=0$ and $T_1-\lambda$ is an invertible operator on $\mathrm{ran}(T-\lambda)^*$. Thus $\lambda \notin \sigma(T_1)$ and therefore λ is an isolated point of $\sigma(T)=\sigma(T_1)\cup\{\lambda\}$. Hence $\sigma(T)\setminus w(T)\subset \pi_{00}(T)$.

Next we show that $\pi_{00}(T)\subset\sigma(T)\setminus w(T)$.

Let $\lambda\in\pi_{00}(T)$. Then $EH=\ker(T-\lambda)$ by Corollary 5.14 and $\mathrm{ran}(T-\lambda)$ is closed by Theorem 5.15. And also $(T-\lambda)H=(1-E)H$ in the proof of Theorem 5.15, where E is a Riesz spectral projection with respect to λ defined by (5.2). Thus we have

$$\ker(T-\lambda)^* \cong H/\mathrm{ran}(T-\lambda)=H/(I-E)H \cong EH=\ker(T-\lambda).$$

This implies that $T-\lambda$ is a Fredholm operator with index 0 which is not invertible. Hence $\lambda\in\sigma(T)\setminus w(T)$.

Next we show that $w(T)$ satisfies the spectral mapping theorem for $f(T)$ and furthermore, the Weyl's theorem holds for $f(T)$ where T is a quasi-isometry and f is analytic on a neighborhood of $\sigma(T)$.

Lemma 5.19. *If T is a quasi-isometry and $T-\lambda$ is Fredholm for some $\lambda\in\mathbb{C}$, then* $\mathrm{ind}(T-\lambda)\le 0$.

Proof. If $\lambda\notin\sigma(T)$, then $T-\lambda$ is invertible, so that $\mathrm{ind}(T-\lambda)=0$.

Suppose $\lambda\in\sigma(T)$.

Case 1. $\lambda=0$. In this case,

$\mathrm{ind}(T-\lambda)\le 0$ since T is Fredholm of finite accent by Theorem 5.3(c) (see (2.3)).

Case 2. $\lambda \neq 0$. In this case,

if λ is an isolated point of $\sigma(T)$, then $\text{ind}(T-\lambda)=0$ (see Theorem 2.6). If λ is not an isolated point of $\sigma(T)$, then $\lambda \notin \partial\sigma(T)$, otherwise $\text{ran}(T-\lambda)$ is not closed (see Theorem 2.8), which is a contradiction to the fact that $T-\lambda$ is Fredlhom. Thus $\lambda \in D$ where $D=\{z \in \mathbb{C} : |z| < 1\}$ by Remark 5.12. So $\lambda \notin \sigma_{ap}(T)$ by Theorem 5.11(b) and $\ker(T-\lambda)=\{0\}$. Thus we must have $\dim\ker(T-\lambda)^* \neq 0$ since $\lambda \in \sigma(T)$. Hence $\text{ind}(T-\lambda)<0$.

Theorem 5.20. *If T is a quasi-isometry and f is analytic on a neighborhood of $\sigma(T)$, then $w(f(T))=f(w(T))$.*

Proof. Let T be a quasi-isometry. Suppose $p(z)$ is any polynomial. Let $p(T)-\lambda=a_0(T-\mu_1)\cdots(T-\mu_n)$ where $p(u_i)-\lambda=0$ $i=1,2,\cdots,n$. We first show that $p(w(T)) \subseteq w(p(T))$. If $\lambda \notin w(p(T))$, then

$$p(T)-\lambda=a_0(T-\mu_1)\cdots(T-\mu_n)$$

is Weyl. Since $T-\mu_i$ commutes each other, every $T-\mu_i$ is Fredlhom. Thus $\text{ind}(T-\mu_i) \leq 0$ for each $i=1,2,\cdots,n$ by Lemma 5.19, so that ind $(T-\mu_i)=0$ since

$$\text{ind}(p(T)-\lambda)=\text{ind}((T-\mu_1))+\cdots+\text{ind}((T-\mu_n))=0.$$

Thus $\mu_i \notin w(T)$ for each $i=1,2,\cdots,n$ and $\lambda \notin p(w(T))$ since $p(\mu_i)=\lambda$ $i=1,2,\cdots,n$. Hence this implies $p(w(T)) \subseteq w(p(T))$.

The converse assertion $p(w(T)) \supseteq w(p(T))$ is trivial (see (2.6)). Hence we have $p(w(T))=w(p(T))$ for any polynomial $p(z)$.

If f is analytic on a neighborhood of $\sigma(T)$, Then by the Runge's theorem, there is a sequence $(p_n(z))$ of polynomials converging uniformly on a neighborhood of $\sigma(T)$ to $f(z)$ so that $p_n(T) \rightarrow f(T)$. Note that the mapping $T \rightarrow w(T)$ is upper semi-continuous. Since each $p_n(T)$ commutes with

$f(T)$, it follows from [30] that

$$f(w(T)) = \lim_{n\to\infty} p_n(w(T)) = \lim_{n\to\infty} w(p_n(T)) = w(f(T)).$$

Hence $w(f(T)) = f(w(T))$.

Corollary 5.21. *If T is a quasi-isometry and f is analytic on a neighborhood of $\sigma(T)$, then the Weyl's theorem holds for $f(T)$.*

Proof. Recall that if $T \in L(H)$ is isoloid, then

$$f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T))$$

for every $f \in H(\sigma(T))$([29], [30]). Since T is a quasi-isometry, T is isoloid by Theorem 5.13. Also the Weyl's theorem holds for T by Theorem 5.18. Thus we have

$$\begin{aligned}\sigma(f(T)) \setminus \pi_{00}(f(T)) &= f(\sigma(T) \setminus \pi_{00}(T)) \\ &= f(w(T)) = w(f(T)) \text{ (Theorem 5.20).}\end{aligned}$$

Therefore $\sigma(f(T)) \setminus \pi_{00}(f(T)) = w(f(T))$, so that the Weyl's theorem holds for $f(T)$.

6. Posiquasi-isometric operators

H. C. Rhaly, Jr. introduced posinormal operators as the class of operators T for which $TT^* = T^*PT$ for some positive operator P ([36]). This is a very large class that includes the hyponormal as well as all invertible operators.

Now, we shall define a new class of posiquasi-isometries which is an extension of the class of quasi-isometries and includes all invertible operators. Its concept is motivated by posinormal operators.

Definition 6.1. An operator $T \in L(H)$ is defined to be a *posiquasi-isometry*, shortened to $T \in PQI$, if there exists a positive operator $P \in L(H)$ called the interrupter, such that $T^*T = T^{*2}PT^2$.

Since $T^*T = T^{*2}PT^2$ if and only if

$$< T^*Tx, x > = < T^{*2}PT^2x, x > = < \sqrt{P}T^2x, \sqrt{P}T^2x >,$$

We can see that $T \in PQI$ if and only if for some positive operator $P \in L(H)$,

$$\| Tx \| = \| \sqrt{P}T^2x \| \text{ for all } x \in H. \tag{6.1}$$

By (6.1), clearly if $T \in PQI$ with interrupter P, then $\| T \| = \| \sqrt{P}T^2 \|$.

Theorem 6.2. *If $T \in PQI$ with interrupter P, then*

(a) $\| Tx \| \leq \sqrt{\| P \|} \, \| T^2x \|$ *for every x in H.*

(b) $\| T^nx \| \leq \sqrt{\| P \|} \, \| T^{n+1}x \|$ *for every $n \geq 1$ and every x in H.*

(c) $T^*T \leq \| P \| \, T^{*2}T^2$.

(d) $1 \leq \| P \| \, \| T \|^2$ *if T is non-zero.*

Proof. (a) Since the interrupter P is positive, $\| \sqrt{P} \| = \sqrt{\| P \|}$. Thus

$\| Tx \| \le \| \sqrt{P} \| \; \| T^2 x \| = \sqrt{\| P \|} \; \| T^2 x \|$ for every $x \in H$ from (6.1).

(b) for $n \ge 2$, by part (a),

$$\| T^n x \| = \| T(T^{n-1})x \| \le \sqrt{\| P \|} \; \| T^2 T^{n-1} x \| = \sqrt{\| P \|} \; \| T^{n+1} x \|,$$

for every x in H. Hence the result follows.

(c) and (d) immediately follow from part (a). because part (a) implies $\| Tx \|^2 \le \| P \| \; \| T^2 x \|^2$ for every $x \in H$ and $\| T \| \le \sqrt{\| P \|} \; \| T \|^2$.

Theorem 6.3. *If $T \in PQI$ with interrupter P and T has dense range, then P is unique.*

Proof. Assume P_1 and P_2 both serve as interrupter for T. Then

$$T^* T = T^{*2} P_1 T^2 = T^{*2} P_2 T^2, \text{ so } T^{*2}(P_1 - P_2) T^2 = 0.$$

Since T has dense range, T^* is one to one. Thus $(P_1 - P_2) T^2 = 0$. Take its adjoint and again applying the fact that T has dense range, then $P_1 - P_2 = 0$, as desired.

Remark 6.4. Let U be a unilateral shift on l_2. Then since U is isometry, U is a posiquasi-isometry and since U have not dense range, the interrupter P for U is not unique. In fact take positive interrupter P to be the diagonal matrix with diagonal entries $p_{11} \ge 0$, $p_{22} \ge 0$ and $p_{kk} = 1$ for $k \ge 3$. Then we have $U^* U = U^{*2} P U^2$ by the direct calculation, which shows the nonuniqueness of P for U.

Theorem 6.5. *If $T \in PQI$ with interrupter P and U is isometry (that is, $U^* U = I$), then $UTU^* \in PQI$ with interrupter UPU^*.*

Proof. Let $T^*T = T^{*2}PT^2$. Since P is positive, $UPU^* \geq 0$ and

$$\begin{aligned}(UTU^*)^*(UTU^*) &= U(T^*T)U^* = U(T^{*2}PT^2)U^* \\ &= UT^{*2}(U^*U)P(U^*U)T^2U^* \\ &= (UTU^*)^{*2}(UPU^*)(UTU^*)^2.\end{aligned}$$

Hence $UTU^* \in PQI$ with interrupter UPU^*.

Theorem 6.6. *For any $T \in PQI$ with interrupter P, the following properties hold.*

(a) *λT is a posiquasi-isometry with interrupter $(1/|\lambda|^2)P$ for each $\lambda \in \mathbb{C}$.*

(b) *If S is unitarily equivalent to T, then $S \in PQI$.*

(c) *If $M \in \mathrm{Lat}(T)$, then $T|M \in PQI$ with interrupter $EP|M$ where E is an orthogonal projection of H onto M.*

(d) *$T \otimes I$ and $I \otimes T$ are both posiquasi-isometry.*

Proof. (a) If $\lambda \neq 0$, then $(\lambda T)^*(\lambda T) = \overline{\lambda}\lambda T^{*2}PT^2 = (\lambda T)^{*2}(1/|\lambda|^2 P)(\lambda T)^2$ and $(1/|\lambda|^2)P$ is positive. Hence $\lambda T \in PQI$.

(b) Let $S = UTU^*$ where U is unitary. Then $S \in PQI$ with interrupter UPU^* by the above Theorem 6.5.

(c) Since $M \in \mathrm{Lat}(T)$, $TE = ETE$ (see Theorem 2.4) or $ET^* = ET^*E$ and $EP|M \geq 0$. So

$$ET^*T = ET^{*2}PT^2 = (ET^*E)T^*PT^2 = (ET^*)(ET^*)(EP)T^2.$$

Hence $(T|M)^*(T|M) = (T|M)^{*2}(EP|M)(T|M)^2$, as desired.

(d) Since $T \in PQI$ with interrupter P, $P \otimes I$ is a positive operator and

$$\begin{aligned}(T\otimes I)^*(T\otimes I) &= (T^*\otimes I)(T\otimes I) \\ &= (T^*T)\otimes I = (T^{*2}PT^2)\otimes I \\ &= (T^{*2}\otimes I)(P\otimes I)(T^2\otimes I) \\ &= (T\otimes I)^{*2}(P\otimes I)(T\otimes I)^2.\end{aligned}$$

Hence $T \otimes I \in PQI$ with interrupter $P \otimes I$. Similarly $I \otimes T \in PQI$ with interrupter $I \otimes P$.

Theorem 6.7. (Douglas [14]) *For any $A, B \in L(H)$, the following statements are equivalent.*

(a) $\operatorname{ran} A \subseteq \operatorname{ran} B$.

(b) $AA^* \leq \lambda^2 BB^*$ *for some* $\lambda \geq 0$.

(c) *there exist a* $T \in L(H)$ *such that* $A = BT$.

Moreover, if (a), (b), *and* (c) *hold, then there is a unique operator* T *such that*

(1) $\| T \|^2 = \inf\{ \mu \mid AA^* \leq \mu BB^* \}$;

(2) $\ker A = \ker T$; *and*

(3) $\operatorname{ran} T \subseteq \overline{(\operatorname{ran} B^*)}$.

Douglas' theorem leads almost immediately to the following result.

Theorem 6.8. *For any $T \in L(H)$, the following statements are equivalent.*

(a) $T \in PQI$.

(b) $T^*T \leq \lambda^2 T^{*2} T^2$ *for some* $\lambda \geq 0$.

(c) $\operatorname{ran} T^* = \operatorname{ran} T^{*2}$.

(d) *there exists a* $A \in L(H)$ *such that* $T^* = T^{*2} A$.

Moreover, if (a), (b), (c) and (d) hold, then there is a unique operator A *such that*

(1) $\| A \|^2 = \inf\{ \mu \mid T^*T \leq \mu T^{*2} T^2 \}$;

(2) $\ker T^* = \ker A$; *and*

(3) $\operatorname{ran} A \subseteq \overline{(\operatorname{ran} T^2)}$.

Proof. (a) implies (b) : If $T \in PQI$, then by Theorem 6.2(c), $T^*T \leq \| P \| T^{*2} T^2$. Put $\lambda = \sqrt{\| P \|}$. Then the result follows.

(b) implies (c) : By hypothesis, since $T^*(T^*)^* \leq \lambda^2 T^{*2}T^2$, $\operatorname{ran} T^* \subseteq \operatorname{ran} T^{*2}$ by Theorem 6.7 and in general $\operatorname{ran} T^* \supseteq \operatorname{ran} T^{*2}$ for any $T \in L(H)$. Hence $\operatorname{ran} T^* = \operatorname{ran} T^{*2}$.

(c) implies (d) : This is trivial by Theorem 6.7.

(d) implies (a) : If $T^* = T^{*2}A$, then

$$\begin{aligned} T^*T &= T^{*2}AT = (T^{*2}A)(A^*T^2) \\ &= T^{*2}(AA^*)T^2 \end{aligned}$$

and $AA^* \geq 0$. Thus $T \in PQI$.

(1), (2), (3) : They immediately follow from Theorem 6.7.

Remark 6.9. (a) If $T \in PQI$, then $\ker T = \ker T^2$ since $\operatorname{ran} T^* = \operatorname{ran} T^{*2}$.
(b) Let U be a unilateral shift on l_2. Then $U \in PQI$, but $U^* \notin PQI$ since $\ker U^* \neq \ker U^{*2}$, so that a posiquasi-isometry need not have a posiquasi-isometry adjoint.

Theorem 6.10. $T \in PQI$ *if and only if there exists a positive operator* $P \in L(H)$ *such that* $T^*T \leq T^{*2}PT^2$.

Proof. It suffices to show that if there exists a positive operator $P \in L(H)$ such that $T^*T \leq T^{*2}PT^2$, then $T \in PQI$. For any $x \in H$,

$$\begin{aligned} < T^*Tx, x > &\leq < T^{*2}PT^2x, x > \\ &= < \sqrt{P}T^2x, \sqrt{P}T^2x > \\ &\leq \|P\| < T^2x, T^2x > \end{aligned}$$

Thus $T^*T \leq \|P\| T^{*2}T^2$. Hence $T \in PQI$ by Theorem 6.8.

Theorem 6.11. *Let* T *and* S *be commuting posiquasi-isometries. Then the product* TS *is a posiquasi-isometry.*

Proof. Let $T \in PQI$ with interrupter P, $S \in PQI$ with interrupter Q and

$TS = ST$. Then by Theorem 6.2(a), we have for each x,

$$\begin{aligned}\| T(Sx) \|^2 &\le \| P \| \ \| T^2(Sx) \|^2 \\ &= \| P \| \ \| S(T^2 x) \|^2 \\ &\le \| P \| \ \| Q \| \ \| S^2(T^2 x) \|^2 \\ &= \| P \| \ \| Q \| \ \| (TS)^2 x \|^2.\end{aligned}$$

Thus $(TS)^* TS \le \lambda^2 (TS)^{*2} (TS)^2$ where $\lambda = \sqrt{\| P \| \| Q \|}$. Hence $TS \in PQI$ by Theorem 6.8.

By the above theorem, any power of a posiquasi-isometry is a posiquasi-isometry. But we will directly prove this fact as following:

Corollary 6.12. *If $T \in PQI$, then $T^n \in PQI$ for every positive integer n.*

Proof. If $T \in PQI$, then by Theorem 6.2(b), for $n \ge 1$,

$$\begin{aligned}\| T^n x \| &\le \| P \|^{1/2 \times 1} \| T^{n+1} x \| \\ &\le \| P \|^{1/2 \times 2} \| T^{n+2} x \| \\ &\le \| P \|^{1/2 \times 3} \| T^{n+3} x \| \\ &\cdots \\ &\le \| P \|^{1/2 \times n} \| T^{n+n} x \|\end{aligned}$$

Hence $\| T^n x \| \le \| P \|^{n/2} \| T^{2n} x \|$ for every x in H. Put $\lambda_n = \| P \|^{n/2}$. Then $(T^n)^* T^n \le \lambda_n^2 (T^n)^{*2} (T^n)^2$ for each n, which implies $T^n \in PQI$ for every positive integer n by Theorem 6.8.

A posiquasi-isometry need not be invertible (see Remark 6.4), but the following theorem tells us that an invertible operator must be a posiquasi-isometry.

Theorem 6.13. *Every invertible operator is a posiquasi-isometry with the unique interrupter P.*

Proof. If T is invertible, then $T^* = T^*(T^*)(T^*)^{-1} = T^{*2}(T^*)^{-1}$. So $T \in PQI$ by Theorem 6.8. Also T has dense range since T is invertible.

Thus the interrupter P is unique by Theorem 6.3.

By Theorem 6.13, we know that if T is invertible, then both T and T^* will be a posiquasi-isometry. Furthermore, T^{-1} and $(T^{-1})^*$ will also be a posiquasi-isometry. The following theorems formalize these relationships in terms of interrupters.

Theorem 6.14. *If T is invertible with interrupter P, then*

(a) *P is invertible and P^{-1} is a positive operator.*

(b) *$B=\sqrt{P}\,T^*\sqrt{P}$ is a posiquasi-isometry with interrupter P^{-1}.*

Proof. (a) By hypothesis, $T^*T=T^{*2}PT^2$ and $P=(T^{-1})^*(T^{-1})$ since T is invertible. Thus P is invertible and $P^{-1}=TT^*$ is positive.

(b) Since $P^{-1}=TT^*$, $TT^*P=I$. Thus $B^*B=\sqrt{P}\,TPT^*\sqrt{P}$ and

$$\begin{aligned} B^{*2}P^{-1}B^2 &= (\sqrt{P}\,T\,PT\,\sqrt{P})P^{-1}(\sqrt{P}\,T^*PT^*\sqrt{P}) \\ &= \sqrt{P}\,TP(TT^*P)T^*\sqrt{P} \\ &= \sqrt{P}\,TPT^*\sqrt{P} \\ &= B^*B. \end{aligned}$$

Hence the result follows.

Theorem 6.15. *If T is invertible and if $T^*\in PQI$ with interrupter P, then P is invertible and P^{-1} serves as the interrupter for the posiquasi -isometry T^{-1}.*

Proof. By Theorem 6.14, P is invertible and P^{-1} is a positive operator. $TT^*=T^2PT^{*2}$ implies $(TT^*)^{-1}=(T^2PT^{*2})^{-1}$. Thus

$$(T^{-1})^*(T^{-1})=(T^{-1})^{*2}P^{-1}(T^{-1})^2,$$

so that T^{-1} is a posiquasi-isometry with interrupter P^{-1}, as desired.

In Remark 5.9, we know that the following statement holds for a non-zero quasi-isometry T,

$$T \text{ is hyponormal if and only if } \|T\|=1. \tag{6.2}$$

The following theorems show the relation between posiquasi-isometry and hyponormal operator, paranormal, and M-paranormal.

Recall that T is M-paranormal if $\|Tx\|^2 \leq M\|T^2x\|$ for any unit vector x in H.

Theorem 6.16. *Let $T \in PQI$ with interrupter P and let $M = \|P\|\ \|T\|^2$. Then the following properties holds.*

(a) *T is M-paranormal and $M \geq 1$.*

(b) *If $\|T\| = \|P\| = 1$, then T is hyponormal.*

Proof. (a) If $T \in PQI$ with interrupter P, then by Theorem 6.2(a), we have

$$\begin{aligned}\|Tx\|^2 &\leq \|P\|\ \|T^2x\|\ \|T^2x\| \\ &\leq \|P\|\ \|T\|^2\|T^2x\|. \\ &= M\|T^2x\|\end{aligned}$$

for any unit vector x in H. Hence T is M-paranormal and $M \geq 1$ by Theorem 6.2(d).

(b) By hypothesis, we have $T^*T \leq \|P\|T^{*2}T^2$ by Theorem 6.2(c). Thus if $\|T\| = \|P\| = 1$, then $T^*T = T^{*2}T^2$, easily checked. In fact $T^*T \leq T^{*2}T^2$ if $\|P\|=1$ and $T^*T \geq T^{*2}T^2$ if $\|T\|=1$. So T is a quasi-isometry with $\|T\|=1$. Hence T is hyponormal by (6.2).

Theorem 6.17. *If T is hyponormal and* $\operatorname{ran} T$ *is closed, then $T \in PQI$.*

Proof. Since T is hyponormal, $\ker T = \ker T^2$ and $(\operatorname{ran} T^*)^\perp = (\operatorname{ran} T^{*2})^\perp$. And also since $\operatorname{ran} T$ is closed, $\operatorname{ran} T^*$ is closed ([10]). Thus we have $\operatorname{ran} T^* = \operatorname{ran} T^{*2}$. Hence $T \in PQI$ by Theorem 6.8.

The following theorem immediately holds by properties of M-paranormal.

Theorem 6.18. ([25]) *Let $T \in PQI$ with interrupter P and $M = \| P \| \ \| T \|^2$. Then we have the followings:*

(a) $M^2 \| T^3 x \| \geq \| T^2 x \| \ \| Tx \|$ *for any unit vector x in H.*

(b) *For every posituve integer k and every unit vector x in H,*

$$M^{2k-1} \| T^{k+1} x \|^2 \geq \| T^k x \|^2 \| T^2 x \| .$$

(c) *If T is a unilateral weighted shift T with non-zero weights $\{\alpha_n\}$, then*

$$|\alpha_n| \leq M |\alpha_{n+1}|.$$

for each positive integer n.

Theorem 6.19. *Let T be a unilateral weighted shift with non-zero weights $\{\alpha_n\}_{n=1}^{\infty}$. Then*

$$T \in PQI \quad \text{if and only if} \quad sup_{n \geq 1} (1/|\alpha_n|) < \infty.$$

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. If $T \in PQI$, then $T^*T \leq \lambda^2 T^{*2} T^2$ for some $\lambda \geq 0$ by Theorem 6.8. So

$$< T^*Te_n, e_n > \ \leq \lambda^2 < T^{*2} T^2 e_n, e_n >$$

for each n. Thus $|\alpha_n|^2 \leq \lambda^2 |\alpha_n|^2 |\alpha_{n+1}|^2$ and $1/|\alpha_{n+1}|^2 \leq \lambda^2$ since α_n is non-zero for each n. Hence $\sup_{n \geq 1} (1/|\alpha_n|) < \infty$.

Conversely, let $\sup_{n \geq 1} (1/|\alpha_n|) < \infty$. Taking a positive operator P to be the diagonal matrix with diagonal entries

$$p_{11} \geq 0 \ , p_{22} \geq 0 \ \text{and} \ p_{nn} = 1/|\alpha_{n-1}|^2 \ \text{for} \ n \geq 3, \qquad (6.3)$$

we have $T^*T = T^{*2} P T^2$. In fact, $T^*Te_n = |\alpha_n|^2 e_n$ for $n \geq 1$ since $Te_n = \alpha_n e_{n+1}$ for $n \geq 1$ and $T^*e_1 = 0$ and $T^*e_n = \overline{\alpha}_{n-1} e_{n-1}$ for $n \geq 2$. On the other hand,

$$T^{*2}PT^2e_n = (\alpha_n\,\alpha_{n+1})\,T^{*2}Pe_{n+2}$$

$$= (\alpha_n\alpha_{n+1})\,(1/|\alpha_{n+1}|^2)\,T^{*2}e_{n+2}$$

$$= \alpha_n\overline{\alpha_n}e_n = |\alpha_n|^2e_n$$

for $n \geq 1$. Hence $T^*Te_n = T^{*2}PT^2e_n$ for $n \geq 1$, as desired.

Corollary 6.20. *If T is a paranormal unilateral weighted shift with non-zero weights $\{\alpha_n\}_{n=1}^{\infty}$, then $T \in PQI$.*

Proof. By hypothesis, non-zero weights $\{\alpha_n\}_{n=1}^{\infty}$ is bounded and $\{|\alpha_n|\}_{n=1}^{\infty}$ is monotonically increasing. So $\{|\alpha_n|\}_{n=1}^{\infty}$ converges to a non-zero limit. Thus $\{1/|\alpha_n|\}_{n=1}^{\infty}$ also converges to a non-zero limit. Hence $\sup_{n\geq 1}(1/|\alpha_n|) < \infty$ and the result follows from Theorem 6.19.

Remark 6.21. A unilateral weighted shift T with weights $\{\alpha_n\}_{n=1}^{\infty}$ is compact if $\{\alpha_n\}_{n=1}^{\infty}$ converges to zero ([17]). But $T \notin PQI$ by Theorem 6.19. Hence a compact operator need not be a posiquasi-isometry.

Let $T \in PQI$ with interrupter P. If $1 = \|P\|\,\|T\|^2$, then T is paranormal by Theorem 6.16(a). But the following example shows that the converse is not true.

Example 6.22. Let T_x be a unilateral weighted shift with non-zero weights

$$\alpha_0 = x\,,\ \alpha_1 = \sqrt{\frac{2}{3}}\,,\ \alpha_2 = \sqrt{\frac{3}{4}}\,,\cdots,\ \alpha_n = \sqrt{\frac{n+1}{n+2}}\,,\cdots.$$

(a) $T_x \in PQI$ for every $x > 0$ since

$$\sup_n\left(\frac{1}{|\alpha_n|}\right) = \max\left\{\frac{1}{x},\ \sqrt{\frac{3}{2}}\right\} < \infty.$$

(b) Let a positive operator P be the diagonal matrix with diagonal entries $p_{11}=0$, $p_{22}=0$ and $p_{nn}=n/n-1$ for $n \geq 3$ from (6.3). Then P is the interrupter for T_x for all $x>0$.

(c) If $0<x \leq \sqrt{\frac{2}{3}}$, then $T_x \in PQI$ with interrupter P and also is a paranormal since $\{\alpha_n\}_{n=0}^{\infty}$ is monotonically increasing. But $1= \| P \| \, \| T_x \|^2$ is failed since $\| T_x \| = \max\{x, 1\} = 1$ and $\| P \| = 3/2$.

The above example gives us that if $x > \sqrt{\frac{2}{3}}$, then $T_x \in PQI$, but T_x is not a paranormal operator. Thus a posiquasi-isometry need not to be a paranormal operator.

In the next theorems we explore several properties of the spectrum of a posiquasi-isometry.

Recall that $T \in L(H)$ is quasinilpotent if $\| T^n \|^{1/n} \to 0$ as $n \to \infty$. Evidently, if T is quasinilpotent, then $\sigma(T)=0$.

Theorem 6.23. *If $T \in PQI$ with interrupter P and if T is quasinilplotent, then $T=0$.*

Proof. By hypothesis, for sufficiently small $\epsilon > 0$, there exits N such that $n \geq N$ implies $\| T^n \|^{1/n} < \epsilon$ since T is quasinilplotent. Using Theorem 6.2(b), we get $\| T \| \leq (\sqrt{\| P \|} \ \epsilon)^{n-1} \epsilon$ for all $n \geq N$ since

$$\begin{aligned} \| T \| &\leq \sqrt{\| P \|} \ \| T^2 \| \\ &\leq (\sqrt{\| P \|})^2 \| T^3 \| \\ &\cdots \\ &\leq (\sqrt{\| P \|})^{n-1} \| T^n \| \\ &\leq (\sqrt{\| P \|})^{n-1} \epsilon^n. \end{aligned}$$

Hence this implies that $T=0$.

Corollary 6.24. *Every quasinilplotent quasi-isometriy T is zero.*

Proof. Since every quasi-isometriy T is a posiquasi-isometry with the interrupter I, the result follows from Theorem 6.23.

Recall that $\pi(T)$ denotes approximate point spectrum of T.

Theorem 6.25. *Let $T \in PQI$ with interrupter P. If $\lambda \in \pi(T) \setminus \{0\}$, then*

$$\frac{1}{\sqrt{\| P \|}} \leq |\lambda| \leq \| T \| .$$

Proof. It is sufficient only to show $\frac{1}{\sqrt{\| P \|}} \leq |\lambda|$. Now if $\lambda \in \pi(T) \setminus \{0\}$, then there exists a sequence (x_n) in H with $\| x_n \| = 1$ for all n such that $\| (T-\lambda) x_n \| \to 0$. So $\| Tx_n \| \leq \sqrt{\| P \|} \; \| T^2 x_n \|$ for every x_n in H by Theorem 6.2(a). Thus $|\lambda| \leq \sqrt{\| P \|} \; |\lambda|^2$ as $n \to \infty$. Since $\lambda \neq 0$, We have $\frac{1}{\sqrt{\| P \|}} \leq |\lambda|$.

Corollary 6.26. *If T is a quasi-isometry, then $\pi(T) \setminus \{0\}$ is a subset of the unit circle.*

Proof. In the proof of Theorem 6.25, using $\| Tx_n \| = \| T^2 x_n \|$ instead of $\| Tx_n \| \leq \sqrt{\| P \|} \; \| T^2 x_n \|$ since T is a quasi-isometry, then the result immediately follows.

Theorem 6.27. *Let $P(H)$ be the set of all posiquasi-isometries on H. Then $P(H)$ is not closed in the operator norm topology on $L(H)$.*

Proof. Let T be a unilateral weighted shift with weights $\{1/(n+1)\}_{n=1}^{\infty}$. Then we have well known that $\sigma(T) = \{0\}$ and T is a compact operator.

Suppose (λ_n) is a sequence converging to 0. Then $T-\lambda_n$ converges to T, but T is not posiquasi-isometry by Theorem 6.23 (or Theorem 6.19), while by Theorem 6.13, each $T-\lambda_n$ is posiquasi-isometry since it is invertible.

Remark 6.28. In the proof of the above theorem we can know the fact that if T is a posiquasi-isometry, then the translate $T-\lambda$ need not be a posiquasi isometry.

Remark 6.29. Consider $T=U-2$ where U is a unilateral shift on l_2. Since 2 is not in $\sigma(U)=\{\lambda : |\lambda| \leq 1\}$, T is a posiquasi-isometry. But the Corollary 6.26 shows that T is not a quasi-isometry because $\sigma(T)=\{\lambda : |\lambda+2| \leq 1\}$ and $\pi(T)\setminus\{0\}$ is not a subset of the unit circle. Thus the following classes are related by proper inclusion :

$$\begin{aligned} \text{Unitary} \subsetneq \text{Isomertry} &\subsetneq \text{Quasi-isometry} \\ &\subsetneq \text{Posiquasi-isometry} \\ &\subsetneq M\text{-paranormal.} \end{aligned}$$

Example 6.30. Let $T=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ be defined on $\mathbb{C}^2$. Then since T is invertible, $T \in PQI$ with unique interrupter $P=\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ by Theorem 6.13. Note that $r(T)=1$ and $\| T \| = \sqrt{2}$. Thus a posiquasi-isometry is not necessarily normaloid.

Example 6.31. Let $T=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ be defined on $\mathbb{C}^2$. Then $T \in PQI$. In fact, T is a quasi-isometry since T is an idempotent operator. Thus we have $\ker T = \ker T^2=\{(0, y) : y \in \mathbb{C}\}$, but $\ker T \subset \ker T^*$ is failed since $\ker T^* =\{(x, -x) : x \in \mathbb{C}\}$. So 0 is not a normal eigenvalue (see (2.1)).

Theorem 6.32. *Let $T \in PQI$ with interrupter P. Then*

(a) *If* $0 \in \sigma_p(T)$, *then* $0 \in \sigma_p(T^*)$.

(b) *If* $0 \in \pi(T)$, *then* $0 \in \pi(T^*)$.

(c) *If T has dense range, then* $\ker T = \ker T^* = \{0\}$.

Proof. (a) Let $0 \in \sigma_p(T)$. If $0 \in \mathbb{C} \setminus \sigma_p(T^*)$, then T^* is one-one. So $T = T^*PT^2$ since $T^*T = T^{*2}PT^2$. Take its adjoint, $T^* = T^{*2}PT$ and again applying the fact that T^* is one-one, we have $I = T^*PT$. This will contradict the fact that $0 \in \sigma_p(T)$.

(b) Let $0 \in \pi(T)$. If $0 \in \mathbb{C} \setminus \pi(T^*)$, then $0 \in \mathbb{C} \setminus \sigma_p(T^*)$ and $I = T^*PT$ by in the proof of part (a). Since $0 \in \pi(T)$, we can choose a sequence (x_n) of unit vectors such that $Tx_n \to 0$. Then $x_n = T^*P(Tx_n)$, so that $\|x_n\| = \|T^*P(Tx_n)\|$ for all n. This is a contradiction since $\|x_n\| = 1$ for all n, and $\|T^*P(Tx_n)\| \to 0$ as $n \to \infty$.

(c) Since T has dense range, T^* is one to one. Thus T is also one to one by part (a), as desired.

If $T \in PQI$ with interrupter P and $1 = \|P\| \, \|T\|^2$, then the Weyl's theorem holds for T since T is paranormal by Theorem 6.16(a). But in general, the following property holds for a posiquasi-isometry.

Theorem 6.33. Let $T \in PQI$ *with interrupter P. Then* $0 \in \sigma(T) \setminus w(T)$ *if and only if* $0 \in \pi_{00}(T)$.

Proof. Let $0 \in \sigma(T) \setminus w(T)$. Then T is a Weyl operator. Hence $\ker T$ is non-zero finite dimensional subspace. Now we only show that 0 is a isolated point in $\sigma(T)$. Since T has finite ascent (see Remark 6.9), T has finite decent (see (2.3)). And hence T is a Browder, so $0 \notin \sigma_b(T) = \sigma_e(T) \cup acc\,\sigma(T)$ (see (2.5)). Hence 0 is a isolated point in $\sigma(T)$.

Conversely, let $0 \in \pi_{00}(T)$. Then we consider Riesz spectral projection E with respect to 0 , $E = \frac{1}{2\pi i}\int_{\partial D}(T-\lambda)^{-1}d\lambda$, where D is an open disk of center 0 which contains no other points of $\sigma(T)$. Then E is a non-zero idempotent operator commuting with T, EH is invariant under the operator T and $\sigma(T|EH) = \{0\}$, $\sigma(T|(1-E)H) = \sigma(T) \setminus \{0\}$ (see Theorem 2.5). Thus $T|EH \in PQI$ by Theorem 6.6(c) and $T|EH = 0$ by Theorem 6.23. Therefore 0 is an eigenvalue of T. And $EH = \ker T$ (see (5.3)). If we use decomposition $H = (1-E)H + EH$, we have

$$TH = T(1-E)H + TEH = (1-E)H$$

since $0 \notin \sigma(T|(1-E)H)$. Hence $\operatorname{ran} T$ is closed. And

$$\dim\ker T^* = \dim(H/\operatorname{ran} T) = \dim EH = \dim\ker T.$$

This implies that T is a Weyl operator which is not invertible. Hence $0 \in \sigma(T) \setminus w(T)$.

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<국 문 초 록>

Q^*-작용소, 2-등거리변환 작용소, 유사-등거리변환 작용소 그리고 양유사-등거리변환 작용소에 관한 연구

본 논문에서 Q-작용소, 2-등거리변환 작용소(2-isometry), 유사-등거리변환 작용소(quasi-isometry) 그리고 새롭게 정의한 Q^*-작용소와 양유사-등거리변환 작용소(posiquasi-isometry)의 대수적 성질과 이들 작용소들의 스펙트럼의 특성을 연구한다. 그리고 이들 작용소들과 하이퍼노말(hyponormal), 파라노말(paranormal)작용소들 등과의 관계를 조사한다. 양유사-등거리변환 작용소들의 집합은 유사-등거리변환 작용소들의 집합의 확장이며 모든 가역적 작용소들을 포함한다.

또한 가중 일단전진이동 작용소(unilateral weighted shift)가 Q-작용소, Q^*-작용소, 2-등거리변환 작용소, 유사-등거리변환 작용소, 양유사-등거리변환 작용소가 되기 위한 필요충분조건을 제시한다. 특히 힐버트 공간에서 유계 선형 작용소 T가 2-등거리변환 작용소 또는 유사-등거리변환 작용소라고 하면 바일정리(Weyl's theorem)가 T에 대하여 성립하고, f가 T의 스펙트럼을 포함하는 개근방에서 정의한 해석적 함수라고 할 때, T의 바일 스펙트럼은 $f(T)$에 대해 스펙트럼 함수 정리(spectral mapping theorem)를 만족시키며 나아가 $f(T)$가 바일정리를 만족한다는 것을 밝힌다.

어떤 작용소가 양유사-등거리변환 작용소가 되기 위한 필요충분조건들을 제시하며, 모든 유사-멱영원(quasinilpotent)이고 양유사-등거리변환 작용소는 영인작용소이며 양유사-등거리변환 작용소의 임의의 거듭제곱은 또한 양유사-등거리변환 작용소임을 밝힌다. 그리고 모든 양유사-등거리변환 작용소들의 집합은 $L(H)$의 작용소 노름 위상(operator norm topology)에서 닫혀있지 않음을 보인다.

감사의 글

지난 4년 동안 어둠의 터널을 지나 이제 막 한 줄기 빛을 보게 되었습니다. 최종 논문 심사를 마치고. 교수님들께 축하의 박수를 받는 순간의 행복감은 이루 말할 수 없었습니다.

만학을 하면서 정신적 육체적으로 힘든 고통의 순간도 있었지만, 얽히고설킨 실타래를 하나씩 풀어 나가는 과정 속에서 수학의 진미를 느끼고 삶에 지혜를 얻을 수 있었던 소중한 시간이었습니다.

박사과정 동안 바쁘신 가운데도 애정을 갖고 용기를 북돋워 주셨으며, 논문 완성에 이르기까지 정성을 다하여 지도를 해주시고 교정까지 꼼꼼히 살펴보아 주신 양영오 교수님께 진심으로 깊은 감사를 드립니다. 그리고 논문의 심사를 맡아 친절하게 지도 조언을 해 주신 방은숙 교수님, 송석준 교수님, 고윤희 교수님, 유상욱 교수님과 그 외 가르침을 주신 수학과 교수님들께도 감사를 드립니다.

이 과정 중에서 서로 의지하고 격려하며 귀중한 시간을 함께한 박권룡 선생님, 문동주 선생님, 김순찬 선생님, 강상진 선생님, 강경훈 선생님, 강문환 선생님, 송미혜 선생님들께 감사의 마음을 전합니다. 그리고 오늘의 결실이 있기까지 부족한 저에게 많은 관심을 가져주시고 힘과 용기를 주신 모든 분들께도 고마운 마음을 전하고 싶습니다.

끝으로, 한평생을 노심초사하시며 항상 아들의 건강을 염려하고 지극한 정성으로 보살펴 주신 어머니, 남편을 믿고 아낌없는 성원과 희생으로 내조해준 사랑하는 아내, 공부하느라 함께할 시간을 많이 할애해 주지 못한 아빠를 인내해준 믿음직한 아들 관엽, 예쁜 딸 민주와 함께 행복의 기쁨을 나누고 싶습니다.

그리고 멀리 계시면서도 가까이에서 우리 가족들의 안녕을 늘 지켜 주시는 아버지의 영전 앞에 삼가 이 한 편의 논문을 드립니다.

2009년 2월