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On Fredholm Operators and Weyl Spectrum

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On Fredholm Operators and Weyl Spectrum

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<국문초록>

FREDHOLM 작용소와 WEYL 스펙트럼에 관한 연구

본 논문에서는 무한차원 힐버트 공간 H 에서 유계 선형작용소 T 의 바일 스펙트럼(*Weyl spectrum*) $w(T)$ 와 *Fredholm* 작용소에 관한 여러가지 성질을 다루었다. 본 논문에서 조사한 주요 내용은 다음과 같다.

(1) 바일 스펙트럼과 진성 스펙트럼(*essential spectrum*) $\sigma_e(T)$ 의 차집합 $w(T) - \sigma_e(T)$ 는 개집합이고 이는 또한 스펙트럼의 도집합(*derived set*) 의 부분집합임을 밝혔다. 또한 바일 스펙트럼의 경계(*boundary*)는 진성스펙트럼의 부분집합임을 보였다.

(2) 바일 스펙트럼과 진성스펙트럼이 일치하기 위한 몇 가지 충분조건을 제시하였다.

(3) 작용소의 바일 스펙트럼 반경을 정의한 후 바일 스펙트럼 반경은 상반연속(*upper semi-continuous*) 임을 밝혔다. 또한 브라우더 스펙트럼이 상반연속임을 다른 방법으로 증명했다.

(4) 작용소 T가 $M-$ *hyponormal* 작용소이고 f 가 T 의 스펙트럼 근방에서 해석적일 때 바일 스펙트럼에 관한 스펙트럼 사상정리 $w(f(T)) = f(w(T))$ 가 성립함을 보이고, 아울러 작용소 T 가 *hyponormal* 이고 p 가 다항식일 때 $p(T)$ 의 바일 정리가 성립함을 보였다. 이는 바일 정리의 성립여부에 관하여 Oberai 가 오래전에 제기한 문제의 해가 된다.

CONTENTS

1. Introduction

Let H denote an infinite-dimensional Hilbert space. If T is an operator, we write $N(T)$ and $R(T)$ for the null space and range of T. We note that $R(T)^{\perp} = N(T^{*})$ for any $T \in B(H)$. An operator T in $B(H)$ is called a Fredholm operator if $N(T) = \ker T = T^{-1}(\{0\})$ is finite-dimensional, $R(T)$ is closed and $R(T)^{\perp}$ is finite dimensional. Write $\mathcal{F}$ and $\mathcal{K}$ for the class of all Fredholm operators and compact operators respectively. The Fredholm spectrum of T, denoted by $\sigma_{\mathcal{F}}(T)$, is the set $\sigma_{\mathcal{F}}(T) = \{ \lambda \in \mathbb{C} : T-\lambda \notin \mathcal{F} \}$. For all $T \in \mathcal{F}$, the index of T, $i(T)$, is defined by $i(T) = \dim\ker T - \dim R(T)^{\perp}$. An operator $T \in B(H)$ is called a Weyl operator if T is Fredholm and $i(T) = 0$. The Weyl spectrum $w(T)$ of T is the set $w(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not a Weyl operator $\}$. The concept of a Weyl spectrum is relevant only for infinite-dimensional space.

In this thesis, we will study properties of Fredholm and Weyl operators, properties of Weyl spectrum, the relations between Weyl spectrum and several spectra and that the Weyl spectrum of hyponormal operator satisfies the spectral mapping theorem. Also we introduce properties(continuity, topological properties, spectral mapping theorem, etc.) of a Weyl spectrum and properties of Fredholm operators and index in detail.

The organization of this thesis is as follows. In section 1, we introduce the basic properties of various spectra (spectrum, point spectrum, approximate point spectrum, etc.) of a linear bounded operator on a Hilbert space H and relations among them.

In section 2, we introduce topological properties of Fredholm operators on

H.

In section 3, we deal with Weyl spectrum of an operator on H. In particular, we show that the boundary of a Weyl spectrum of an operator is a subset of the essential spectrum of an operator and the Weyl spectrum is invariant under similarity.

Let $\theta(T)$ be the set of complex number λ such that $T-\lambda$ is Fredholm of nonzero index. We show that $\theta(T)$ is an open set of $B(H)$ and $\theta(T) \subset$ acc$\sigma(T)$.

In section 4, we introduce continuities of $\sigma_i(T)$ $i = 1, 2, 3, 4, 5$, which are defined in Definition 5.12. We see that the mapping $T \to w(T)$ is upper semi-continuous but not continuous. In particular we give some conditions under which the mapping $T \to w(T)$ is continuous.

In section 5, we deal with the spectral mapping theorems. In particular we show that the Weyl spectrum of $M-$hyponormal operator satisfies the spectral mapping theorem. Also we show that if T is hyponrmal, then for any polynomial p on a neighborhood of $\sigma(T)$, Weyl's theorem holds for $p(T)$.

2. Basic Properties of Spectra

Let H be a Hilbert space and let $B(H)$ be the set of all bounded linear operators on H. Denote the kernel of T and the range of T by $\ker T(= N(T))$ and $R(T)$ respectively. Write $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$, $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda) \neq (0)\}$. Let $\pi_{0f}(T)$ be the set of eigenvalues of finite multiplicity, $\pi_{00}(T)$ the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity and $\rho(T) = \sigma(T)^c$ the resolvent of T.

An operator valued funtion $T(\lambda)$ which maps a subset of $\mathbb{C}$ into $B(H)$ is said to be *analytic* at λ_0 if $T(\lambda) = T_0 + (\lambda - \lambda_0)T_1 + \cdots$, where $T_k \in B(H)$ for each k and the series converges on each λ in some neighborhood of λ.

Lemma 2.1. ([3], [7], [9]) *The function $\rho(\lambda) = (T - \lambda)^{-1}$ is analytic on $\rho(T)$.*

Proof. Suppose $\lambda_0 \in \rho(T)$. Since $\rho(T)$ is open, choose $\varepsilon > 0$ such that $|\lambda - \lambda_0| < \varepsilon$, $\lambda \in \rho(T)$ and $\|(\lambda - \lambda_0)\rho(\lambda_0)\| < 1$. Also $T - \lambda = (T - \lambda_0) - (\lambda - \lambda_0) = (T - \lambda_0)(1 - (T - \lambda_0)^{-1}(\lambda - \lambda_0))$. We note that if $\|A\| < 1$, then $(1 - A)^{-1} = 1 + A + A^2 + A^3 + \cdots$. Since $\|(\lambda - \lambda_0)\rho(\lambda_0)\| < 1$, $1 - (\lambda - \lambda_0)(T - \lambda_0)^{-1}$ is invertible and so $(1 - (\lambda - \lambda_0)(T - \lambda_0)^{-1})^{-1} = \sum_{k=0}^{\infty}(\lambda - \lambda_0)^k(T - \lambda_0)^{-k}$. Thus

$$
\begin{aligned}
(T - \lambda)^{-1} &= (T - \lambda_o)^{-1}\sum_{k=0}^{\infty}(\lambda - \lambda_0)^k(T - \lambda_0)^{-k} \\
&= \sum_{k=0}^{\infty}(\lambda - \lambda_0)^k(T - \lambda_0)^{-(k+1)}.
\end{aligned}
$$

Hence ρ is analytic on $\rho(T)$. □

Theorem 2.2. ([3], [7], [9]) *For any operator $T \in B(H)$, $\sigma(T) \neq \phi$.*

Proof. Suppose $\sigma(T) = \phi$. Then $\rho(T) = \mathbb{C}$ and the function $\rho : \mathbb{C} \to B(H)$ defined by $\rho(\lambda) = (T - \lambda)^{-1}$ is analytic on $\mathbb{C}$. Also $\rho(\frac{1}{\lambda}) = (T - \frac{1}{\lambda})^{-1} = -\lambda(1 - \lambda T)^{-1} \to 0$ as $\lambda \to 0$, i.e., $\rho(\infty) = 0$. For all $x, y \in H$, $h(\lambda) = < (T - \lambda)^{-1}x, y >$ is anayltic on $\mathbb{C}$. Since $\rho(\infty) = 0$, h is bounded on $\mathbb{C}$. By Liouville's theorem, h is constant. Since $\rho(\infty) = 0$, $h(\lambda) \to 0$ as $|\lambda| \to \infty$. Thus h=0. Take $(T - \lambda)x$ and x in place of x and y. Then $h(\lambda) = < (T - \lambda)^{-1}(T - \lambda)x, x > = < x, x > > 0$. This is a contradiction. Hence $\sigma(T) \neq \phi$. □

Definition 2.3. ([3], [12]) $\sigma_p(T) = \{ \lambda \in \mathbb{C} : T - \lambda$ *is not injective*$\}$ *is called the* ***point spectrum*** *of T and $\sigma_{com}(T) = \{ \lambda \in \mathbb{C} : R(T - \lambda)$ is not dense in $H\}$ is called the* ***compression spectrum*** *of T.*

Lemma 2.4. ([3]) *For any operator $T \in B(H)$,*

(1) $\sigma_p(T) = \{ \lambda \in \mathbb{C} : T - \lambda$ *is a left divisor of zero in* $B(H)\}$.

(2) $\sigma_{com}(T) = \{ \lambda \in \mathbb{C} : T - \lambda$ *is a right divisor of zero in* $B(H)\}$.

Proof. (1) It suffices to show that T is a left divisor of zero in $B(H)$ iff T is not injective, by Theorem 6.1 (1).

($\Leftarrow$) Suppose T is not injective, i.e., $\ker T \neq (0)$. Then there exists $y(\neq 0) \in H$ such that Ty=0. Let f be a nonzero continuous linear form on H (by Hahn-Banach Theorem). Define $S \in B(H)$ by $Sx = f(x)y$ for all $x \in H$. Then $S \neq 0$ and $TSx = T(f(x)y) = f(x)Ty = 0$ for all $x \in H$. Hence $TS = 0$, i.e., T is a left divisor of zero in $B(H)$.

($\Rightarrow$) Suppose that there exists $S \in B(H)$, $S \neq 0$ such that $TS = 0$. Since $S \neq 0$, $A = \{x \in H | Sx \neq 0\} \neq (0)$. Since $TS = 0$, $TS(A) = 0$ and so

$S(A) \subset \ker T$, i.e., $\ker T \neq (0)$.

(2) It suffices to show that T is a right divisor of zero iff $T(H)$ is not dense in H by Theorem 6.1 (3).

($\Leftarrow$) Suppose that $T(H)$ is not dense in H, i.e., $\overline{T(H)} \neq H$. Then there exists $y \in H$ such that $y \notin \overline{T(H)}$. Let f be a continuous linear form on H such that $f(y) \neq 0$ and $f(\overline{T(H)}) = 0$ and let z be a nonzero vector in H. Define $S \in B(H)$ by $Sx = f(x)z$ for all $x \in H$. Then $STx = f(Tx)z = 0$ for all $x \in H$, but $Sy = f(y)z \neq 0$, i.e., $S \neq 0$. Thus T is a right divisor of zero.

($\Rightarrow$) suppose T is a right divisor of zero in $B(H)$. Then there exists $S \in B(H)$ such that $S \neq 0$ and $ST = 0$. If $\overline{T(H)} = H$, then for all $x \in H$, there exists a sequence $\{x_n\}$ in H such that $x = \lim T(x_n)$. Since $ST = 0$, $STx_n = 0$ for all n. Thus $0 = \lim_{n\to\infty} STx_n = S(\lim_{n\to\infty} Tx_n) = Sx$, i.e., $S = 0$. This is a contradiction, and so $\overline{T(H)} \neq H$. □

Theorem 2.5. *For any $T \in B(H)$, $\sigma_p(T) \subset \sigma(T)$ and $\sigma_{com}(T) \subset \sigma(T)$.*

Proof. If $\lambda \in \sigma_p(T)$, then $\ker(T - \lambda) \neq (0)$, i.e., $T - \lambda$ is not injective and so $T - \lambda$ is not invertible. Thus $\lambda \in \sigma(T)$.

Let $\lambda \in \sigma_{com}(T)$. Then $T - \lambda$ is a right divisor of zero in $B(H)$. Thus there exists $S \neq 0$ in $B(H)$ such that $S(T - \lambda) = 0$. If $T - \lambda$ is invertible, then $S = 0$. This is a contradiction. Thus $T - \lambda$ is not invertible and so $\lambda \in \sigma(T)$. □

Definition 2.6. ([3], [7], [9]) *$\lambda \in \mathbb{C}$ is said to be an approximate eigenvalue of T if there exists a sequence $\{x_n\}$ with $\|x_n\| = 1$ such that $Tx_n - \lambda x_n \to 0$, i.e., $(T - \lambda)x_n \to 0$. Let*

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} \ : \ \lambda \text{ is an approximate eigenvalue of } T\}.$$

Then $\sigma_{ap}(T)$ is called the approximate point spectrum of T.

Lemma 2.7. *Let $T \in B(H)$. Then the following conditions are equivalent.*

(1) *T is a left TDZ in $B(H)$.*

(2) *There exists a sequence $\{x_n\}$ in H with $\|x_n\| = 1$ such that $Tx_n \to 0$, i.e., $0 \in \sigma_p(T)$.*

(3) *T is not bounded below.*

Proof. (1) $\Rightarrow$ (2). Let S_n be a sequence in $B(H)$ such that $\|S_n\| = 1$ and $\|TS_n\| \to 0$. Since $\|S_n\| = 1$ for each n, we can choose a unit vector $y_n \in H$ such that $\|S_n y_n\| \geq \frac{1}{2}$. Put $x_n = \|S_n y_n\|^{-1} S_n y_n$. Then $\|x_n\| = 1$ and

$$\begin{aligned}\|Tx_n\| = \|T(\|S_n y_n\|^{-1} S_n y_n)\| = \|S_n y_n\|^{-1} \|T(S_n y_n)\| \leq 2\|TS_n y_n\| \\ \leq 2\|TS_n\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

(2) $\Rightarrow$ (1). Let $\{x_n\}$ be a sequence such that $\|x_n\| = 1$ and $Tx_n \to 0$. Let $f \in H'$ be a linear functional on H with $\|f\| = 1$. Define $S_n \in B(H)$ by $S_n x = f(x)x_n \ {}^{\forall}x \in H, {}^{\forall}n \in N$. Then

$$\|S_n\| = \sup_{\|x\|=1} \|S_n x\| = \sup_{\|x\|=1} \|f(x)x_n\| = \sup_{\|x\|=1} \|f(x)\| = 1,$$

$TS_n x = f(x)Tx_n$ $(x \in H)$ and

$$\begin{aligned}\|TS_n\| &= \sup_{\|x\|=1} \|f(x)Tx_n\| = \sup_{\|x\|=1} |f(x)|\|Tx_n\| \\ &= \|Tx_n\| \sup_{\|x\|=1} |f(x)| = \|Tx_n\|\|f\| \to 0\end{aligned}$$

as $n \to \infty$. Since $Tx_n \to 0$, $\|TS_n\| \to 0$. Thus T is a left TDZ in $B(H)$.

(2) $\Rightarrow$ (3). If T is bounded below, then there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in H$. Since $\|x_n\| = 1$ and $\|Tx_n\| \geq c$ for all x_n, $Tx_n \not\to 0$. Thus T is bounded below.

(3) $\Rightarrow$ (2). We can choose $x'_n \in H$ such that $\|Tx_n\| < \frac{1}{n}\|x'_n\|$. Thus $\frac{\|Tx'_n\|}{\|x'_n\|} < \frac{1}{n}$ and so $\|T(\frac{x'_n}{\|x'_n\|})\| < \frac{1}{n}$. Put $\frac{x'_n}{\|x'_n\|} = x_n$. Then $\|x_n\| = 1$ and $\|Tx_n\| \to 0$, i.e., $Tx_n \to 0$ as $n \to \infty$. □

Theorem 2.8. ([3], [7]) *Let T be any operator in $B(H)$. Then*

(1) $\sigma_p(T) \subset \sigma_{ap}(T)$ *and*

(2) $\sigma_{ap}(T) \subset \sigma(T)$.

Proof. (1) If $\lambda \notin \sigma_{ap}(T)$, then $T - \lambda$ is bounded below. Thus there exists $c > 0$ such that $\|(T - \lambda)x\| \geq c\|x\|$ for all $x \neq 0$. If $x \in \ker(T - \lambda)$, then $0 = \|(T - \lambda)x\| \geq c\|x\| \geq 0$ and so $\|x\| = 0$. Thus $\ker(T - \lambda) = (0)$ and so $\lambda \notin \sigma_p(T)$.

(2) If $\lambda \notin \sigma(T)$, then $T - \lambda$ is invertible. Thus $\ker(T - \lambda) = (0)$ and so for all $x \in H\,(x \neq 0)$, $\|(T - \lambda)x\| > 0$. So there doesn't exist $\{x_n\}$ in H with $\|x_n\| = 1$ such that $(T - \lambda)x_n \to 0$. Hence we have that $\lambda \notin \sigma_{ap}(T)$, i.e., $\sigma_{ap}(T) \subset \sigma(T)$. □

Theorem 2.9. ([3]) *Let $T \in B(H)$ be any operator. The followings are equivalent:*

(1) *T is singular.*

(2) *T is either a right divisor of zero or a left TDZ in $B(H)$.*

Proof. (1) $\Rightarrow$ (2). Suppose that T is neither a right divisor of zero nor a left TDZ in $B(H)$, then $T(H)$ is dense in H by Lemma 2.4 and T is bounded below by Lemma 2.7. Since T is bounded below, there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in H$. If $T(x) = T(y)$, then $0 = \|Tx - Ty\| = \|T(x - y)\| \geq c\|x - y\|$. Thus $x = y$. If $y \in \overline{T(H)}$, then there exists a sequence $\{x_n\}$ in H such that $y = \lim_{n\to\infty} Tx_n$. Since $\{Tx_n\}$ converges, $\{Tx_n\}$ is a Cauchy sequence in H. Since $c\|x_n - x_m\| \geq \|T(x_n - x_m)\| = \|Tx_n - Tx_m\| \to 0$, $= \|x_n - x_m\| \to 0$ and so $\{x_n\}$ is a Cauchy sequence in H. Since H is complete, $\{x_n\}$ converges. Put $\lim_{n\to\infty} x_n = x$. Then $y = \lim T(x_n) = T(\lim x_n) = T(x)$, i.e., $y \in T(H)$. Thus $\overline{T(H)} = T(H)$. Since $T(H)$ is dense in H, $\overline{T(H)} = H$. Thus $T(H) = H$, i.e., T is onto. Since T is one-one and onto, there exists $T^{-1}x$ for all $x \in H$ and so $\|T(T^{-1}x)\| \geq c\|T^{-1}x\| \Rightarrow \|x\| \geq c\|T^{-1}x\|$. Thus $\|T^{-1}x\| \leq \frac{1}{c}\|x\|$ for all $x \in H$ and so $\|T^{-1}\| \leq \frac{1}{c}$. Thus T^{-1} is invertible. This is a contradiciton to (1). Hence (2) holds.

(2) $\Rightarrow$ (1). If T is a right divisor of zero, then there exists $S \neq 0$ such that $TS = 0$, i.e., $TSx = 0$ for all $x \in H$. Since $S \neq 0$, there exists a nonzero vector $x \in H$ such that $Sx \neq 0$. Thus $\ker T \neq (0)$ and so T is not singualr. If T is a left TDZ in $B(H)$, by Lemma 2.7, there exists $\{x_n\}$ in H such that $\|x_n\| = 1$ and $Tx_n \to 0$, i.e., $0 \in \sigma_{ap}(T)$. Since $\sigma_{ap}(T) \subset \sigma(T)$, $0 \in \sigma(T)$. Thus T is not invertible. □

From Theorem 2.9, we can know that $\sigma(A) = \sigma_{ap}(A) \cup \sigma_{com}(A)$.

Lemma 2.10. *Let $A \in B(H)$ be any operator. Then $\sigma_{ap}(A)$ is closed.*

Proof. Let $\lambda_0 \in \sigma_{ap}(A)^c$. Then $A - \lambda_o$ is bounded below, i.e., there exists $c > 0$ such that $\|(A - \lambda_0)x\| \geq c\|x\|$. Since $\|Ax - \lambda_0 x\| = \|Ax - \lambda_0 x + \lambda x - \lambda x\| \leq \|Ax - \lambda x\| + \|\lambda x - \lambda_0 x\|$, $\|Ax - \lambda_0 x\| - \|\lambda x - \lambda_0 x\| \leq \|Ax - \lambda x\|$. Thus $c\|x\| - |\lambda - \lambda_0|\|x\| \leq \|(A - \lambda_0)x\| - |\lambda - \lambda_0|\|x\| \leq \|Ax - \lambda x\|$ for all x. Choose $\delta > 0$ such that $c - \delta > 0$ and $c - |\lambda - \lambda_0| > c - \delta > 0$ for all λ with $|\lambda - \lambda_0| < \delta$. So $(c - \delta)\|x\| < (c - |\lambda - \lambda_0|)\|x\| < \|(A - \lambda)x\|$ for all $x \in H$.

Thus $A-\lambda$ is bounded below and so $\lambda \notin \sigma_{ap}(A)$. Hence $\sigma_{ap}^c(A)$ is open and thus $\sigma_{ap}(A)$ is closed. $\square$

Theorem 2.11. ([7]) *$\partial\sigma(T) \subset \sigma_{ap}(T)$ for any $T \in B(H)$.*

Proof. First we will show that if $\{A_n\}$ is a sequence of invertible operators and $\|A_n - A\| \rightarrow 0$ where A is not invertible, then $0 \in \sigma_{ap}(A)$. Since A is not invertible, then $0 \in \sigma(A) = \sigma_{ap}(A) \cup \sigma_{com}(A)$, i.e., $0 \in \sigma_{ap}(A)$ or $0 \in \sigma_{com}(A)$. If $0 \in \sigma_{ap}(A)$, then we are done. If $0 \in \sigma_{com}(A)$, then $\overline{R(A)} \neq H$ and so there exists $x \neq 0$ such that $x \perp R(A)$. Put $x_n = \frac{A_n{}^{-1}x}{\|A_n{}^{-1}x\|}$. Then $\|x_n\| = 1$, $A_n x_n = \frac{x}{\|A_n{}^{-1}x\|}$ and $A_n x_n \perp R(A)$. Since $\|(A_n - A)x_n\| \leq \|A_n - A\| \rightarrow 0$, $\|(A_n - A)x_n\| \rightarrow 0$. Since $Ax_n \in R(A)$ and $A_n x_n \perp R(A)$, $< Ax_n, A_n x_n >= 0$. Thus $\|Ax_n\|^2 \leq \|A_n x_n\|^2 + \|Ax_n\|^2 = \|A_n x_n - Ax_n\|^2 \rightarrow 0$, i.e., $\|Ax_n\| \rightarrow 0$. Therefore A is not bounded below, i.e., $0 \in \sigma_{ap}(A)$. Now let $\lambda \in \partial\sigma(T)$. Since $\sigma(T)$ is closed, $\lambda \in \sigma(T)$, i.e., $T - \lambda$ is not invertible. Since $\lambda \in \partial\sigma(T)$, there exists $\{\lambda_n\}$ with $\lambda_n \notin \sigma(T)$ such that $\lambda_n \rightarrow \lambda$. Thus $T - \lambda_n$ is invertible for all n. Since $\|(T - \lambda_n) - (T - \lambda)\| = |\lambda_n - \lambda| \rightarrow 0$, by the above argument, $0 \in \sigma_{ap}(T - \lambda)$, i.e., $T - \lambda$ is not bounded below. Hence $\lambda \in \sigma_{ap}(T)$ and so $\partial\sigma(T) \subset \sigma_{ap}(T)$. $\square$

Let $T \in B(H)$ be any operator. Since $\sigma(T)$ is closed, $\partial\sigma(T) \neq \emptyset$.

Theorem 2.12. ([7]) *Let $T \in B(H)$ be any operator. The following conditions are equivalent :*

(1) $\lambda \notin \sigma_{ap}(T)$.

(2) *$R(T - \lambda)$ is closed and* $\dim\ker(T - \lambda) = 0$.

(3) *$\lambda \notin \sigma_l(T)$, the left spectrum of T.*

(4) *$\bar{\lambda} \notin \sigma_r(T^*)$, the right spectrum of T^*.*

(5) $R(T^* - \bar{\lambda}) = H$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\lambda \notin \sigma_{ap}(T)$, i.e., $T - \lambda$ is bounded below. Then there exists $c > 0$ such that $\|(T-\lambda)x\| \geq c\|x\|$ for all $x \in H$. Let $y \in \overline{R(T-\lambda)}$. Then $y = \lim_{n\to\infty}(T-\lambda)x_n$ where $x_n \in H$. Since $c\|x_n - x_m\| \geq \|(T-\lambda)(x_n - x_m)\| = \|(T-\lambda)x_n - (T-\lambda)x_m\| \to 0$, $\{x_n\}$ is a Cauchy sequence in H and hence $\{x_n\}$ converges. Let $\lim x_n = x$. Then $y = \lim(T-\lambda)x_n = (T-\lambda)(\lim x_n) = (T-\lambda)x$, i.e., $y \in R(T-\lambda)$. Thus $R(T-\lambda)$ is closed. If $(T-\lambda)x = (T-\lambda)y$, then $(T-\lambda)(x-y) = 0$ and so $0 = \|(T-\lambda)(x-y)\| \geq c\|x-y\|$. Therefore $x - y = 0$, i.e., $x = y$ and so $\ker(T-\lambda) = (0)$.

(2) $\Rightarrow$ (1). $T - \lambda : H \to R(T-\lambda)$ is a continuous bijection since $\ker(T-\lambda) = (0)$. By the inverse mapping theorem, there is a bounded operator $S : R(T-\lambda) \to H$ such that $S(T-\lambda)x = x$ for all $x \in H$. Thus if $\|x\| = 1$, then $1 = \|S(T-\lambda)x\| \leq \|S\|\|(T-\lambda)x\|$. That is, $\|(T-\lambda)x\| \geq \|S\|^{-1}$ whenever $\|x\| = 1$. Hence $\lambda \notin \sigma_{ap}(T)$.

(2) $\Rightarrow$ (3). Let $M = R(T-\lambda)$. Define $S : H \to M$ by $Sh = (T-\lambda)h$. Since $\ker(T-\lambda) = (0)$, S is one-one and clearly S is onto. Thus there exists $S^{-1} : M \to H$. Since M is closed in H, $H = M \oplus M^\perp$. Define $B : M \oplus M^\perp \to H$ by $B|_M = S^{-1}$ and $B(M^\perp) = (0)$. Then for all $h \in H$, $B(T-\lambda)h = BSh = S^{-1}(Sh) = h$. Thus $B(T-\lambda) = I$, i.e., $T - \lambda$ is left invertible. Hence $\lambda \notin \sigma_l(T)$.

(3) $\Leftrightarrow$ (4). $S(T-\lambda) = I$ iff $(T^* - \bar{\lambda})S^* = I$.

(4) $\Rightarrow$ (5). Assume that $\bar{\lambda} \notin \sigma_r(T^*)$. Then $(T^* - \bar{\lambda})$ is right invertible and so there exists C in $B(H)$ such that $(T^* - \bar{\lambda})C = I$. Thus $H = ((T^* - \bar{\lambda})C)(H) \subset R(T^* - \bar{\lambda}) \subset H$. Thus $R(T^* - \bar{\lambda}) = H$.

(5) $\Rightarrow$ (1). Let $\ker(T^* - \bar{\lambda})^{\perp} = N$. Define $S : N \to H$ by $Sh = (T^* - \bar{\lambda})h$. If $Sh = 0$, then $(T^* - \bar{\lambda})h = 0$. Thus $h \in \ker(T^* - \bar{\lambda})$. But since $h \in \ker(T^* - \bar{\lambda})^{\perp}$, $h = 0$. So $\ker S = (0)$. For all $h \in H = R(T^* - \bar{\lambda})$, there exists $x \in H$ such that $(T^* - \bar{\lambda})x = h$. For if $h = 0$, take $x = 0 \in N$ and if $h \neq 0$, take $x \notin \ker(T^* - \bar{\lambda})$. Then $x \in N$. Thus S is onto and so S is invertible. Define $C : H \to H$ by $Ch = S^{-1}h$. Then $C(H) = N$ and $(T^* - \bar{\lambda})C(h) = (T^* - \bar{\lambda})(S^{-1}h) = S(S^{-1}h) = h$. Thus $C^*(T - \lambda) = I$. Also $1 = \| C^*(T - \lambda)\| \leq \|C^*\|\|T - \lambda\|$, i.e., $\|C^*\|^{-1} \leq \|T - \lambda\|$, i.e., $T - \lambda$ is bounded below. Hence $\lambda \notin \sigma_{ap}(T)$. □

From Theorem 2.12, we can know $\sigma_{ap}(T) = \sigma_l(T) = \sigma_r(T^*)^*$.

Lemma 2.13. *$\partial\sigma(T) \subset \sigma_l(T) \cap \sigma_r(T)$ for any $T \in B(H)$.*

Proof. If $\lambda \in \partial\sigma(T)$, then by Theorem 2.11, $\lambda \in \sigma_{ap}(T)$. By Theorem 2.12 $\lambda \in \sigma_l(T)$. Since $\bar{\lambda} \in \partial\sigma(T^*)$, $\bar{\lambda} \in \sigma_{ap}(T^*)$ and so $\bar{\lambda} \in \sigma_l(T^*) = \sigma_r(T)^*$. Since $\sigma_r(T^*) = \sigma_r(T)^*$, $\lambda \in \sigma_r(T)$. Thus $\lambda \in \sigma_l(T) \cap \sigma_r(T)$. □

3. Fredholm Operators

Let H be an infinite-dimensional Hilbert space. If T is an operator, we write $\ker T$ and $R(T)$ for the null space and range of T respectively. We note that $R(T)^{\perp} = \ker T^*$ for any operator $T \in B(H)$.

Definition 3.1. ([7], [8]) *An operator T is called a Fredholm operator if $N(T) = \ker T$ is finite-dimensional, $R(T)$ is closed and $\ker T^* = R(T)^{\perp}$ is finite dimensional. The Fredholm spectrum of T, denoted by $\sigma_{\mathcal{F}}(T)$, is the set $\sigma_{\mathcal{F}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda$ is not Fredholm operator $\}$.*

We write $\mathcal{F}$ and $\mathcal{K}$ for the class of all Fredholm operators and compact operators respectively. We note that if $T \in B(H)$ and $K \in \mathcal{K}$, then $TK \in \mathcal{K}$ and $KT \in \mathcal{K}$, i.e., $\mathcal{K}$ is an ideal in $B(H)$. $B(H)/\mathcal{K}$ is called the Calkin algebra of H. Let π denote the natural map from $B(H)$ onto $B(H)/\mathcal{K}$. We note that the function $\pi : B(H) \to B(H)/\mathcal{K}$ is continuous.

Thoerem 3.2. *(Atkinson's theorem [12]) Let H be a Hilbert space and let $T \in B(H)$. The following conditions on T are equivalent:*

(1) *An operator T is a Fredholm operator.*

(2) *T is invertible modulo the ideal of operators of finite rank.*

(3) *T is invertible modulo the ideal of compact operators.*

From Theorem 3.2, $\sigma_{\mathcal{F}}(T) = \sigma(\hat{T})$.

Definition 3.3. ([7], [8]) *For all $T \in \mathcal{F}$, the index of T, denoted by $i(T)$, is defined by $i(T) = \dim \ker T - \dim R(T)^{\perp}$.*

Since $R(T)^{\perp} = \ker T^*$ for any $T \in B(H)$, $i(T) = \dim \ker T - \dim \ker T^*$.

For examples, if S_r is the unilateral shift on l_2, then $i(S_r) = -1$.

Lemma 3.4.

(1) If $T \in B(H)$ is normal, i.e., $TT^* = T^*T$, then $i(T) = 0$.

(2) If $T \in B(H)$ is hyponormal, i.e., $TT^* \leq T^*T$, then $i(T) \leq 0$.

Proof. (1) If T is normal, then $\|Tx\| = \|T^*x\|$ for all $x \in H$ and so $\ker T = \ker T^*$. Since $\ker T^* = R(T)^{\perp}$, $i(T) = 0$.

(2) If T is hyponormal, then $\|T^*x\| \leq \|Tx\|$ for all $x \in H$ and so $\ker T \subset \ker T^*$. Thus $\dim \ker T \leq \dim \ker T^*$ and so $i(T) \leq 0$. □

Definition 3.5. ([7], [8]) *Let $T \in B(H)$ be an operator. T is a left Fredholm operator if $\pi(T) = \hat{T}$ is left invertible in $B(H)/\mathcal{K}$, and T is a right Fredholm operator if $\pi(T)$ is right invertible in $B(H)/\mathcal{K}$. Let $\mathcal{F}_l$, $\mathcal{F}_r$ denote the set of all left Fredholm, right Fredholm operators respectively. Clearly $\mathcal{F} = \mathcal{F}_l \cap \mathcal{F}_r$. Operators in the set $S\mathcal{F} = \mathcal{F}_l \cup \mathcal{F}_r$ are called semi-Fredholm operators.*

Definition 3.6. ([7], [8]) *If $T \in B(H)$, then the essential spectrum of T is the spectrum of $\pi(T) = \hat{T}$ in $B(H)/\mathcal{K}$, denoted by $\sigma_e(T)$. Similarly the left and right essential spectrum of T are defined by $\sigma_e^l(T) = \sigma^l(\pi(T))$ and $\sigma_e^r(T) = \sigma^r(\pi(T))$.*

It is obvious from Theorem 3.2 that $\sigma_e(T) = \sigma(\pi(T)) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{F}\} = \sigma_{\mathcal{F}}(T), \sigma_e^l(T) = \sigma^l(\pi(T)) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{F}_l\}$ and $\sigma_e^r(T) = \sigma^r(\pi(T)) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{F}_r\}$.

Lemma 3.7. ([7]) *If T and S are commuting operators and $TS \in \mathcal{F}$, then $T \in \mathcal{F}$ and $S \in \mathcal{F}$.*

Proof. Since $\ker T \cup \ker S \subset \ker TS$ ($\because$ $h \in \ker T \cup \ker S \rightarrow Th = 0$

or $Sh = 0$. If $Th = 0$, then $(TS)h = (ST)h = 0$ and if $Sh = 0$, then $(TS)h = 0$. Thus $h \in \ker TS$.), $\dim\ker T \leq \dim\ker TS < \infty$ and $\dim\ker S \leq \dim\ker TS < \infty$. Similarly since $\ker T^* \cup \ker S^* \subset \ker T^*S^* = \ker(ST)^* = \ker(TS)^*$, $\dim\ker T^* \leq \dim\ker(TS)^* < \infty$ and $\dim\ker S^* \leq \dim\ker(TS)^* < \infty$. Thus $\dim\ker T < \infty$, $\dim\ker T^* < \infty$, $\dim\ker S < \infty$ and $\dim\ker S^* < \infty$. If $R(T)$ is not closed, then there exists $z \in H$ such that $z = \lim z_n$, $z_n \in R(T)$, but $z \notin R(T)$. Since $z_n \in R(T)$, $Sz_n \in S(R(T)) = R(ST)$, i.e., $Sz_n \in R(ST)$. Since S is continuous, $Sz = S(\lim z_n) = \lim Sz_n$ and $Sz_n \in R(ST)$. Since $R(ST)$ is closed, $Sz \in R(ST) = S(R(T))$, i.e., $z \in R(T)$. This is a contradiction. Thus $R(T)$ is closed. Similarly $R(S)$ is closed. Hence $T, S \in \mathcal{F}$. □

We note that the function $\pi : B(H) \to B(H)/\mathcal{K}$ is continuous. Let Δ be the set of invertible operator in $B(H)/\mathcal{K}$. Since Δ is open, $\pi^{-1}\Delta$ is open. Thus $\mathcal{F}$ is open in $B(H)$. If $R(A)$ is closed, then $R(A^*) = R(A)^*$ is also closed ([7]) and $i(A) = -i(A^*)$. Hence if $A \in \mathcal{F}$, then $A^* \in \mathcal{F}$. Also if $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $AB \in \mathcal{F}$.

Theorem 3.8. ([8]) *If H is a Hilbert space, then $\mathcal{F}$ is an open subset of $B(H)$, which is self adjoint, closed under multiplication and invariant under compact perturbations*

Proof. If Δ denote a group of invertible elements in $B(H)/\mathcal{K}$, then Δ is open and hence $\mathcal{F} = \pi^{-1}(\Delta)$ is open since the natural homomorphism $\pi : B(H) \to B(H)/\mathcal{K}$ is continuous and onto. Since π is multiplicative and Δ is a group, $\mathcal{F}$ is closed under multiplication($\because$ $S, T \in \mathcal{F} \Rightarrow \pi(S), \pi(T)$; invertible $\Rightarrow \pi(S)\pi(T) = \pi(ST) =$ invertible $\Rightarrow ST \in \mathcal{F}$). Further if $T \in \mathcal{F}$ and K is compact, then by Theorem 3.2, $T+K$ is in $\mathcal{F}$ since $\pi(T) = \pi(T+K)$. Finally if T is in $\mathcal{F}$, then there exist S in $B(H)$ and compact operators K_1 and K_2 such that $ST = I + K_1$ and $TS = I + K_2$. Taking adjoint, $T^*S^* = I + K_1^*$ and $S^*T^* = I + K_2^*$ and so $\pi(T^*)$ is invertible in the Calkin algebra. Hence $\mathcal{F}$ is self-adjoint. □

Theorem 3.9. ([8]) *If H is a Hilbert space, then each of the sets $\mathcal{F}_n$ is open in $B(H)$. Thus $\bigcup_{n\neq 0}\mathcal{F}_n$ is open in $B(H)$, where $\mathcal{F}_n = \{\, T : T \in \mathcal{F},\ i(T) \neq n \,\}$.*

Proof. If T is a Fredholm operator not in $\mathcal{F}_0$, then there exists a finite rank operator F such that $T+F$ is either left or right invertible(Lemma 5.20 [8]). By Proposition 2.7([8]), there exists $\varepsilon > 0$ such that if S is an operator in $B(H)$ such that $\|T - (S - F)\| = \|T + F - S\| < \varepsilon$, then S is either left or right invertible but not invertible. Thus S is a Fredholm operator of index not equal to 0 and therefore so is $S - F$. Hence $\cup_{n\neq 0}\mathcal{F}_n$ is also an open subset of $B(H)$. □

Theorem 3.10. ([6],[7],[17])

(1) *If $A \in \mathcal{F}$, then there exists $\delta > 0$ such that $\|B - A\| < \delta \rightarrow i(B) = i(A)$ and $B \in \mathcal{F}$. Furthermore, for all $A \in \mathcal{SF}$, there exists $\delta > 0$ such that $\|B - A\| < \delta \rightarrow B \in \mathcal{SF}$ and $i(B) = i(A)$. Thus $\mathcal{SF}$ is open in $B(H)$.*

(2) *(The index product theorem) If A, $B \in \mathcal{F}$, then $i(AB) = i(A)+i(B)$.*

4. Weyl Operator and Weyl Spectrum

Definition 4.1. ([2]) *$T \in B(H)$ is called a Weyl operator if T is Fredholm and $i(T) = 0$. The Weyl spectrum $w(T)$ of T is the set*

$$w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Weyl operator}\}.$$

Let $\mathcal{I}$, $\mathcal{F}_0$ and $\mathcal{F}$ be the classes of invertible, Weyl and Fredholm operators respectively. Then since $\mathcal{I} \subset \mathcal{F}_0 \subset \mathcal{F}$, $\sigma_{\mathcal{F}}(T) \subset w(T) \subset \sigma(T)$.

Remark 4.2. *The concept of a Weyl spectrum is relevant only for infinite-dimensional space. Indeed, when $\dim(H) < \infty$, $R(T)$ is finite dimensional in Hilbert space and so $R(T)$ is closed. Clearly, $\dim \ker T < \infty$ and $\dim R(T)^{\perp} < \infty$. Hence all operators are Weyl operators.*

Lemma 4.3. ([2]) *Let T be a Weyl operator. Then T is invertible iff T is one-to-one iff T is onto.*

Proof. Since T is a Weyl operator, $\dim \ker T = \dim R(T)^{\perp}$. Hence the proof is complete by the fact that T is injective iff $0 = \dim \ker T = \dim R(T)^{\perp}$ iff $R(T)^{\perp} = (0)$, i.e., $R(T) = H$. □

By the Fredholm theory of compact operator ([10]), we have the following result:

Lemma 4.4. ([9]) *If K is any copmact operator, then $I - K$ is a Weyl operator.*

Lemma 4.5. ([2]) *If S is invertible and K is compact, then $S + K$ is Weyl.*

Proof. Note that $S + K = S(I + S^{-1}K)$. Since K is compact, $S^{-1}K$ is also compact. By Lemma 4.4, $I + S^{-1}K$ is Weyl. Since S is invertible,

$\dim\ker(S(I+S^{-1}K)) = \dim\ker(I+S^{-1}K) < \infty$ and $\dim[R(S(I+S^{-1}K))]^{\perp} = \dim[R(I+S^{-1}K)]^{\perp} < \infty$ and $R(S(I+S^{-1}K)) = R(I+S^{-1}K)$ is closed. Hence $S+K$ is Weyl. □

Theorem 4.6. ([2]) *If $T \in B(H)$ is Weyl, then there exists an operator K of finite rank such that $T+K$ is invertible.*

Proof. Assume that T is Weyl. By hypothesis, $i(T) = 0$ and $\dim\ker(T) = \dim\ker(T^*) = \dim R(T)^{\perp} < \infty$. Since $H = (\ker T)^{\perp} \oplus \ker T = R(T)^{\perp} \oplus R(T)$, there exists an invertible operator $F_0 : \ker T \to R(T)^{\perp} = \ker T^*$. Define $F = F_0(I-P)$ where P is the projection of H onto $(\ker T)^{\perp}$. Then F is of finite rank since $\dim\ker T^* < \infty$. We show that $T+F$ is invertible. First we show that $T+F$ is injective, i.e., $(T+F)x = 0$ $(x \in H)$ implies $x = 0$.

Case 1. If $x \in \ker T$, then $0 = (T+F)x = Fx$. Since $Fx = F_0(I-P)x = F_0(x-Px) = F_0x$, and F_0 is injective, $x = 0$.

Case 2. If $x \in (\ker T)^{\perp}$, then $Fx = F_0(I-P)x = F_0(x-Px) = F_0(x-x) = 0$. Hence $0 = (T+F)x = Tx$, i.e., $x \in \ker T$. Since $\ker T \cap (\ker T)^{\perp} = (0)$, $x = 0$.

From case 1 and 2, $T+F$ is injective.

Secondly, we show that $T+F$ is onto. If $x \in H$, then $x = u+v$ where $u \in R(T)$ and $v \in R(T)^{\perp}$ since $H = R(T) \oplus R(T)^{\perp}$. So $u = Tp$ for some $p \in (\ker T)^{\perp}$ and $v = F_0q$ for some $q \in \ker T$ since F_0 is one-to-one and onto. Thus $x = u+v = Tp+F_0q$. Put $h = p+q \in (\ker T)^{\perp} \oplus \ker T = H$. Then $Fq = F_0(I-P)q = F_0q$, $\quad Fp = F_0(I-P)p = F_0(p-p) = 0$ and so $Fh = Fp+Fq = Fq = F_0q$. Thus $x = Tp+F_0q = Th+Fh = (T+F)h$ and hence $T+F$ is onto. □

Corollary 4.7. ([2]) *The following conditions on an operator T are equivalent:*

(1) $T =$ *Weyl.*

(2) $T = S + F$, *with S invertible and F of finite rank.*

(3) $T = S + K$, *with S invertible and K compact.*

Proof. (1) $\Rightarrow$ (2). If T is Weyl, then by the above theorem there exists K of finite rank such that $T + K$ is invertible. Thus $T = (T + K) - K = (T + K) + (-K)$.

(2) $\Rightarrow$ (3). Any operator of finite rank is compact.

(3) $\Rightarrow$ (1) . It follows from Lemma 4.5. □

Lemma 4.8. *If T is any operator and K is a compact operator, then $w(T) \subset \sigma(T + K)$.*

Proof. If $\lambda \notin \sigma(T + K)$, then $(T + K) - \lambda$ is invertible and so $T - \lambda = ((T + K) - \lambda) - K$ is a Weyl operator. Thus $\lambda \notin w(T)$. □

Theorem 4.9. ([2], [7]) *For all $T \in B(H)$, $w(T) = \bigcap_{K \in \mathcal{K}} \sigma(T + K)$.*

Proof. By Lemma 4.8, $w(T) \subset \sigma(T + K)$ for all $K \in \mathcal{K}$. Thus $w(T) \subset \bigcap_{K \in \mathcal{K}} \sigma(T + K)$. If $\lambda \notin w(T)$, then $T - \lambda$ is a Weyl operator. By Corollary 4.7, there exists a finite rank operator K such that $T - \lambda + K$ is invertible, i.e., $(T + K) - \lambda$ is invertible. Thus $\lambda \notin \sigma(T + K)$. Therefore $\lambda \notin \bigcap_{K \in \mathcal{K}} \sigma(T + K)$ and so $\bigcap_{K \in \mathcal{K}} \sigma(T + K) \subset w(T)$. Hence $w(T) = \cap_{K \in \mathcal{K}} \sigma(T + K)$. □

Theorem 4.10. ([2]) *For any operator $T \in B(H)$, $w(T)$ is a nonempty compact subset of $\sigma(T)$.*

Proof. From Theorem 4.9, $w(T) = \bigcap_{K \in \mathcal{K}} \sigma(T + K)$. Since $\sigma(T)$ is compact and $w(T) \subset \sigma(T)$, $w(T)$ is bounded. Since $\sigma(T+K)$ is bounded and closed for any compact K, $\bigcap_{K\in\mathcal{K}} \sigma(T+K)$ is also closed. Hence $w(T)$ is compact. □

Corollary 4.11. *If T is any operator, then $w(T+K) = w(T)$ for all compact operator K.*

Proof. The proof is easy by the fact that $w(T+K) = \bigcap_{K'\in\mathcal{K}} \sigma(T+K+K') = \bigcap_{K\in\mathcal{K}} \sigma(T+K) = w(T)$. □

Theorem 4.12. *For any operator $T \in B(H)$, $w(T^*) = w(T)^*$.*

Proof. If $\lambda \notin w(T^*)$, then $T^* - \lambda$ is a Weyl operator. By Corollary 4.7, $T^* - \lambda = S + K$, where S is invertible and K is a compact operator. Thus $(T - \bar{\lambda})^* = S + K$ and so $T - \bar{\lambda} = (S+K)^* = S^* + K^*$. Since S^* is invertible and K^* is a compact operator, $T - \bar{\lambda}$ is a Weyl operator. Hence $\bar{\lambda} \notin w(T)$ and so $w(T)^* \subset w(T^*)$.

Similarly, we obtain $w(T^*) \subset w(T)^*$. □

Theorem 4.13. ([2]) *For any $T \in B(H)$, $\sigma(T) - w(T) \subset \pi_{0f}(T)$ or equivalently $\sigma_{(}T) - \pi_{0f}(T) \subset w(T)$.*

Proof. If $\lambda \in \sigma(T) - w(T)$, then $T - \lambda$ is not invertible and $T - \lambda$ is a Weyl operator. Since $T - \lambda$ is a Weyl operator, by Lemma 4.3, $T - \lambda$ is not one-to-one, i.e., $\ker(T-\lambda) \neq (0)$. So $0 < \dim\ker(T-\lambda) < \infty$. Thus $\lambda \in \pi_{0f}(T)$.

Equivalently, if $\lambda \in \sigma(T) - \pi_{0f}(T)$, then $T - \lambda$ is not invertible and $\dim\ker(T-\lambda) = \infty$. Thus $T - \lambda$ is not a Weyl operator. Hence we have $\lambda \in w(T)$. □

Corollary 4.14. *For all $T \in B(H)$ and for any compact operator K, $\sigma(T) - \pi_{0f}(T) \subset \sigma(T+K)$.*

Since $\mathcal{I} \subset \mathcal{F}_0 \subset \mathcal{F}$, $\sigma_{\mathcal{F}} \subset w(T) \subset \sigma(T)$.

Lemma 4.15. *Let $T \in B(H)$ be any operator. Then*

(1) *(Schechter) $w(T) = \sigma_{\mathcal{F}}(T) \cup \{\lambda : T - \lambda \in \mathcal{F}$ and $i(T-\lambda) \neq 0\}$.*

(2) *$\{\lambda : T - \lambda \in \mathcal{F},\ i(T-\lambda) \neq 0\}$ is open in $B(H)$.*

Proof. (1) See Theorem 10.8([6]).

(2) Let $\lambda \in \theta(T) = \{\ \lambda\ :\ T - \lambda \in \mathcal{F}$ and $i(T-\lambda) \neq 0\ \}$. Since $T - \lambda \in \mathcal{F}$, there exists $\varepsilon > 0$ such that

$$\|(T-\lambda) - S\| < \varepsilon \Rightarrow S \in \mathcal{F} \ \text{ and } \ i(T-\lambda) = i(S). \tag{4.1}$$

For all $\mu \in B(\lambda\ ;\ \varepsilon)$, $\|(T-\lambda)-(T-\mu)\| = |\lambda - \mu| < \varepsilon$. By (4.1), $T - \mu \in \mathcal{F}$ and $i(T-\lambda) = i(T-\mu) \neq 0$. Hence $\mu \in \theta(T)$ and so $\theta(T)$ is open. □

Theorem 4.16. *If T is normal, then $w(T) = \sigma_{\mathcal{F}}(T)$.*

Proof. If T is normal, then $T - \lambda$ is also normal for all $\lambda \in \mathbb{C}$. Since $\|(T-\lambda)x\| = \|(T-\lambda)^* x\|$ for all $x \in H$, $\ker(T-\lambda) = \ker(T-\lambda)^*$. Thus $i(T-\lambda) = \dim\ker(T-\lambda) - \dim\ker(T-\lambda)^* = 0$ and so $\{\lambda \in \mathbb{C} : T - \lambda \in \mathcal{F}$ and $i(T-\lambda) \neq 0\} = \emptyset$. Therefore $w(T) = \sigma_{\mathcal{F}}(T)$. □

Theorem 4.17. *If S is invertible, then $w(S^{-1}TS) = w(T)$ and $i(S^{-1}TS) = i(T)$. Thus the Weyl spectrum and index are invariant under similarity.*

Proof. ($\subset$) Let $\lambda \notin w(T)$. Then $T - \lambda$ is Weyl. Thus by Corollary 4.7, there exists A invertible and B compact such that $T - \lambda = A + B$. Since S is invertible, $S^{-1}(T-\lambda)S = S^{-1}(A+B)S$, i.e., $S^{-1}TS - \lambda = S^{-1}AS +$

$S^{-1}BS$ and since $S^{-1}AS$ is invertible and $S^{-1}BS$ is compact by Corollary 4.7, $S^{-1}TS-\lambda$ is Weyl, i.e., $\lambda \notin w(S^{-1}TS)$. Hence $w(S^{-1}TS) \subset w(T)$.

($\supset$) Let $\lambda \notin w(S^{-1}TS)$. Then $S^{-1}TS-\lambda$ is Weyl. Thus by Corollary 4.7, there exists A invertible and B compact such that $S^{-1}TS-\lambda = A+B$. Since $TS - S\lambda = SA+SB$, $T-(S\lambda)S^{-1} = SAS^{-1}+SBS^{-1}$ and so $T-\lambda$ is Weyl, i.e, $\lambda \notin w(T)$. Thus $w(T) \subset w(S^{-1}TS)$. By the index theorem, $i(S^{-1}TS) = i(S^{-1}) + i(T) + i(S) = i(T)$ since $\mathcal{I} \subset \mathcal{F}_o$. □

Lemma 4.18. *Let $\mathbb{F}$ be the class of all finite rank operators and $\mathcal{K}$ be the class of all compact operators. Then $w(T) = \bigcap_{F\in\mathbb{F}} \sigma(T+F)$ for any $T \in B(H)$.*

Proof. We note that $w(T) = \bigcap_{K\in\mathcal{K}} \sigma(T+K)$. We claim that $\bigcap_{K\in\mathcal{K}} \sigma(T+K) = \bigcap_{F\in\mathbb{F}} \sigma(T+F)$. Since $\mathbb{F} \subset \mathcal{K}$,

$$\bigcap_{K\in\mathcal{K}} \sigma(T+K) \subset \bigcap_{F\in\mathbb{F}} \sigma(T+F). \tag{4.2}$$

Let $\lambda \notin \bigcap_{K\in\mathcal{K}} \sigma(T+K)$. Then there exists $K' \in \mathcal{K}$ such that $\lambda \notin \sigma(T+K')$. Since K' is compact, there exists a sequence $\{F_n\}$ of finite rank operators such that $\lim F_n = K'$ and $F_n \in \mathbb{F}$. Thus $(\lim_{n\to\infty} F_n) + T = K' + T$, i.e., $\lim_{n\to\infty}(F_n + T) = K' + T$. Since spectrum is upper semi-continuous(in Theorem 5.2), $\limsup \sigma(F_n+T) \subset \sigma(K'+T)$. Thus if $\lambda \notin \sigma(T+K')$, then $\lambda \notin \limsup \sigma(F_n+T)$, i.e.,

$$\lambda \notin \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} \sigma(F_k+T)\right).$$

So $\lambda \notin \bigcup_{k=m}^{\infty} \sigma(F_k+T)$ for some m. $\lambda \notin \bigcap_{F\in\mathcal{K}} \sigma(F_k+T)$. Hence

$$\bigcap_{F\in\mathbb{F}} \sigma(F_k+T) \subset \bigcap_{K\in\mathcal{K}} \sigma(T+K). \tag{4.3}$$

By (4.2) and (4.3), $w(T) = \bigcap_{F \in \mathcal{F}} \sigma(T+F)$. □

Theorem 4.19. ([7],[8]) *If H is a Hilbert space, then the set $\mathcal{F}_0$ is open in $B(H)$.*

Proof. Let T be in $\mathcal{F}_0$. Then by Corollary 4.7, there exists a finite rank operator F such that $T+F$ is invertible. Then if S is an operator in $B(H)$ which satisfies $\|T-S\| < 1/\|(T+F)^{-1}\|$, then $S+F$ is invertible and hence $S \in \mathcal{F}_0$. Since $\pi(S) = \pi(S+F)$ is invertible, $\mathcal{F}_0$ is an open set. □

Theorem 4.20. *For any operator T in $B(H)$, $\partial\omega(T) \subset \sigma_e(T)$ where ∂K denotes the boundary of K.*

Proof. If $\lambda \in \partial\omega(T) - \sigma_e(T)$, then $T - \lambda I$ is Fredholm since $\lambda \notin \sigma_e(T) \Longleftrightarrow \pi(T) - \lambda\hat{I} =$ invertible $\Longleftrightarrow T - \lambda I =$ Fredholm. Also since $\lambda \in \partial w(T)$, there exists a sequence $\{\lambda_n\}$ of points in the plane such that $\lambda_n \to \lambda$ and $T - \lambda_n$ is Fredholm of index 0 for each n. By the continuity of the index, $T - \lambda I$ must have index 0 and so $\lambda \notin \omega(T)$. This is a contradiction since $\omega(T)$ is compact indeed, $\omega(T)$ is compact $\Longrightarrow \omega(T)$ is closed $\Longrightarrow \lambda \in \omega(T)$. Hence $\partial\omega(T) \subset \sigma_e(T)$. □

Corollary 4.21. *If T is in $B(H)$, then $\omega(T)$ and $\sigma(\pi(T))$ have identical boundaries and convex hulls.*

By Lemma 4.15, $\sigma(\pi(T)) = \omega(T)$ if and only if the open set $\theta(T)$ is empty where $\theta(T) = \{\lambda : T - \lambda \in \mathcal{F} \text{ and } i(T-\lambda) \neq 0\}$.

Corollary 4.22. *If any of the following conditions holds for T in $B(H)$, then $\sigma(\pi(T)) = \omega(T)$:*

(1) T is normal.

(2) the point spectra of T and T^* are countable.

(3) the complement of $\sigma(\pi(T))$ is connected.

Proof. For any T in $B(H)$, $\lambda \in \theta(T)$ implies that $\dim\ker(T-\lambda) \neq \dim\ker(T^* - \overline{\lambda})$.

(1) If T is normal, then by Theorem 4.16, $\theta(T)$ is empty and so $w(T) = \sigma_e(T)$.

(2) If λ is in $\theta(T)$, by the above fact, either λ is an eigenvalue of T, i.e., $\lambda \in \sigma_p(T)$ or $\overline{\lambda}$ is an eigenvalue of T^*. Thus if T and T^* have countable point spetra, $\theta(T)$ is countable and open, therefore empty.

(3) Since $\sigma(\pi(T)) \subset w(T)$, $\partial w(T) \subset \sigma_e(T)$ and by Lemma 4.15 (2), condition (3) clearly implies that $\theta(T)$ is empty. □

Example 4.23. *If U is the simple unilateral shift, then $w(U) = \{\lambda : |\lambda| \leq 1\}$ and $\sigma(\pi(U)) = \{\lambda : |\lambda| = 1\}$. Thus $\theta(U) = \{\lambda : |\lambda| < 1\}$.*

Lemma 4.24. *If λ is an isolated point of $\sigma(T)$ and $T - \lambda \in \mathcal{F}$, then $T - \lambda$ is Weyl.*

Proof. Since $T - \lambda \in \mathcal{F}$, by the continuity of index, there exists $\delta_1 > 0$ such that

$$\|(T - \lambda) - S\| < \delta_1 \Rightarrow i(T - \lambda) = i(S). \tag{4.4}$$

Since λ is an isolated point of $\sigma(T)$, there exists $\delta_2 > 0$ such that $B(\lambda, \delta_2) \cap \sigma(T) = \{\lambda\}$. Put $\delta = \min\{\delta_1, \delta_2\}$. Then for all $\mu \in B(\lambda, \delta)$ with $\mu \neq \lambda$, $\|(T - \lambda) - (T - \mu)\| = |\lambda - \mu| < \delta$. Thus $T - \mu$ is invertible and by (4.4), $i(T - \mu) = i(T - \lambda)$. Since $T - \mu$ is invertible, $i(T - \lambda) = i(T - \mu) = 0$. Thus $T - \lambda$ is Weyl. □

Lemma 4.25. *For any operator* $T \in B(H)$, $\theta(T) \subsetneq \mathrm{acc}\sigma(T)$ *where* $\mathrm{acc}K$ *denotes the set of all accumulation points of* K.

Proof. Suppose that $\theta(T) - \mathrm{acc}\,\sigma(T) \neq \emptyset$. Then there exists $\lambda \in \theta(T) - \mathrm{acc}\,\sigma(T)$, i.e., $\lambda \in \theta(T)$ and $\lambda \notin \mathrm{acc}\sigma(T)$. Since $\lambda \in \theta(T)$, $T - \lambda \in \mathcal{F}$ and $i(T-\lambda) \neq 0$. Since $\lambda \notin \mathrm{acc}\,\sigma(T)$, then λ is an isolated point of $\sigma(T)$. Since $T - \lambda \in \mathcal{F}$, by Lemma 4.24, $T - \lambda$ is Weyl and so $i(T-\lambda) = 0$. This is a contradiction. Thus $\theta(T) - \mathrm{acc}\,\sigma(T) = \emptyset$, i.e., $\theta(T) \subset \mathrm{acc}\,\sigma(T)$. Since $\theta(T)$ is open and $\sigma(T)$ is closed, $\theta(T) \subsetneq \mathrm{acc}\,\sigma(T)$. □

5. Continuities of Several Spectra

Lemma 5.1. ([12]) *The following two definitions of upper semicontinuity for a set-valued function are equivalent:*

(1) *(Metric definition) For each open set Λ_0 containing $\sigma(A)$, there exists $\varepsilon > 0$ such that $\|A - B\| < \varepsilon \Rightarrow \sigma(B) \subset \Lambda_0$.*

(2) *(Sequential definition) For all $A_n \to A$, $\limsup \sigma(A_n) \subset \sigma(A)$.*

Proof. (1) $\Rightarrow$ (2). Let $A_n \to A$ and let $\lambda \notin \sigma(A)$. Then there exists disjoint open sets U and V such that $\lambda \in U$ and $\sigma(A) \subset V$. By (1), there exists $\varepsilon > 0$ such that

$$\|A - B\| < \varepsilon \Rightarrow \sigma(B) \subset V. \tag{5.1}$$

Since $A_n \to A$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0 \Rightarrow \|A_n - A\| < \varepsilon$. By (5.1), $\sigma(A_n) \subset V$ for all $n \geq n_0$. Since $\lambda \in U$, $\sigma(A_n) \subset V$ for all $n \geq n_0$ and $U \cap V = \emptyset$. Hence $\lambda \notin \limsup \sigma(A_n)$. and so $\limsup \sigma(A_n) \subset \sigma(A)$.

(2) $\Rightarrow$ (1). Suppose that (1) does not hold. Then there exists open set Λ_0 containing $\sigma(A)$ such that for all $\varepsilon > 0$, there exists B such that $\|A - B\| < \varepsilon$ and $\sigma(B) \cap \Lambda_0^c \neq \emptyset$. Put $\varepsilon = \frac{1}{n}$. Then there exists A_n such that $\|A_n - A\| < \frac{1}{n}$ and $\sigma(A_n) \cap \Lambda_0^c \neq \emptyset$. Thus $A_n \to A$ but $\Lambda_0^c \subset \sigma(A)^c$ since $\sigma(A) \subset \Lambda_0$. Since $\sigma(A_n) \cap \Lambda_0^c \neq \emptyset$, $\sigma(A_n) \cap \sigma(A)^c \neq \emptyset$ and so $\sigma(A_n) \not\subset \sigma(A)$. Thus $\limsup \sigma(A_n) \not\subset \sigma(A)$. This is a contradiction. Hence (1) holds. □

Theorem 5.2. ([12]) *Spectrum is upper semi-continuous.*

Proof. Let G^c be the set of all singular operators on H, i.e., G^c is the set of noninvertible operators. For all $A \in B(H)$, define $\varphi(\lambda) = d(A - \lambda, G^c)$. Then we can show easily that $\varphi : \mathbb{C} \to \mathbb{R}^+$ is continuous. Let Λ_0 be an open set containing $\sigma(A)$ and let $\Delta = \bar{B}(0; 1 + \|A\|)$ denote the closed ball with center 0 and radius $1 + \|A\|$. If $\lambda \in \Delta - \Lambda_0$, then $\lambda \notin \sigma(A)$ and so $A - \lambda$ is invertible, i.e., $A - \lambda \notin G^c$. Since G is open, G^c is closed. Thus $d(A - \lambda, G^c) > 0$, i.e., $\varphi(\lambda) > 0$. Since $\Delta - \Lambda_0 = \Delta \cap \Lambda_0^c$ is a closed subset of Δ and Δ is compact, $\Delta - \Lambda_0$ is compact. Since $\varphi(\lambda)$ is continuous on $\Delta - \Lambda_0$ and $\varphi(\lambda) > 0$ for all $\lambda \in \Delta - \Lambda_0$, there exists $\varepsilon > 0$ such that

$$\varphi(\lambda) \geq \varepsilon \quad \text{for all} \quad \lambda \in \Delta - \Lambda_0. \tag{5.2}$$

Suppose that $\|A - B\| < \varepsilon < 1$. We claim that $\sigma(B) \subset \Lambda_0$. If $\lambda \in \Delta - \Lambda_0$, then by (5.2), $\|(A - \lambda) - (B - \lambda)\| < \varepsilon \leq \varphi(\lambda) = d(A - \lambda, G^c)$. Thus $\|(A-\lambda)-(B-\lambda)\| < d(A-\lambda, G^c)$. If $B-\lambda \in G^c$, then $\|(A-\lambda)-(B-\lambda)\| \geq d(A - \lambda, G^c)$. Thus $B - \lambda \notin G^c$ and so $B - \lambda$ is invertible. Thus $\lambda \notin \sigma(B)$, i.e.,

$$\lambda \in \Delta - \Lambda_0 \implies \lambda \notin \sigma(B). \tag{5.3}$$

For all $\lambda \in \sigma(B)$, $|\lambda| \leq \|B\| \leq \|A\| + \|A - B\| < \|A\| + 1$ and so $\lambda \in \Delta$, i.e., $\sigma(B) \subset \Delta$. Hence $\sigma(B) \subset \Lambda_0$ by (5.3). □

Definition 5.3. *The spectral radius of an operator A, denoted by $r(A)$, is defined by $r(A) = \sup\{ |\lambda| \ : \ \lambda \in \sigma(A) \}$.*
The Weyl spectral radius of A, denoted by $r_w(A)$ is defined by $r_w(A) = \sup\{ |\lambda| \ : \ \lambda \in w(A) \}$.

Corollary 5.4. *Spectral radius is upper semi-continuous. That is, to each operator A and for all $\delta > 0$, there exists $\varepsilon > 0$ such that $\|A - B\| < \varepsilon \Rightarrow r(B) < r(A) + \delta$.*

Proof. Let $\delta > 0$ be given. Let $r_\delta = r(A) + \delta > r(A)$. For all $\lambda \in \sigma(A)$, $|\lambda| \leq r(A)$ and so $|\lambda| \leq r(A) + \delta$. Thus $\lambda \in B(0, r_\delta)$, i.e., $\sigma(A) \subset B(0, r_\delta)$.

Since spectrum is upper semi-continuous, there exists $\varepsilon > 0$ such that $\|B - A\| < \varepsilon \implies \sigma(B) \subset B(0, r_\delta)$. For all $\lambda \in \sigma(B)$, $|\lambda| < r_\delta$, i.e., $|\lambda| < r(A) + \delta$. Thus $\sup\{\, |\lambda| : \lambda \in \sigma(B)\} \leq r(A) + \delta$ and so $r(B) \leq r(A) + \delta$. □

Theorem 5.5. ([12]) *Let T_n, T be normal operators and $T_n \to T$. Then $\lim \sigma(T_n) = \sigma(T)$, i.e., the restriction of spectrum to the normal is continuous.*

Example 5.6. *(Istratescu) Let $H = l_2$ and for each $x = (x_1, x_2, \cdots) \in H$, we define $T_n(x) = (x_1, x_2, \cdots, x_n, 0, 0, \cdots)$. It is clear that $T_n \to I$ pointwise, i.e., $T_n(x) = I(x) = x$ as $n \to \infty$ for any $x \in H$ and T_n is of finite rank. Thus T_n are compact and $w(T_n) = w(T_n + 0) = w(0) = \{0\}$ for all n. Also 1 is the only eigenvalue of I and not of finite multiplicity. Since $\sigma(I) - w(I) \subset \pi_{0f}(I) = \emptyset$, $\sigma(I) - w(I) = \emptyset$, i.e., $w(I) = \sigma(I) = \{1\}$. In fact,*

$$\begin{aligned} \|T_n - I\| &= \sup_{\|x\|=1} \|T_n x - x\| \\ &= \sup_{\|x\|=1} \|(0, 0, \cdots, 0, x_{n+1}, x_{n+2}, \cdots)\| = 1 \not\to 0 \end{aligned}$$

as $n \to \infty$, i.e., $T_n \not\to I$ in norm. Define the shift operator operator $Sx = (0, x_1, x_2, \cdots)$ and put $A_n = ST_n$. Then $A_n \to S$ pointwise since

$$\begin{aligned} A_n x = (ST_n)x &= S(x_1, x_2, \cdots, x_n, 0, 0, \cdots) \\ &= (0, x_1, x_2, \cdots, x_n, 0, 0, \cdots) \to Sx \end{aligned}$$

as $n \to \infty$. Also A_n is compact since A_n is of finite rank and thus $w(A_n) = \{0\}$. Since S is the unilateral shift, we know that $w(S) = \{z : |z| \leq 1\}$. Clearly the function $T \to w(T)$ is not continuous.

Theorem 5.7. ([20]) *The mapping $T \rightarrow w(T)$ is upper semi-continuous.*

Proof. It suffices to show that $T_n \rightarrow T \Rightarrow \limsup w(T_n) \subset w(T)$.
Let $\lambda \notin w(T)$. Then $T - \lambda$ is Weyl. By Theorem 5.5 ([17]), there exists an $\eta > 0$ such that if $S \in B(H)$ and $\|(\lambda - T) - S\| < \eta$, then S is Weyl. Since $T_n \rightarrow T$, there exists an integer N such that $\|(\lambda I - T) - (\lambda I - T_n)\| = \|T_n - T\| < \frac{\eta}{2}$ for any $n \geq N$. Let V be an open $\frac{\eta}{2}$- neighborhood of λ, i.e., $V = B(\lambda : \frac{\eta}{2})$. We have for any $\mu \in V$ and $n \geq N$,

$$\begin{aligned}
\|(\lambda - T) - (\mu I - T_n)\| &= \|(\lambda - T) - (\mu - T_n) + (\lambda - T_n) - (\lambda - T_n)\| \\
&\leq \|(\lambda - T) - (\lambda - T_n)\| + \|(\mu - T_n) - (\lambda - T_n)\| \\
&= \|T_n - T\| + \|(\mu - \lambda)I\| = \|T_n - T\| + |\mu - \lambda| \\
&\leq \frac{\eta}{2} + \frac{\eta}{2} = \eta
\end{aligned}$$

so that $\mu - T_n$ is Weyl, i.e., $\mu \notin w(T_n)$ for $n \geq N$, i.e., for $n \geq N$,
$V \cap w(T_n) = \emptyset$. So $\lambda \notin \limsup w(T_n)$. Hence $\limsup w(T_n) \subset w(T)$. □

Corollary 5.8. *The Weyl spectral radius is upper semi-continuous. That is, to each operator A, for all $\delta > 0$, there exists $\varepsilon > 0$ such that $\|A - B\| < \varepsilon \Rightarrow r_w(B) < r_w(A) + \delta$.*

Proof. Let $\delta > 0$ be given. Let $r_\delta = r_w(A) + \delta > r_w(A)$. For all $\lambda \in w(A)$, $|\lambda| \leq r_w(A)$ and so $|\lambda| \leq r_w(A) + \delta$. Thus $\lambda \in B(0, r_\delta)$, i.e., $w(A) \subset B(0, r_\delta)$. Since Weyl spectrum is upper semi-continuous, there exists $\varepsilon > 0$ such that $\|B - A\| < \varepsilon \Longrightarrow w(B) \subset B(0, r_\delta)$. For all $\lambda \in w(B)$, $|\lambda| < r_\delta$, i.e., $|\lambda| < r_w(A) + \delta$. Thus $\sup\{\, |\lambda| : \lambda \in w(B)\} \leq r_w(A) + \delta$ and so $r_w(B) \leq r_w(A) + \delta$. □

Theorem 5.9. ([20]) *Let* $T_n \to T \in B(H)$. *Then* $\lim w(T_n) = w(T)$ *if* $\lim \sigma(\hat{T}_n) = \sigma(\hat{T})$.

Proof. By the above theorem, $\limsup w(T_n) \subset w(T)$. It is enough to show that $w(T) \subset \liminf w(T_n)$. Let $\lambda \notin \liminf w(T_n)$. Then there is a neighborhood V of λ which does not intersect infinitely many $w(T_n)$. Since $\sigma(\hat{T}) \subset w(T_n)$ for any n, V does not intersect infinitely many $\sigma(\hat{T}_n)$, i.e., $\lambda \notin \liminf \sigma(\hat{T}_n) = \lim \sigma(\hat{T}_n) = \sigma(\hat{T})$. This shows that $\lambda - \hat{T}$ is invertible, i.e., $\lambda - T$ is Fredholm. By using Theorem 5.5([17]), it is easy to see that $i(\lambda - T) = 0$. Therefore $\lambda - T \neq$ Weyl, i.e., $\lambda \notin w(T)$. □

We recall that if the mapping $T \to \sigma_e(T)$ is continuous, then the mapping $T \to w(T)$ is continuous.

Corollary 5.10. *Let* $T_n \to T$. *Then* $\lim w(T_n) = w(T)$ *if one of the following cases holds.*

(1) $T_nT = TT_n$ *for all* n.

(2) $\sigma(T)$ *is totally disconnected.*

(3) T_n *and* T *are normal operators.*

Proof. By [18], each one of the above conditions implies $\lim \sigma_e(T_n) = \sigma_e(T)$. By Theorem 5.8, our result holds. □

Corollary 5.11. *Let* $T_n \to T$ *and* $w(T_n) = \sigma_e(T_n)$ *for all* n. *Then* $w(T) = \sigma_e(T)$ *if one of the following cases holds.*

(1) $T_nT = TT_n$ *for all* n.

(2) $\sigma(T)$ *is totally disconnected.*

Proof. By Corollary 5.10 and [18], each the above conditions implies that $\sigma_e(T) = \lim \sigma_e(T_n) = \lim w(T_n) = w(T)$. □

Definition 5.12. ([21]) *For $T \in B(H)$, the essential spectra $\sigma_i(T)$ are defined by*

$$\sigma_1(T) = \{\lambda \in \mathbb{C} : R(T-\lambda) \text{ is not closed. }\},$$

$$\sigma_2(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not semi-Frddholm }\},$$

$$\sigma_3(T) = \sigma_e(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Fredholm}\},$$

$$\sigma_4(T) = w(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Weyl}\} \text{ and}$$

$$\sigma_5(T) = \sigma_4(T) \cup \{\text{ limit points of } \sigma(T)\}$$

; *the Browder's limit point spectrum of T.*

Clearly $\sigma_1(T) \subset \sigma_2(T) \subset \sigma_3(T) \subset \sigma_4(T) \subset \sigma_5(T) \subset \sigma(T)$. We note that $\sigma_1(T)$ may be empty, e.g., take $T = 0$. $\sigma_4(T) = w(T) = \sigma_3(T) \cup \{\lambda \in \mathbb{C} : T-\lambda \in \mathcal{F}$ and $i(T-\lambda) \neq 0\}$. Put $\sigma_4'(T) = \sigma_3(T) \cup \operatorname{acc}\sigma(T)$, where $\operatorname{acc}\sigma(T)$ is the set of accummulation points of $\sigma(T)$. By Lemma 4.25, $\theta(T) \subset \operatorname{acc}\sigma(T)$. Thus $\sigma_4'(T) = \sigma_5(T)$.

Theorem 5.13. ([21]) *Let $T \in B(H)$. Then the mapping $T \to \sigma_2(T)$ is upper semi-continuous.*

Proof. Let $T_n \to T$. We show that $\limsup \sigma_2(T) \subset \sigma_2(T)$. Let $\lambda \notin \sigma_2(T)$. Then $T-\lambda$ is semi-Fredholm. By the continuity of index, there exists $\varepsilon > 0$ such that $\|(T-\lambda)-S\| < \varepsilon \Rightarrow S \in \mathcal{SF}$ and $i(T-\lambda) = i(S)$. Since $T_n \to T$, there exists N_0 such that for all $n \geq N_0$, $\|T_n - T\| < \frac{\varepsilon}{2}$. Thus for all $n \geq N_o$ and for all μ with $|\mu - \lambda| < \frac{\varepsilon}{2}$,

$$\begin{aligned}\|(\lambda - T) - (\mu - T_n)\| &= \|(\mu-\lambda)I + T_n - T\| \\ &\leq |\mu - \lambda| + \|T_n - T\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus $\mu - T_n \in \mathcal{SF}$ and $i(T-\lambda) = i(T_n - \mu)$. Since for all μ with $|\mu - \lambda| < \frac{\varepsilon}{2}$ and for all $n \geq N_0$ $\mu \notin \sigma_2(T_n)$, $\lambda \notin \limsup \sigma_2(T)$. □

Theorem 5.14. *Let $T \in B(H)$. The mapping $T \to \sigma_3(T)$ is upper semi-continuous.*

Proof. Let $\lambda \notin \sigma_3(T)$. Then $T - \lambda$ is Fredholm. By the continuity of index, there exists $\varepsilon > 0$ such that $\|(T-\lambda) - S\| < \varepsilon \Rightarrow S \in \mathcal{SF}$ and $i(S) = i(T-\lambda)$. Since $T - \lambda \in \mathcal{F}$, $\dim\ker(T-\lambda) < \infty$ and $\dim R(T-\lambda)^{\perp} < \infty$. Since $i(S) = i(T-\lambda)$, $\dim\ker S - \dim R(S)^{\perp} = \dim\ker(T-\lambda) - \dim R(T-\lambda)^{\perp}$. Thus $\dim\ker S < \infty$ and $\dim R(S)^{\perp} < \infty$. Since $S \in \mathcal{SF}$, $R(S)$ is closed. Thus $S \in \mathcal{F}$. Therefore

$$\|(T-\lambda) - S\| < \varepsilon \Rightarrow S \in \mathcal{F}. \tag{5.4}$$

Since $T_n \to T$, there exists a positive integer N such that for all $n \geq N$, $\|T_n - T\| < \frac{\varepsilon}{2}$. Now for all $\mu \in B(\lambda, \frac{\varepsilon}{2})$ with $|\mu - \lambda| < \frac{\varepsilon}{2}$ and for all $n \geq N$, $\|(\lambda - T) - (\mu - T_n)\| \leq \|(\mu - \lambda)I\| + \|T_n - T\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. By (5.4), $\mu - T_n \in \mathcal{F}$. Therefore $\mu \notin \sigma_3(T)$ for all $n \geq N$ and for all $\mu \in B(\lambda, \frac{\varepsilon}{2})$. Thus $\lambda \notin \limsup \sigma_3(T_n)$. Hence $\limsup \sigma_3(T_n) \subset \sigma_3(T)$. □

Remark 5.15. *The mapping $T \to \sigma_1(T)$ is not upper semi-continuous.*

Proof. Let T be any operator where range is not closed and let $T_n = \frac{1}{n}T$. Then $\|T_n\| = \frac{1}{n}\|T\| \to 0$ as $n \to \infty$, i.e., $T_n \to 0$. Since $\sigma_1(0) = \emptyset$, $0 \notin \sigma_1(0)$. But since for all n, $R(T_n) = R(\frac{1}{n}T) = \frac{1}{n}R(T)$ is not closed, $0 \in \sigma_1(T_n)$. Thus $0 \in \limsup \sigma_1(T)$. Therefore $\limsup \sigma_1(T_n) \not\subset \sigma_1(T)$ where $T_n \to T$. □

In [22], Oberai have showed that the mapping $T \to \sigma_5(T)$ is upper semi-continuous. We note that $w(T) = \sigma_e(T) \cup \theta(T)$ and $\theta(T) \subset \mathrm{acc}\sigma(T)$ in Lemma 4.25. Since $\sigma_5(T) = w(T) \cup \mathrm{acc}\sigma(T)$, $\sigma_5(T) = \sigma_e(T) \cup \theta(T) \cup \mathrm{acc}\sigma(T) = \sigma_e(T) \cup \mathrm{acc}\sigma(T)$. Using this fact, we reprove that the mapping $T \to \sigma_5(T)$ is upper semi-continuous.

Theorem 5.16. *The mapping $T \to \sigma_5(T)$ is upper semi-continuous.*

Proof. We note that $\sigma_4'(T) = \sigma_5(T)$. Let $T_n \to T$. We show that $\limsup \sigma_4'(T) \subset \sigma_4'(T)$. Let $\lambda \notin \sigma_4'(T)$. If $\lambda \notin \sigma(T)$, then $\lambda \notin \limsup \sigma_4'(T)$ since $\limsup \sigma_4'(T_n) \subset \limsup \sigma(T_n) \subset \sigma(T)$. Let $\lambda \in \sigma(T) - \sigma_4'(T)$. Then $\lambda \notin \sigma_3(T)$ and $\lambda \notin \mathrm{acc}\,\sigma(T)$. Hence $T - \lambda$ is Fredholm and λ is an isolated point of $\sigma(T)$. So there exists $\varepsilon_1 > 0$ such that

$$\|(T - \lambda) - S\| < \varepsilon_1 \;\Rightarrow\; S \quad \text{is Fredholm.} \tag{5.5}$$

Since $T_n \to T$, there exists N_1 such that $\|T_n - T\| < \varepsilon_1$ for all $n \geq N_1$. Thus $\|(T_n - \lambda) - (T - \lambda)\| = \|T_n - T\| < \varepsilon_1$ for all $n \geq N_1$. By (5.5),

$$T_n - \lambda \text{ is Fredholm for all } n \geq N_1 \,. \tag{5.6}$$

Since λ is an isolated point of $\sigma(T)$, there exists $\varepsilon_2 > 0$ such that $\sigma(T) \cap \{\mu : |\mu - \lambda| < \varepsilon_2\} = \{\lambda\}$. Put $\varepsilon = \min\{\varepsilon_1\, \varepsilon_2\}$. For all μ with $|\mu - \lambda| < \varepsilon$, $\mu \notin \sigma(T)$ and so $\mu \notin \limsup \sigma(T_n) = \bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} \sigma(T_k))$. Thus $\mu \notin \bigcup_{k=m}^{\infty} \sigma(T_k)$ for some m, i.e.,

$$\mu \notin \sigma(T_k) \quad \text{for all} \quad k \geq m \,. \tag{5.7}$$

Let $N = \max\{m, N_1\}$. If $\lambda \notin \limsup \sigma(T_n)$, then $\lambda \notin \limsup \sigma_4'(T_n)$ since $\limsup \sigma_4'(T_n) \subset \limsup \sigma(T_n)$. If $\lambda \in \limsup \sigma(T_n)$, then $\lambda \in \bigcup_{k=n}^{\infty} \sigma(T_k)$

for all n. Thus $\lambda \in \bigcup_{k=N}^{\infty} \sigma(T_k) \Rightarrow \lambda \in \sigma(T_{k_1})$ for some $k_1 \geq N$ and $\lambda \in \bigcup_{k=N+1}^{\infty} \sigma(T_k) \Rightarrow \lambda \in \sigma(T_{k_2})$ for some $k_2 \geq k_1 \geq N$. There exists a sequence $\{k_n\}$ such that $\lambda \in \sigma(T_{k_n})$ for all n, $k_n \geq N$. By (5.5), $T_{k_n} - \lambda$ is Fredholm. By (5.7), λ is an isolated point of $\sigma(T_{k_n})$ for all n. Hence $\lambda \notin \sigma_4'(T_{k_n})$ for all n and so $\lambda \notin \limsup \sigma_4'(T_n)$. □

6. Spectral Mapping Theorem

Theorem 6.1. ([12]) *For any operator A and for any polynomial p,*

(1) $p(\pi_0(A)) = \pi_o(p(A))$,

(2) $\sigma_{ap}(p(A)) = p(\sigma_{ap}(A))$,

(3) $\sigma_{com}(p(A)) = p(\sigma_{com}(A))$ *and*

(4) $p(\pi_0(A)) = \pi_o(p(A))$ *if A is invertible and* $p(z) = \frac{1}{z}$.

Proof. First, we show that if the product of a finite number of operators has of the following properties:

a) nonzero kernel,

b) it is not bounded below and

c) it has a range that is not dense,

then at least one factor of the product must have the same property.

Let AB be the product of A and B. If $\ker(AB) \neq \emptyset$, then $\ker A \neq \emptyset$ since $\ker(AB) \subset \ker A$. Thus a) holds.

Let AB be not bounded below. If B is bounded below, then there exists $\{x_n\}$ such that $\|(AB)x_n\| \to 0$ and $\|Bx_n\| \geq c\|x_n\|$ for some $c > 0$. Put $y_n = \frac{Bx_n}{\|Bx_n\|}$. Then $\|y_n\| = 1$ and $\|Ay_n\| \to 0$. Thus A is not bounded below. and so b) holds.

Let AB have a range which is not dense. Then $\overline{R(AB)} \neq H$. Let $p(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_0$, $a_i \in \mathbb{R}$. Then $p(\lambda) - p(\lambda_0) = (\lambda - \lambda_0)Q(\lambda, \lambda_0)$, i.e., $p(\lambda) - p(\lambda_0)$ is divisible by $\lambda - \lambda_0$.

(1) Let $\lambda_0 \in \pi_0(A)$. Then $\ker(A - \lambda_0) \neq (0)$. Since $p(A) - p(\lambda_0 I) = (A - \lambda_0 I)Q(A, \lambda_0)$, $\ker(A - \lambda_0) \subset \ker(p(A) - p(\lambda_0 I))$. Since $\ker(A - \lambda_0) \neq (0)$, $\ker(p(A) - p(\lambda_0 I)) \neq (0)$. Thus $p(\lambda_0) \in \pi_o(p(A))$ and so $p(\pi_0(A)) \subset$

$\pi_0(p(A))$. For all $\alpha \in \mathbb{C}$, let $p(\lambda) - \alpha = (\lambda - \lambda_0)(\lambda - \lambda_1)\cdots(\lambda - \lambda_{n-1})$ and $\alpha \in \pi_0(p(A))$. Then $\ker(p(A) - \alpha I) \neq (0)$ and $p(A) - \alpha I = (A - \lambda_0)(A - \lambda_1)\cdots(A - \lambda_{n-1})$. By a), $\ker(A - \lambda_k I) \neq (0)$ for some k, i.e., $\lambda_k \in \pi_0(A)$. Thus $\pi_0(p(A)) \subset p(\pi_0(A))$. Therefore $p(\pi_0(A)) = \pi_0(p(A))$.

(2) Let $\lambda_0 \in \sigma_{ap}(A)$. Then $A - \lambda_0$ is not bounded below and so there exists a sequence $\{x_n\}$, $\|x_n\| = 1$ such that $(A - \lambda_0)x_n \to 0$. Since $p(A) - p(\lambda_0) = (A - \lambda_0)Q(A, \lambda_0)$, $(p(A) - p(\lambda_0))x_n \to 0$ and so $p(\lambda_0) \in \sigma_{ap}(p(A))$. Thus $p(\sigma_{ap}(A)) \subset \sigma_{ap}(p(A))$. Let $\alpha \in \sigma_{ap}(p(A))$, i.e., $p(A) - \alpha$ is not bounded below. By b), $A - \lambda_k$ is not bounded below for some k. Thus $\lambda_k \in \sigma_{ap}(A)$. Since $p(\lambda_k) - \alpha = 0$, $\alpha = p(\lambda_k) \in p(\sigma_{ap}(A))$ and so $\sigma_{ap}(p(A)) \subset p(\sigma_{ap}(A))$. Hence $\sigma_{ap}(p(A)) = p(\sigma_{ap}(A))$.

(3) Since $p(\pi_0(A^*)) = \pi_0(p(A^*))$ by (1), $\sigma_{com}(p(A)) = \pi_0(p(A)^*)^* = p(\pi_0(A^*)^*) = p(\sigma_{com}(A))$.

(4) If A is invertible and $Ax = \lambda x$ with $x \neq 0$, then $\lambda \neq 0$. Applying A^{-1} to both sides of the equation and dividing by λ, $A^{-1}x = \frac{1}{\lambda}x$. Thus $\frac{1}{\sigma_p(A)} \subset \sigma_p(A^{-1})$. Replacing A by A^{-1}, $\frac{1}{\sigma_p(A^{-1})} \subset \sigma_p((A^{-1})^{-1}) = \sigma_p(A)$ and so $\sigma_p(A^{-1}) \subset \frac{1}{\sigma_p(A)}$. Hence $\sigma_p(A^{-1}) = \frac{1}{\sigma_p(A)}$. □

Theorem 6.2. *For any operator T and for all polynomial p, $w(p(T))$ is a proper subset of $p(w(T))$, i.e., $w(p(T)) \subsetneq p(w(T))$.*

Proof. Let $\mu \notin p(w(T))$ and $p(\lambda) - \mu = a(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$. Then $p(T) - \mu I = a(T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n)$ and for all j, $p(\lambda_j) - \mu = 0$. Thus $\mu = p(\lambda_j) \notin p(w(T))$ and so $\lambda_j \notin w(T)$. Therefore $T - \lambda_j \in \mathcal{F}_0$ for all j. By Theorem 3.8 and index prduct theorem, $(T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n) \in \mathcal{F}_0$ and so $p(T) - \mu I \in \mathcal{F}_0$. Hence $\mu \notin w(p(T))$.

Let $T = U \oplus (U^* + 2I)$ where U is the unilateral shift operator and let

$p(\lambda) = \lambda(\lambda - 2)$. Then $R(U) = \{(0, x_0, x_1, \cdots) : (x_0, x_1, x_2, \cdots) \in l_2\}$ and so $R(U)$ is closed. Also $\ker(U) = (0)$ and $R(U)^{\perp} = \{(x, 0, 0, \cdots) : x \in \mathbb{C}\}$. Thus $i(U) = 0 - 1 = -1$. For all $x = (x_n)$, $y = (y_n) \in H$, $< Ux, y > = < (0, x_1, x_2, , x_n, \cdots), (y_1, y_2, \cdots,) > = x_1\overline{y_2} + x_2\overline{y_3} + \cdots = < (x_1, x_2, \cdots,), (y_2, y_3, \cdots,) > = < x, U^*y >$. Thus $U^*(y_1, y_2, \cdots,) = (y_2, y_3, \cdots,)$. Also $R(U^*) = l_2$, $\ker(U^*) = \{(x, 0, 0, \cdots,) : x \in \mathbb{C}\}$, and $R(U^*)^{\perp} = (0)$. Thus $i(U^*) = 1 - 0 = 1$. Since $-2 \notin \sigma(U^*)$ and $2 \notin \sigma(U)$, $U^* + 2I$ and $U - 2I$ are invertible and so $i(U^* + 2I) = 0$ and $i(U - 2I) = 0$. Since $p(\lambda) = \lambda(\lambda - 2)$, $p(T) = T(T - 2I) = [U \oplus (U^* + 2I)][U \oplus (U^* + 2I) - 2I] = [U \oplus (U^* + 2I)][(U - 2I) \oplus U^*]$. We note that $i(A \oplus B) = i(A) + i(B)$ and $i(AB) = i(A) + i(B)$ (see Proposition 3.7 and Theorem 3.8, [7]). Thus $i(T) = i(U \oplus (U^* + 2I)) = -1$ and $i(T - 2I) = i((U - 2I) \oplus U^*) = 1$. So $i(p(T)) = i(T(T - 2I) = i(T) + i(T - 2I) = 0$. Therefore $p(T) \in \mathcal{F}_0$ and so $0 \notin w(p(T))$. Since $i(T) = -1$, $T \notin \mathcal{F}_o$, i.e., $0 \in w(T)$. Since $0 = P(0)$, $0 \in P(w(T))$. □

Theorem 6.3. *Let $T \in B(H)$. Then for any polynomial $p(t)$, we have $\sigma(p(T)) - \pi_{00}(p(T)) \subset p(\sigma(T) - \pi_{00}(T))$.*

Proof. Let $\lambda \in \sigma(p(T)) - \pi_{00}(p(T)) = p(\sigma(T)) - \pi_{00}(p(T))$.

Case 1. λ is not an isolated point of $p(\sigma(T)) = \sigma(p(T))$. Then there exists a sequence $\{\lambda_n\}$ such that $\lambda_n \in p(\sigma(T))$ and $\lambda_n \to \lambda$, and so there exists a sequence $\{\mu_n\}$ in $\sigma(T)$ such that $\lambda_n = p(\mu_n) \to \lambda$. Since $\lim p(\mu_n) = \lambda$, $\{p(\mu_n)\}$ is bounded and so $\{\mu_n\}$ is bounded. Thus $\{\mu_n\}$ has a convergent subsequence. Let $\mu_0 = \lim \mu_{n_k}$. Then $p(\mu_0) = p(\lim \mu_{n_k}) = \lim p(\mu_{n_k}) = \lambda$, i.e., $p(\mu_0) = \lambda$. Since $\lim \mu_{n_k} = \mu_0$ and $\mu_{n_k} \in \sigma(T)$, μ_0 is not an isolated point of $\sigma(T)$. Since $\sigma(T)$ is closed, $\mu_0 \in \sigma(T)$. Thus $\mu_0 \in \sigma(T) - \pi_{00}(T)$

and so $\lambda = p(\mu_0) \in p(\sigma(T) - \pi_{00}(T))$.

Case 2. λ is an isolated point of $\sigma(p(T))$. Since $\lambda \notin \pi_{00}(p(T))$, either $\lambda \notin \pi_0(p(T))$ or λ is an eigenvalue of infinite multiplicity. Let $p(x) - \lambda = a_0(x - \mu_1)(x-\mu_2)\cdots(x-\mu_n)$. Then $p(T)-\lambda I = a_0(T-\mu_1 I)(T-\mu_2 I)\cdots(T-\mu_n I)$. If $\lambda \notin \pi_0(p(T))$, then $\ker(p(T) - \lambda) = (0)$. For all j, $\ker(T - \mu_j) = (0)$, i.e., $\mu_j \notin \pi_0(T)$ for all j. If $\mu_j \notin \sigma(T)$ for all j, then $T - \mu_j$ is invertible, and so $p(T) - \lambda$ is invertible. This is a contradiction to the fact that $\lambda \in \sigma(p(T))$. Thus $\mu_k \in \sigma(T)$ for some k and $\mu_k \notin \pi_0(T)$. Therefore $\mu_k \in \sigma(T)-\pi_0(T)$ and so $\mu_k \in \sigma(T) - \pi_{00}(T)$. Since $p(\mu_k) - \lambda = 0$, $\lambda = p(\mu_k) \in p(\sigma(T) - \pi_{00}(T))$.

If λ is an eigenvalue of $p(T)$ with infinite multiplicity, then $\ker(p(T)-\lambda) = \bigcup_{n=1}^{\infty} \ker(T - \mu_k)$. Since $\dim\ker(p(T) - \lambda) = \infty$, $\dim\ker(T - \mu_k) = \infty$ for some k. Thus μ_k is an eigenvalue of T with infinite multiplicity, i.e., $\mu_k \notin \pi_{0f}(T)$ and so $\mu_k \notin \pi_{00}(T)$. Hence $\mu_k \in \sigma(T) - \pi_{00}(T)$ and $p(\mu_k) = \lambda \in p(\sigma(T) - \pi_{00}(T))$. □

Theorem 6.4. *If either $\pi_{0f}(T) = \phi$ or $\pi_{0f}(T^*) = \phi$, then $w(f(T)) = f(w(T))$ for every holomorphic function f.*

Proof. Suppose $\pi_{0f}(T) = \phi$. Since $\pi_{0f}(f(S)) \subset f(\pi_{0f}(S))$ for every operator S and any holomorphic f. Thus $\pi_{0f}(T) = \phi$ implies $\pi_{0f}(f(T)) = \phi$. Therefore $w(T) = \sigma(T)$ and $w(f(T)) = \sigma(f(T))$ by Theorem 4.13. Since $\sigma(f(T)) = f(\sigma(T))$ by the usual holomorphic spectral mapping formula, $w(f(T)) = \sigma(f(T)) = f(\sigma(T)) = f(w(T))$. Similarly if $\pi_{0f}(T^*) = \phi$, then $w(f(T)) = f(w(T))$ since $w(T^*) = w(T)^*$. □

Note that if A is hyponormal, then $\|A^*x\| \leq \|Ax\|$ for each $x \in H$. Thus $\ker A \subset \ker A^*$.

Definition 6.5. ([24]) *An operator T is M-hyponormal if there exists $M > 0$ such that $\|(T-z)^*x\| \leq M\|(T-z)x\|$ for all $x \in H$ and $z \in \mathbb{C}$.*

Every hyponormal operator is clearly 1-hyponormal.

Theorem 6.6. *If T and S are commuting M-hyponormal and TS is a Weyl operator, then T and S are Weyl operators.*

Proof. If T is M-hyponormal, then there exists $M > 0$ such that $\|T^*x\| \leq M\|Tx\|$ for all $x \in H$ and so $\ker T \subset \ker T^*$. Thus $\dim\ker T \leq \dim\ker T^*$ and so $i(T) \leq 0$. If TS is a Weyl operator, then $TS \in \mathcal{F}$. Thus by Lemma 3.7, $T \in \mathcal{F}$ and $S \in \mathcal{F}$. Since T and S are M-hyponormal, $i(T) \leq 0$ and $i(S) \leq 0$. Since $0 = i(TS) = i(T) + i(S)$, $i(T) = 0$ and $i(S) = 0$. Thus $T \in \mathcal{F}_0$ and $S \in \mathcal{F}_0$. □

Corollary 6.7. *If T and S are commuting hyponormal operators and TS is a Weyl operator, then T and S are Weyl operators.*

If the "hyponormal" condition is dropped in the above theorem, then the theorem may fail even though T_1 and T_2 commute. For example, if U is the unilateral shift on l_2, consider the following operators on $l_2 \oplus l_2$: $T_1 = U \oplus I$ and $T_2 = I \oplus U^*$. Then $T_1T_2 = (U \oplus I)(I \oplus U^*) = U \oplus U^* = T_2T_1$, $i(T_1) = i(U \oplus I) = i(U) + i(I) = -1 + 0 = -1$, and $i(T_2) = i(I \oplus U^*) = i(I) + i(U^*) = 0 + 1 = 1$. So T_1 and T_2 are not Weyl. But $i(T_1T_2) = i(U \oplus U^*) = i(U) + i(U^*) = -1 + 1 = 0$, so T_1T_2 is Weyl.

Theorem 6.8. *If T is M-hyponormal and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

Proof. Suppose that p is any polynomial. Let $p(T) - \lambda I = a_0(T - \mu_1 I)\cdots(T-$

$\mu_n I$). Since T is M-hyponormal, $T - \mu_i I$, $i = 1, 2, \cdots, n$ are commuting M-hyponormal operators. Thus

$$\begin{aligned}
\lambda \notin \omega(p(T)) &\Longleftrightarrow p(T) - \lambda I = \text{ Weyl} \\
&\Longleftrightarrow a_0(T - \mu_1 I) \cdots (T - \mu_n I) = \text{ Weyl} \\
&\Longleftrightarrow T - \mu_i I = \text{ Weyl for each } i = 1, 2, \cdots, n \\
&\Longleftrightarrow \mu_i \notin \omega(T) \text{ for each } i = 1, 2, \cdots, n \\
&\Longleftrightarrow \lambda \notin p(\omega(T)),
\end{aligned}$$

which says that $\omega(p(T)) = p(\omega(T))$. If f is analytic on a neighborhood of $\sigma(T)$, then by Runge's theorem ([6]), there is a sequence $\{p_n\}$ of polynomials such that $p_n \to f$ uniformly on $\sigma(T)$. Since $p_n(T)$ commutes with $f(T)$, by Corollary 5.9, $f(\omega(T)) = \lim p_n(\omega(T)) = \lim \omega(p_n(T)) = \omega(f(T))$. □

Corollary 6.9. *If T is hyponormal and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

We say that *Weyl's theorem holds for T* if $\omega(T) = \sigma(T) - \pi_{00}(T)$. There are several classes of operators including normal and hyponormal operators for which Weyl's theorem holds. Oberai has raised the following question. Does there exist a hyponormal operator T such that Weyl's theorem does not hold for T^2 ? Note that T^2 may not be hyponormal even if T is hyponormal(Problem 209 [12]). We will show that Weyl's theorem holds for $p(T)$ when T is hyponormal.

Definition 6.10. *An operator T is called isoloid if isolated points of $\sigma(T)$ are eigenvalues of T.*

Theorem 6.11. *Let $T \in B(H)$ be isoloid. Then for any polynomial $p(t)$, $p(\sigma(T) - \pi_{00}(T)) = p(\sigma(T)) - \pi_{00}(p(T))$.*

Proof. Since $p(\sigma(T)) - \pi_{00}(p(T)) \subset p(\sigma(T) - \pi_{00}(T))$ by Theorem 6.3, we will show that $p(\sigma(T) - \pi_{00}(T)) \subset p(\sigma(T)) - \pi_{00}(p(T))$. Let $\lambda \in p(\sigma(T) - \pi_{00}(T))$. Then there exists $\mu \in \sigma(T) - \pi_{00}(T)$ such that $\lambda = p(\mu)$. Suppose that $\lambda \in \pi_{00}(p(T))$, i.e., λ is an isolated point of $p(\sigma(T)) = \sigma(p(T))$ and an eigenvalue of $p(T)$ of infinite multiplicity. Let $p(x) - \lambda = a_0(x - \mu_1)(x - \mu_2) \cdots (x - \mu_n)$. Then $\mu = \mu_k$ for some k since $\lambda = p(\mu)$. Since $\lambda = p(\mu)$ and $\mu \in \sigma(T) - \pi_{00}(T)$, $\lambda = p(\mu_k)$ where $\mu_k \in \sigma(T) - \pi_{00}(T)$. Hence μ_k must be an isolated point of $\sigma(T)$. In fact, if μ_k is not isolated, then there exists $\{\xi_n\}$ in $\sigma(T)$ such that $\lim \xi_n = \mu_k$. Thus $\lambda = p(\mu_k) = p(\lim \xi_n) = \lim p(\xi_n)$ and $p(\xi_n) \in p(\sigma(T))$. Thus λ is not an isolated point of $p(\sigma(T)) = \sigma(p(T))$. This is a contradiction to the fact $\lambda \in \pi_{00}(p(T))$. Since T is isoloid, μ_k is an eigenvalue of T. Since $\ker(T - \mu_k) \subset \ker(p(T) - \lambda)$ and $\dim\ker(p(T) - \lambda) < \infty$, $\dim\ker(T - \mu_k) < \infty$. Thus $\mu_k \in \pi_{00}(T)$. This contradicts the fact that $\lambda = p(\mu_k) \in p(\sigma(T) - \pi_{00}(T))$. Hence $\lambda = p(\mu_k) \in \pi_{00}(p(T))$ and $\lambda \in \sigma(p(T)) - \pi_{00}(p(T))$. □

Corollary 6.12. *If $T \in B(H)$ is hyponormal, then for any polynomial p on a neighborhood of $\sigma(T)$, Weyl's theorem holds for $p(T)$.*

Proof. By [8], T is isoloid and Weyl's theorem holds for any hyponormal operator. Thus by Corollary 6.9 and Theorem 6.11, $w(p(T)) = p(w(T)) = p(\sigma(T) - \pi_{00}(T)) = \sigma(p(T)) - \pi_{00}(p(T))$. Hence Weyl's theorem holds for $p(T)$. □

Theorem 6.13. ([2]) *If T is a normal operator, then $w(f(T)) = f(w(T))$*

for every continuous complex-valued fuction f on $\sigma(T)$.

Proof. If T is normal, then $\hat{T}$ is normal in $B/\mathcal{K}$. By standard C^*-algebra theory, $f(\hat{T})$ exists and $f(\hat{T}) = \widehat{f(T)}$. If T is normal, then $f(T)$ is normal. We note that if S is normal, $w(S) = \sigma(\hat{S})$. Hence $w(f(T)) = \sigma(\widehat{f(T)}) = \sigma(f(\hat{T})) = f(\sigma(\hat{T})) = f(w(T))$. □

Theorem 6.14. ([21]) *For any $T \in B(H)$ and for any polynomial $p(t)$, $p(\sigma_2(T)) \subsetneq \sigma_2(p(T))$.*

Proof. Let $\lambda \in \sigma_2(T)$ and $p(T) - p(\lambda)I = (T - \lambda I)(T - \lambda_1 I)\cdots(T - \lambda_{n-1} I)$. Since $T - \lambda$ is not semi-Fredholm, $p(T) - p(\lambda)I$ is not semi-Fredholm. Thus $p(\lambda) \in \sigma_2(p(T))$ and so $p(\sigma_2(T)) \subset \sigma_2(p(T))$. Define $S : l_2 \to l_2$ by $S(x_1, x_2, \cdots,) = (0, x_1, 0, x_2, 0, x_3, \cdots)$. Let $T = S \oplus (S^* + 2I)$ and $p(t) = t(t-2)$. Then $p(T) = T(T - 2I) = [S \oplus (S^* + 2I)][(S \oplus (S^* + 2I)) - 2I] = [S \oplus (S^* + 2I)][(S - 2I) \oplus ((S^* + 2I) - 2I)] = [S \oplus (S^* + 2I)][(S - 2I) \oplus S^*]$ and $\ker S^* = R(S)^\perp = \{ (y_1, 0, y_2, 0, y_3, 0, \cdots) : (y_1, y_2, y_3, \cdots) \in l_2 \}$. Thus $\dim \ker S^* = \dim R(S)^\perp = \infty$. Also $\ker[(S - 2I) \oplus S^*)] \subset \ker p(T) = \ker[(S \oplus (S^* + 2I))((S - 2I) \oplus S^*)]$. Note that $(S - 2I)(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_2, 0, x_3, \cdots) - (2x_1, 2x_2, 2x_3, \cdots) = (-2x_1, x_1 - 2x_2, -2x_3, x_2 - 2x_4, \cdots) = (0, 0, 0, \cdots)$ iff $x_1 = 0, x_2 = 0, \cdots$, i.e., $x = 0$. Thus $\ker(S - 2I) = (0)$. Since $S - 2I$ is invertible, $\ker[(S - 2I) \oplus S^*] = \ker S^*$. Thus $\dim \ker[(S - 2I) \oplus S^*] = \infty$ and so $\dim \ker p(T) = \infty$. Also $R(p(T)) = R[(S \oplus (S^* + 2I))((S - 2I) \oplus S^*)] \subset R(S \oplus (S^* + 2I))$ and so $R(S \oplus (S^* + 2I))^\perp \subset R(p(T))^\perp$. Since $R[(S \oplus (S^* + 2I)] \supset R(S \oplus 0) \approx R(S)$, $R(S)^\perp \subset R(S \oplus (S^* + 2I)^\perp$. Since $\dim R(S))^\perp = \infty$, $\dim R(S \oplus (S^* + 2I))^\perp = \infty$. Thus $p(T)$ is not semi Fredholm and so $0 \in \sigma_2(p(T))$. Note that $T = S \oplus (S^* + 2I)$, and

$\ker T = 0$. Also $R(S) = \{ (0, x_1, 0, x_2, 0, x_3, \cdots) \,|\, (x_1, x_2, x_3, \cdots) \in l_2\}$ and $R(S^* + 2I) = \{(x_2 + 2x_1, x_3 + 2x_1, \cdots) \,|\, (x_1, x_2, x_3, \cdots) \in l_2\}$ are closed. Indeed, if $y \in \overline{R(S^* + 2I)}$, then $y = \lim y_n$ where $y_n \in R(S^* + 2I)$. Thus there exists x_n such that $y_n = (S^* + 2I)x_n$ and $y = \lim y_n$. Since $-2 \notin \sigma(S^*)$, $S^* + 2I$ is invertible. Thus $y = \lim y_n = \lim(S^* + 2I)x_n = (S^* + 2I)(\lim x_n)$, and so there exists $\lim x_n = (S^* + 2I)^{-1} y = z$. Thus $y = (S^* + 2I)z$ and so $y \in R(S^* + 2I)$. Therefore $R(T)$ is closed. Thus T is semi-Fredholm and so $0 \notin \sigma_2(T)$. Hence $p(0) = 0 \notin p(\sigma_2(T))$. □

Corollary 6.15. *Let f be a holomorphic function defined on a neighborhood of $\sigma(T)$. Then $f(\sigma_2(T)) \subset \sigma_2(f(T))$. If f is univalent, then $f(\sigma_2(T)) = \sigma_2(f(T))$.*

Proof. Since f is analytic, there exists a sequence $(p_n(t))$ of polynomials such that $\lim p_n(t) = f(t)$ uniformly. Thus by Theorem 5.13 and 6.14, $f(\sigma_2(T)) = \lim p_n(\sigma_2(T)) \subset \limsup \sigma_2(p_n(T)) \subset \sigma_2(f(T))$. Since f is univalent (term for one-to-one), $f^{-1}(\sigma_2(f(T))) \subset \sigma_2(f^{-1}(f(T))) = \sigma_2(T)$, and so $\sigma_2(f(T)) \subset f(\sigma_2(T))$. Hence $\sigma_2(f(T)) = f(\sigma_2(T))$. □

If $\sigma_2(T) = \sigma_3(T)$, then $f(\sigma_2(T)) = \sigma_2(f(T))$ and in this case, $\sigma_2(f(T)) = \sigma_3(f(T))$. Indeed, $\sigma_2(f(T)) \subset \sigma_3(f(T)) \subset f(\sigma_3(T)) = f(\sigma_2(T)) \subset \sigma_2(f(T))$, $\sigma_2(f(T)) = f(\sigma_2(T))$.

Remark 6.16. ([21]) *$p(\sigma_1(T)) \not\subset \sigma_1(p(T))$ and $\sigma_1(p(T)) \not\subset p(\sigma_1(T))$.*

Recall that $\sigma_1(T) = \{\lambda \in \mathbb{C} : R(T - \lambda) \text{ is not closed}\}$. We show that there exists T and S such that for the polynomial $p(t) = t^2$, $p(\sigma_1(T)) \not\subset \sigma_1(p(T))$ and $\sigma_1(p(S)) \not\subset p(\sigma_1(S))$. Define $T : l_2 \to l_2$ by $T(x_1, x_2, \cdots,) = (0, x_1, 0, \frac{x_3}{3}, 0, \frac{x_5}{5}, \cdots,)$. Put $T(1, 0, 0, \cdots) = (0, 1, 0, 0, \cdots,) = y_1$, $T(1, 0, 1,$

$0,\cdots) = (0,1,0,\frac{1}{3},0,\cdots,) = y_2,\cdots$ and $T(1,0,1,\cdots,1,0,\cdots) = (0,1,0,\frac{1}{3}, 0,\frac{1}{5},0,\cdots,0,\frac{1}{2n-1},0,\cdots) = y_n$. Then for all n, $y_n \in l_2$, $y_n \in R(T)$ and $y_n \to (0,1,0,\frac{1}{3},0,\frac{1}{5},0,\cdots,0,\frac{1}{2n-1},\cdots) = y$ as $n \to \infty$. Also $y \in l_2$ since $\|y\|^2 = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} < \infty$ and since $x = (1,0,1,0,\cdots)$, $Tx = y$. But since $\|x\|^2 = \sum_{k=1}^{\infty} 1^2 = \infty$, $x \notin l_2$ and so $y \notin R(T)$. Thus $R(T)$ is not closed. Since $0 \in \sigma_2(T)$, $0 = p(0) \in p(\sigma_2(T))$. But $p(T) = T^2 = 0[\because$ For all $(x_1,x_2,x_3,\cdots) \in l_2$, $T^2(x_1,x_2,x_3,\cdots) = T(0,x_1,0,\frac{x_3}{3},\cdots) = (0,0,\cdots)$]. Since $R(T^2)$=(0) is closed, $0 \notin \sigma_2(T^2) = \sigma_2(p(T))$.

Theorem 6.17. *Let $T \in B(H)$. Then for any polynmial $p(t)$, $p(\sigma_4'(T)) = \sigma_4'(p(T))$.*

Proof. Let $\mu \in \sigma_4'(p(T))$.

Case 1. μ is not an isolated point of $\sigma(p(T)) = p(\sigma(T))$. Then there exists a sequence $\{\lambda_n\}$ in $\sigma(T)$ such that $\mu = \lim p(\lambda_n)$. Thus $\{p(\lambda_n)\}$ is bounded and so $\{\lambda_n\}$ is also bounded. Therefore $\{\lambda_n\}$ has a convergent subsequence $\{\lambda_{n_k}\}$. Let $\lim \lambda_{n_k} = \lambda$. Then $\lambda \in \sigma_4'(T)$. Since $p(\lambda) = p(\lim \lambda_{n_k}) = \lim p(\lambda_{n_k}) = \mu$, $\mu = p(\lambda) \in p(\sigma_4'(T))$.

Case 2. μ is an isolated point of $\sigma(p(T)) = p(\sigma(T))$. Then by definition of $\sigma_4'(T)$, $\mu \in \sigma_3(p(T))$, i.e., $p(T) - \mu = (T-\lambda_1)(T-\lambda_2)\cdots(T-\lambda_n)$ is not Fredholm. Then for some k, $T - \lambda_k$ is not Fredholm. Thus $\lambda_k \in \sigma_3(T)$ and so $\mu = p(\lambda_k) \in p(\sigma_3(T)) \subset p(\sigma_4'(T))$ since $\sigma_3(T) \subset \sigma_4'(T)$.

By case 1 and 2, $\sigma_4'(p(T)) \subset p(\sigma_4'(T))$.

Let $\lambda \in \sigma_4'(T)$. If λ is not an isolated point of $\sigma(T)$, then $p(\lambda)$ is also not an isolated point of $\sigma(p(T))$. $p(\lambda) \in \sigma_4'(p(T))$. If λ is an isolated point of $\sigma(T)$, then $\lambda \in \sigma_3(T)$, i.e., $T - \lambda$ is not Fredholm and also $p(T) - p(\lambda I) = (T-\lambda_1)(T-\lambda_2)\cdots(T-\lambda_n)$ is not Fredholm. Note that if $TS = ST$ and T is not Fredholm, then TS is not Fredholm. Hence $p(\lambda) \in \sigma_3(p(T)) \subset \sigma_4'(p(T))$ and so $p(\sigma_4'(T)) \subset \sigma_4'(p(T))$. □

Theorem 6.18. *If T is isoloid and $\pi_0(T) = \pi_{0f}(T)$, then for every polynomial $p(t)$, $\pi_{00}(p(T)) = p(\pi_{00}(T))$.*

Proof. Let $\lambda \in \pi_{00}(p(T))$. Then we note that

(1) λ is an isolated point of $\sigma(p(T)) = p(\sigma(T))$ and

(2) $0 < \dim \ker(p(T) - \lambda) < \infty$.

Thus $\lambda = p(\mu)$, $\mu \in \sigma(T)$ and μ is also an isolated point of $\sigma(T)$. Indeed, if μ is not an isolated point, then there exists a sequence $\{\mu_n\}$ in $\sigma(T)$ such that $\lim \mu_n = \mu$. Thus $\lambda = p(\mu) = p(\lim \mu_n) = \lim p(\mu_n)$. Since $p(\mu_n) \in \sigma(p(T))$, this is a contradiction to (1). Since T is isoloid, μ is an eigenvalue of T. Let $p(T) - \lambda I = (T - \mu)(T - \mu_1) \cdots (T - \mu_{n-1})$. Then $\ker(T - \mu) \subset \ker(p(T) - \lambda)$. By (2), $\dim \ker(T - \mu) < \infty$ and so $\mu \in \pi_{oo}(T)$, i.e., $\lambda = p(\mu) \in p(\pi_{00}(T))$. Therefore $\pi_{00}(p(T)) \subset p(\pi_{00}(T))$.

Conversely, let $\lambda \in p(\pi_{00}(T)) \subset p(\pi_0(T))$. Since $\pi_0(p(T)) = p(\pi_0(T))$, $\lambda \in \pi_0(p(T))$ and so $\ker(p(T) - \lambda) \neq (0)$. Let $p(T) - \lambda I = (T - \mu_1)(T - \mu_2) \cdots (T - \mu_n)$. Then since $\ker(p(T) - \lambda I) = \bigcup_{j=1}^{n} \ker(T - \mu_j)$, $\ker(T - \mu_k) \neq (0)$ for some k. Note that if $\ker(T - \mu_k) = (0)$, $\mu_k \notin \pi_o(T)$. Without loss of generality, we can assume that $\ker(T - \mu_j) \neq (0)$ for all $j = 1, 2, \cdots, n$. Since $\pi_0(T) = \pi_{0f}(T)$, $\dim \ker(T - \mu_j) < \infty$ for all $j = 1, 2, \cdots, n$ and so $\dim \ker(p(T) - \lambda I) < \infty$. Thus $\lambda \in \pi_{of}(p(T))$. Since $\lambda \in p(\pi_{00}(T))$ and $\lambda = p(\mu_j)$ where $\mu_j \in \pi_o(T)$ for all $j = 1, 2, \cdots, n$,

$$\mu_j \in \pi_{00}(T) \quad \text{for all } j = 1, 2, \cdots, n. \tag{6.1}$$

If λ is not an isolated point of $\sigma(p(T))$, then there exists $\{\lambda_n\}$ in $\sigma(p(T))$ such that $\lim \lambda_n = \lambda$. Let $\lambda_n = p(\mu_n)$ where $\mu_n \in \sigma(T)$. Thus $\lambda = \lim p(\mu_n) = p(\lim \mu_n)$ and so $\lim \mu_n$ is a solution of $p(t) - \lambda = 0$. Hence $\lim \mu_n = \mu_k$ for some k and so μ_k is not an isolated point of $\sigma(T)$. This is a contradiction to (6.1) and so λ is an isolated point of $\sigma(p(T))$. Hence $\lambda \in \pi_{00}(p(T))$. □

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< Abstract>

On Fredholm operator and Weyl spcetrum

In this thesis, we deal with the Weyl spectrum $w(T)$ of a bounded linear operator T on an infinite dimensional Hilbert space H. The followings are the main results of this thesis.

(1) We show that the set $w(T)-\sigma_e(T)$ is an open subset and is a subset of $\mathrm{acc}\,\sigma(T)$ where $\sigma_e(T)$ denotes the essential spectrum of T and $\mathrm{acc}\,K$ denotes the set of all accumulation points of K. Also we show that the boundary of the Weyl spectrum is a subset of the essential spectrum.

(2) We give some sufficient conditions on which the Weyl spectrum and the essential spectrum are equal.

(3) We define the spectral radius and prove that the Weyl spectral radius of an operator is upper semicontinuous. Also we give a different proof of upper semicontinuity of the Browder spectrum.

(4) We show that the spectral mapping theorem $w(f(T)) = f(w(T))$ holds for any $M-$hyponormal operator T and any analytic function f on a neighborhood of $\sigma(T)$ and that Weyl's theorem can be extended to $p(T)$ for any polynomial p and for any hypornormal operator T which is an answer for an old question of Oberai.

감 사 의 글

우선 본 논문이 나올 수 있게 처음부터 끝까지 세심한 지도를 아끼지 않으시고, 용기를 북돋워 주신 양영오 교수님께 감사드립니다. 대학원 과정 동안 틈틈이 격려와 조언을 해주신 송석준 교수님, 방은숙 교수님, 양성호 교수님, 김철수 교수님께 고마움을 전합니다. 그리고 학문적 기초를 넓히게 좋은 강의를 해주신 현진오 교수님, 고봉수 교수님, 고윤희 교수님, 정승달 교수님, 그리고 학업에 열중할 수 있게 배려를 해주신 김도현 교수님과 김한일 교수님께 고마운 마음을 전합니다. 바쁘신 와중에도 세심하게 논문교정을 보아주신 윤용식 교수님께도 고마움을 전합니다.

대학원 4학기 동안 서로 의지하고 어려운 일에 협조를 아끼지 않은 김인영,최희봉, 그리고 김희선 동기들에게 고마운 마음을 전합니다. 주위에서 따뜻한 마음으로 저를 지켜봐준 여러분들께도 아울러 고마운 마음을 전합니다.

많은 어려움에도 불구하고 무사히 대학원을 다닐 수 있게 항상 옆에서 따뜻한 말씀을 해주신 어머니와 언제나 믿음직한 동생 현우와 귀여운 항우에게 사랑을 전하며, 이 기쁨을 같이 하고자 합니다.

1995년 12월