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On Comparison of Maximal Column Rank of Anti-Negative Matrices

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指導敎授　宋 錫 準

金 希 宣

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On Comparison of Maximal Column Rank of Anti-Negative Matrices

Hui-Seon Kim
(Supervised by professor Seok-Zun Song)

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<국문초록>

非陰 行列上의 極大 列階數들의 比較에 관한 硏究

주어진 行列에 대해 一次 獨立關係에 있는 列들의 최대 개수를 極大 列階數라 한다. 이 極大列階數는 일반적인 列階數와 體 위에서는 의미가 같아지나, 半環상의 行列에서는 이 두 槪念이 서로 다르다.

앞선 연구자들의 연구에서 列階數에 관한 연구가 있었다. 本 연구는 앞선 연구를 참조하여 極大列階數들을 비교 분석하였다. 곧 包含關係에 있는 두 半環에 대하여 작은 半環상의 行列의 極大列階數는 일반적으로 큰 半環上의 행렬들의 極大 列階數보다 같거나 크다. 또한, 이 論文은 서로 관련된 非陰半環上의 極大 列階數들의 값이 서로 같아지는 때의 값의 최대값을 구하는 연구로 이들을 비교 분석하였다. 그 결과로서 連結고리 半環과 일반적인 부울대수에서는 極大 列階數 값이 항상 같았고, 非陰의 실수에서는 최대 3 까지 極大 列階數 값이 같고 그 이상에서는 같지않음을 밝혔다.

CONTENTS

I. Introduction

Suppose that $\mathbb{A}$ is a field and let $\mathcal{M}_{m,n}(\mathbb{A})$ denote the set of $m \times n$ matrices with entries in $\mathbb{A}$. If $B \in \mathcal{M}_{m,n}(\mathbb{A})$, recall that the column rank of B is the column space of B while the factor rank of B is the smallest integer k such that B can be factored as $B = XY$ where $X \in \mathcal{M}_{m,n}(\mathbb{A})$ and $Y \in \mathcal{M}_{m,n}(\mathbb{A})$. It is well known that the column rank is the factor rank in this situation.

However, we can also consider matrices whose entries come from another kind of an algebraic system, such as a semiring or Boolean algebra. In this different context, the notions of column rank and factor rank of a matrix can be still defined, but the two ranks do not necessarily agree. Indeed, Beasley and Pullman [3] compared the column rank and the factor rank for matrices over anti-negative semirings, Boolean algebra and chain semirings and found that except small value of m and n, the two ranks did not agree in general. Later Beasley and Song [7] extended this study to include real rank where the algebraic system was a subsemiring of the real numbers. On the other hand, the maximal column rank was introduced in [9], by S. G. Hwang, S. J. Kim and S. Z. Song at 1994. For a matrix A over a semirings, the maximal column rank of a matrix A is the number of the largest linearly independent columns over the given semiring. Then we know that the maximal column rank of A is larger than or equal to the column rank of A. For any matrix, the two ranks are different in general.

In this thesis, we will continue the study of factor rank, column rank and the maximal column rank, but instead of fixing the algebraic system and comparing the three ranks, we will fix the type of rank and compare its values when the matrix is considered over different algebraic systems. Recently, Beasley, Kirkland and Shader [6] proved the comparisons of factor rank or column rank between various semiring and their extended semirings. We will correct the mistakes of the above results for factor rank and column rank, and show the new result for the maximal column rank. Now, given semirings $\mathbb{K}$ and $\mathbb{L}$, suppose that A is a matrix which can be considered either as a matrix over $\mathbb{K}$ or as a matrix over $\mathbb{L}$. Under what circumstances is factor rank, column rank and maximal column rank over $\mathbb{K}$ equal to the factor rank, column rank and maximal column rank respectively over $\mathbb{L}$? Less than? Is there any relationship ? We will investigate these questions for several well studied semirings, including the reals, the integers module a, finitly generated Boolean algebra, and chain semirings.

In chapter II, we give necessary definitions and preliminary results. In chapter III, IV and V, we will investigate some general inequalities for the factor rank, column rank and maximal column rank, respectively of any matrix between the comparable semirings. In particular, each section will establish some equality cases for rank functions.

We shall adopt the convention that for $\mathcal{M}_{m,n}(\mathbb{S})$, $\mathbb{S}$ denotes a semiring, and we assume $m \leq n$ in this thesis.

II. Definition and Preliminaries

A *semiring* is algebraic system which satisfies all the axioms of a ring with identity except that not all elements need have an additive inverse. Many combinatorially interesting semirings have the property that zero is the only element with an additive inverse. These semirings are called ***anti-negative semirings***. That is, in such a semiring $\mathbb{S}$, if $x + y = 0$ for $x, y \in \mathbb{S}$ then $x = y = 0$. Examples of anti-negative are reals, $\mathbf{R}^+$ and the nonnegative rationals, $\mathbf{Q}^+$.

The *Boolean algebra* of subsets of a k- set, denoted $\mathbf{B}_k$, is also an anti-negative semiring, where addition corresponds to set union and multiplication corresponds to set intersection. In particular, we assume that $\mathbf{B}_k$ is the set $0, 1$, where arithmetic in $\mathbf{B}_k$ is the same usual rules except that $1 + 1 = 1$. In the sequl, we will often want to consider $\mathbf{B}_k$ to be a subsemiring of $\mathbf{B}_j$ when $k \leq j$. This is easily accomplished by considering the j -set for $\mathbf{B}_j$ to be $\{a_1, a_2, \cdots, a_n\}$ and then associating $\mathbf{B}_k$ with the isomorphic subsemiring of $\mathbf{B}_j$ consisting of the set of all unions and intersections of $\{a_1\}, \{a_2\}, \cdots, \{a_{k-1}\}$, and $\{a_k, \cdots, a_j\}$. Henceforth we will assume that $\mathbf{B}_k$ is a subsemiring of $\mathbf{B}_j$ whenever $j \leq k$.

Let $\mathbf{Z}$ be a set of two or more elements which is totally ordered by $\leq$. Further, suppose that $\mathbf{Z}$ contans both a universal lower bound and a universal upper bound. If for each $x, y \in \mathbf{Z}$ we define addition and multiplication by $x + y = \max(x, y)$ and $xy = \min(x, y)$, then the resulting algegraic structure is a *chain semiring*. In particular, the chain semiring generated by the numbers in the interval $[0, 1]$ is denoted by $\mathbf{F}$. Evidently, a chain semiring is another example of an anti-negative semiring, and as above for the Boolean semirings, a chain semiring that is a subset of another may be considered a subsemiring by appeending the zero and identity of the larger to the smaller. Henceforth we will assume that a chain semiring that is a subset of another is a subsemiring.

Given any semiring $\mathbb{S}$ we denote the set of $m \times n$ matrices with entries in $\mathbb{S}$ by $\mathcal{M}_{m,n}(\mathbb{S})$. Addition of vectors ($m \times 1$ matrices), addition and multiplication of matrices, and scalar multiplication are defined as if $\mathbb{S}$ were a field. A set of vectors is a *semimodule* if it is closed under addition and scalar multiplication (others, including Beasley and Pullman refer to such a set as a ***vector space***). A subset $\mathcal{W}$ of a semimodule $\mathcal{V}$ is a spanning set if each vector in $\mathcal{V}$ can be written as a sum of scalar multiples (i.e. a linear combination) of elements of $\mathcal{W}$.

As for semirings $\mathbb{S}$ we can define two notions of rank for a matrix $A \in \mathcal{M}_{m,n}(\mathbb{S})$. The column space of a matrix A is the semimodule spanned by the columns of A. Since the column space is spanned by a finite set of vectors, it contains a spanning set of minimum cardinality ; the cardinality is the column rank of A, $\chi_S(A)$ (others, including Beasley and Pullman refer to a spanning set of the column space, and to the cardinality of a basis as the dimension of the column space). The factor rank of A, $\phi_S(A)$, is the minimum integer k such that A can be factored as $A = BC$, where $B \in \mathcal{M}_{m,n}(\mathbb{S})$ and $C \in \mathcal{M}_{m,n}(\mathbb{S})$. The maximal column rank of A, $\psi_S(A)$ is the number of the largest linearly independent columns of A over $\mathbb{S}$.

Our goal here is to compare the values of $\psi_S(A)$ as $\mathbb{S}$ varies over some familiar semirings such as $\mathbf{R}$, $\mathbf{R}^+$, $\mathbf{Z}$, $\mathbf{Z}^+$, $\mathbf{Z}_a$, and $\mathbf{B}_k$. First, let's begin seeing some example which column, maximal column and factor ranks are different one another, and necessary basic proposition.

Example 2.1. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then A is considered as a matrix over $\mathbf{R}$, the real numbers, or as a matrix over $\mathbf{Z}_2$, the integer module 2. Considered as a matrix over $\mathbf{R}$, the rank (column rank and factor rank) of A is three, while considered as a matrix over $\mathbf{Z}_2$, the rank of A is two (the third column is the sum of the first two and note $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$).

Example 2.2. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then A can be considered as being in any of $\mathcal{M}_{2,2}(\mathbf{R})$, $\mathcal{M}_{2,2}(\mathbf{R}^+)$, $\mathcal{M}_{2,2}(\mathbf{Z}^+)$, or $\mathcal{M}_{2,2}(\mathbf{Z}_a)$ for any $a \geq 4$. We have $\phi_S(A) = 2$ if $\mathbb{S}$ is $\mathbf{R}$, $\mathbf{R}^+$, $\mathbf{Z}$, $\mathbf{Z}^+$, or $\mathbf{Z}_a$ if a is relatively prime to 6. But $\phi_{Z_6}(A)$ since over $\mathbf{Z}_6$,

$$A = \begin{bmatrix} 2 \\ 3 \end{bmatrix} [4 \quad 3].$$

Note that $\chi_{Z^+}(A) = 1$.

Example 2.3. Let a semiring $\mathbb{S}$ be $\mathbf{Z}^+$ and $A = [1 \quad 2 \quad 3 \quad 4 \quad 5]$. Then the column rank of A, $\chi_{Z^+}(A)$ is 1 since the first column generates the others. But the maximal column rank of A, $\psi_{Z^+}(A)$ is 3 since the last three columns are the largest linearly independent columns of A.

Example 2.4. Let a semiring $\mathbb{S}$ be $\mathbf{R}^+$ and consider the matrix

$$A = \begin{bmatrix} 0 & 4 & 1 & 8 \\ 1 & 3 & 1 & 7 \\ 2 & 2 & 1 & 6 \end{bmatrix}.$$

Then the column rank of A, $\chi_{R^+}(A)$ is two since $\frac{\mathbf{a}_1+\mathbf{a}_2}{2} = \mathbf{a}_3$ and $\mathbf{a}_1 + 2\mathbf{a}_2 = \mathbf{a}_4$, while the maximal column rank of A, $\psi_{R^+}(A)$ is three since the columns $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4$ are the largest linearly independent columns of A (i.e. $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ is the linearly independent set and $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ is the linearly dependent set).

The following is the proposition which shall be used in each chapter.

Proposition 2.5. *Suppose that* $\mathbb{S}$ *is a semiring and* A *is* $p \times q$ *matrix over* $\mathbb{S}$. *If* $A = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$, *where the zero block on the diagonal has arbitrary dimensions, then*

(1) $\phi_S(A) = \phi_S(B)$.
(2) $\chi_S(A) = \chi_S(B)$.
(3) $\psi_S(A) = \psi_S(B)$.

Proof. Case (1). $\Rightarrow$) Let $\phi_S(A) = k$. Then for some $X \in \mathcal{M}_{p,k}(\mathbb{S})$ and $Y \in \mathcal{M}_{k,q}(\mathbb{S})$, $A = XY$. If B is a $m \times n$ martrix with $m < p$ and $n < q$, then we claim that X is the form $\begin{bmatrix} X_0 \\ 0 \end{bmatrix}$ and Y is the form $[Y_0 \quad 0]$ where X_0 is the $m \times k$ matrix, Y_0 is the $k \times n$ matrix and 0 is the zero block. If not, we may assume that X_0 is $i \times k$ matrix for some $i > m$. Then $A = [X_0 \quad 0][Y_0 \quad 0] = \begin{bmatrix} X_0Y_0 & 0 \\ 0 & 0 \end{bmatrix}$ and X_0Y_0 is a $i \times n$ matrix for some $i > m$. That is, the nonzero block of A is a $i \times n$ matrix for $i > m$. This is a contradiction to the fact that B is the nonzero block of A, as $m \times n$ matrix. For $i < m$, it is similar. Therefore $B = X_0Y_0$ for $X_0 \in \mathcal{M}_{m,k}(\mathbb{S})$ and $Y_0 \in \mathcal{M}_{k,n}(\mathbb{S})$, and so $\phi_S(B) \leq k = \phi_S(A)$.

$\Leftarrow$) Suppose that $\phi_S(B) = k$ and $B \in \mathcal{M}_{m,n}(\mathbb{S})$. Then for some $Z \in$

$\mathcal{M}_{m,k}(\mathbb{S})$ and $W \in \mathcal{M}_{k,n}(\mathbb{S})$, $B = ZW$. Since

$$A = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ZW & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Z \\ 0 \end{bmatrix} [\, W \quad 0 \,],$$

let $X = [\, Z \quad 0 \,]$ and let $Y = [\, W \quad 0 \,]$. Then $X \in \mathcal{M}_{p,k}(\mathbb{S})$ and $Y \in \mathcal{M}_{k,q}(\mathbb{S})$. Therefore $\phi_S(A) \leq k = \phi_S(B)$ by definition.

Case (2). The minimum cardinality of column space of A and that of B are the same since zero columns do not act on the cardinality of column space.

Case (3). The number of largest linearly independent columns of A is equal to that of B as the case 2. □

III. The Comparisons of Factor Rank

In this section, we establish some general theorems about the factor rank of matrices whose entries lie in two related semirings.

3.1. Factor Rank Inequalities

Suppose that $\mathbb{K}$ and $\mathbb{L}$ are semirings and that $\xi : \mathbb{K} \to \mathbb{L}$ is a semiring homomorphism. We identify an $m \times n$ matrix $A = [a_{ij}]$ whose entries lie in $\mathbf{K}$, with the $m \times n$ matrix $\Xi(A)$ whose (i, j) th entry equals $\xi(a_{ij})$. Thus $\Xi : \mathcal{M}_{m,n}(\mathbf{K}) \to \mathcal{M}_{m,n}(\mathbb{L})$ and any matrix, $A \in \mathcal{M}_{m,n}(\mathbb{K})$ can be viewed as a matrix $\Xi(A) \in \mathcal{M}_{m,n}(\mathbb{L})$.

Theorem 3.1.1. *Let $\mathbb{K}$ and $\mathbb{L}$ be semirings and $\xi : \mathbb{K} \to \mathbb{L}$ be semiring homomorphism. Then, $\phi_K(A) \geq \phi_L(\Xi(A))$ for every matrix $A \in \mathcal{M}_{m,n}(\mathbb{K})$.*

Proof. Let $A \in \mathcal{M}_{m,n}(\mathbb{K})$ with $\phi_K(A) = k$. Then there exist matrices $B \in \mathcal{M}_{k,n}(\mathbb{K})$ and $C \in \mathcal{M}_{k,n}(\mathbb{K})$ satisfying $A = BC$. Since ξ is a semiring homomorphism,

$$\begin{aligned}\Xi(B)\Xi(C) &= \left[\textstyle\sum_{r=1}^{k} \xi(b_{ir})\xi(c_{rj})\right] = \left[(\textstyle\sum_{r=1}^{k} \xi(b_{ir}c_{rj}))\right] \\ &= \left[\xi(\textstyle\sum_{r=1}^{k} b_{ir}c_{rj})\right] = \Xi(BC) = \Xi(A).\end{aligned}$$

Hence the proof is complete. □

If $\mathbb{K}$ is a subsemiring of $\mathbb{L}$, then the canonical injection of $\mathbb{K}$ into $\mathbb{L}$ is a homomorphism, and hence by Theorem 3.1.1, $\phi_K(A) \geq \phi_L(A)$ for each matrix $A \in \mathcal{M}_{m,n}(\mathbb{K})$. In this case, we abbrebiate the above to $\phi_K \geq \phi_L$.

Corollary 3.1.2. *If $\mathbb{K}$ is a subsemiring of $\mathbb{L}$, then $\phi_K \geq \phi_L$. In particular,*

(1) *if $j \geq k$ then $\phi_{B_k} \geq \phi_{B_j}$.*

(2) *$\phi_{Z^+} \geq \phi_Z$, $\phi_Z \geq \phi_P$, and $\phi_P \geq \phi_R$ for any subring ,$\mathbf{P}$, of the reals with identity.*

(3) *$\phi_{Z^+} \geq \phi_{P^+}$ and $\phi_{P^+} \geq \phi_{R^+}$ for any subsemiring with identity ,$\mathbf{P}^+$, of $\mathbf{R}^+$.*

Let $A = [a_{ij}]$ be an $m \times n$ matrix whose entries belong to a semiring $\mathbb{S}$. We define the pattern matrix of A to be the $m \times n$ matrix $\bar{A} = [\bar{a}_{ij}]$ where $\bar{a}_{ij} = 0$ if $a_{ij} = 0$ and $\bar{a}_{ij}$ equals to the multiplicative identity of $\mathbb{S}$ otherwise.

Corollary 3.1.3. *Let $\mathbb{S}$ be an anti-negative semiring with multiplicative identity and let $\mathbb{S}$ have no zero divisor. Then $\phi_{B_1}(A) \leq \phi_S(A)$ for all matrices $A \in \mathcal{M}_{m,n}(\mathbb{S})$. In particular, if A is any (0, 1) matrix, then $\phi_{B_1}(A) \leq \phi_S(A)$.*

Proof. Define the map $\xi : \mathbb{S} \to \mathbf{B}_1$ by

$$\xi(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a \neq 0. \end{cases}$$

Then $\xi(a + b) = \xi(a) + \xi(b)$ for all $a, b \in \mathbb{S}$. If a and b are all zero then $\xi(a + b) = 0 = \xi(a) + \xi(b)$. If one of two is zero and the other is not, then $\xi(a + b) = 1 = \xi(a) + \xi(b)$. And if neither a nor b is zero then $\xi(a) + \xi(b) = 1 + 1 = 1$ over $\mathbf{B}_k$. Thus $\xi(a + b) = \xi(a + b) = \xi(a) + \xi(b)$. Futhermore, $\xi(ab) = \xi(a)\xi(b) = \xi(a)\xi(b)$ in the similar manner. Therefore ξ is a semiring homorphism and so from Theorem 3.1, $\phi_{B_1}(A) \leq \phi_S(A)$. □

Corollary 3.1.4. *Let a and b be integers with $a, b \geq 0$. Suppose A is a matrix with entries in $\{0, 1, 2, \cdots, a-1\}$. Then $\phi_{Z_a}(A) \leq \phi_{Z_{ab}}(A) \leq \phi_Z(A)$.*

Proof. Define the map $\xi : \mathbf{Z} \to \mathbf{Z}_{ab}$ by $\xi(n) = k$ with $n = k(\text{mod } ab)$. Then the map ξ is a homomorphism of ring into factor ring. Similarly the map $\zeta : \mathbf{Z}_{ab} \to Z_a$ defind by $\zeta(k) = p$ with $k = p(\text{mod } a)$. Then ξ is a homomorphism of ring into factor ring. □

3.2. The Case of Equality in Factor Rank

It turns out that equality holds in certain cases.

Theorem 3.2.1. *Suppose that* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are chain semirings and that* $\mathbf{C}_1$ *is a subsemiring of* $\mathbf{C}_2$. *If* $A \in \mathcal{M}_{m,n}(\mathbf{C}_1)$, *then* $\phi_{C_1}(A) = \phi_{C_2}(A)$.

Proof. Since $\mathbf{C}_1 \subset \mathbf{C}_2$, we have $\phi_{C_1}(A) \geq \phi_{C_2}(A)$ by Corollary 3.1.2. Let $\mathbf{C}(A)$ be the chain semirimg consisting of 0, 1 and the entries in A (i.e., 0 and 1 are the additive and the multiplicative identities of $\mathbf{C}(A)$, respectively). If A can be factored as $A = BC$ where both B and C have their entries in $\mathbf{C}(A)$, then B is also in $\mathcal{M}_{m,k}(\mathbf{C}_1)$ and C is also in $\mathcal{M}_{k,n}(\mathbf{C}_1)$ since $\mathbf{C}(A)$ is a subsemiring of $\mathbf{C}_1$. It follows that $\phi_{C(A)}(A) \geq \phi_{C_1}(A)$. Let the map $\xi : \mathbf{C}_2 \to \mathbf{C}(A)$ given by $\xi(x) = \sum_{y \in S(x)} y$ where $S(x) = \{y \in \mathbf{C}(A) : y \leq x\}$. If $x_1, x_2 \in \mathbf{C}_2$ with $x_1 \leq x_2$, then we find that $S(x_1) \subseteq S(x_2)$. Hence $\xi(x_1) \leq \xi(x_2)$ by the definition of chain semiring. Therefore we will easily prove that ξ is a homomorphism. Note that if $x_1, x_2 \in \mathbf{C}_2$ then $x_1 + x_2 \equiv \max(x_1, x_2) = x_2$. Also $\xi(x_1) \leq \xi(x_2)$ implies $\xi(x_1) + \xi(x_2) \equiv \max(\xi(x_1), \xi(x_2)) = \xi(x_2)$. Thus $\xi(x_1 + x_2) = \xi(x_2) = \xi(x_1) + \xi(x_2)$. Also for the above x_1, x_2, note that $x_1 \cdot x_2 \equiv \min(x_1, x_2) = x_1$ and $\xi(x_1) \cdot \xi(x_2) \equiv \min(\xi(x_1), \xi(x_2)) = \xi(x_1)$. Thus $\xi(x_1 \cdot x_2) = \xi(x_1) = \xi(x_1) \cdot \xi(x_2)$. That is, therefore for any $x_1, x_2 \in \mathbf{C}_2, \xi(x_1 + x_2) = \xi(x_1) + \xi(x_2)$ and $\xi(x_1 \cdot x_2) = \xi(x_1) \cdot \xi(x_2)$. Hence

ξ is a homomorphism. Evidently $\Xi(A) = A$, since all entries in A are in $\mathbf{C}(A)$. So, by Theorem 3.1.1, $\phi_{C_2}(A) \geq \phi_{C(A)}(A)$. But $\phi_{C_2}(A) \geq \phi_{C_1}(A)$. Hence $\phi_{C_2}(A) \geq \phi_{C_1}(A)$. Since $\mathbf{C}_1$ is a subsemiring of $\mathbf{C}_2$, it follows that $\phi_{C_2}(A) = \phi_{C_2}(A)$. □

Theorem 3.2.2. *Suppose that $j \leq k$, so that $\mathbf{B}_j \subset \mathbf{B}_k$. If $A \in \mathcal{M}_{m,n}(\mathbf{B}_j)$, then $\phi_{B_j}(A) = \phi_{B_k}(A)$.*

Proof. Since $\mathbf{B}_j \subset \mathbf{B}_k$, $\phi_{B_j}(A)$. Suppose that $\mathbf{C}(A)$ be a chain semiring of 0(empty set), 1(univesal set, $\mathbf{B}_k$) and all entries in A. Then in the similar method as above theorem, it is proved that $\phi_{B_j}(A) = \phi_{B_k}(A)$. □

Corollary 3.2.3. *For any (0, 1) matrix A, any chain semiring $\mathbf{C}$ which contains 0 and 1, and any integer with $k \geq 1$, we have $\phi_{B_1}(A) = \phi_{B_k}(A) = \phi_C(A) = \phi_F(A)$.*

Proof. By Theorem 3.2.2, $\phi_{B_1}(A) = \phi_{B_2}(A)$ for any $k \geq 1$. The Boolean algebra $\mathbf{B}_k$ is also a chain semiring $\mathbf{C}$, and in particular, in the fuzzy numbers $\mathbf{F}$. Thus by Theorem 3.2.1, both $\phi_C(A)$ and $\phi_F(A)$ are equal to $\phi_{B_1}(A)$. □

Suppose that $\mathbb{K}$ is a subsemiring of a semiring $\mathbb{L}$. Let $\Phi(\mathbb{K}, \mathbb{L}, m, n)$ denote the maximum integer j such that there exists a matrix in $\mathcal{M}_{m,n}(\mathbb{K})$ with $\phi_K(A) \leq j$, we have $\phi_K(A) = \phi_L(A)$. Note that $\Phi \geq 0$, since for any semiring $\mathbb{S}$, $\phi_S(A) = 0$ if and only if A is the zero matrix. The equalities $\phi_{B_j} = \phi_{B_k}$ for any $j \leq k$ and $\phi_{C_1} = \phi_{C_2}$ for any chain semirings $\mathbf{C}_1$ and $\mathbf{C}_2$ with $\mathbf{C}_1 \subset \mathbf{C}_2$ have been established. Further, from the results in the preceding section, we have that for any matrix $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$,

$$\phi_{B_1} \leq \phi_{R^+}, \tag{3.1}$$

$$\phi_{R^+} \leq \phi_{Z^+}, \tag{3.2}$$

$$\phi_R \leq \phi_{R^+}, \tag{3.3}$$

$$\phi_{Z_a} \leq \phi_{Z_{ab}}, \tag{3.4}$$

for any positive integers a and b, and

$$\phi_{Z_a} \leq \phi_Z, \tag{3.5}$$

for any positive integer a. The equality does not hold in general for any of (3.1)–(3.3). First we will look at (3.1).

Example 3.2.4. Let $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Then $\phi_{B_1}(M) = 2$ since M is evidently not rank 1 and $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (here, the arithmetic is Boolean). However, considered as a real matrix, a straight forward calculation reveals that $\phi_R(M) = 3$. Thus, by inequality (3.3), $\phi_{R^+}(M) = 3$.

The below theorem follows from Example 3.2.4 and Proposition 2.5.

Theorem 3.2.5. *Suppose that* $A \in \mathcal{M}_{m,n}(\mathbf{B}_k)$. *If* $\min(m,n) \leq 2$, *then* $\phi_{B_1}(A) = \phi_{R^+}(A)$. *On the other hand, if* $m, n \geq 3$, *there is a matrix* $M \in \mathcal{M}_{m,n}(\mathbf{R}^+)$ *such that* $\phi_{B_1}(M) < \phi_{R^+}(M)$.

Our next example gives some insight into inequality (3.2).

Example 3.2.6. Let $M = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 1 & 4 \\ 1 & 3 & 9 \end{bmatrix}$. Then $M = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{5}{2} \end{bmatrix}$ and from which it follows that $\phi_{Z^+}(M) \geq 2$, we claim that in fact $\phi_{Z^+}(M) = 3$. To see the claim, suppose that M can be factored over $\mathbf{Z}^+$ as $[\mathbf{v_1}\, \mathbf{v_2}]W$, where W is 2×3. If $W = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$, then $M = [\mathbf{v_1} \quad \mathbf{v_2}]W = [c_{11}\mathbf{v_1} + c_{21}\mathbf{v_2} | c_{12}\mathbf{v_1} + c_{22}\mathbf{v_2} | c_{13}\mathbf{v_2}]$. Thus the vector $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ is a linear combination (over $\mathbf{Z}^+$) of $\mathbf{v_1}$ and $\mathbf{v_2}$. We can suppose, without loss of generality, that $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \geq \mathbf{v}_1$ (where the inequality holds entrywise). Then the first entry of $\mathbf{v}_1$ is 0, so that the first entry of $\mathbf{v}_2$ cannot be 0 since if both $\mathbf{v}_1$ and $\mathbf{v}_2$ has the first zero entry then the first row of M is a zero row, a contradiction. Therefore $\mathbf{v}_1$ and $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ are the same. Since $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is also a linear combination (over Z^+) of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then note that $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 = c_{11}\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + c_{21}\mathbf{v}_2$. Thus $c_{11} = 0$ and it implies that $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. But then we must have $\begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix} = x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ with $x, y \in Z^+$, which is impossible. Thus we see that $\phi_{Z^+}(M) = 3 > \phi_{R^+}(M)$.

The above example will help us establish the following.

Theorem 3.2.7.

$$\Phi(\mathbf{Z}^+, \mathbf{R}^+, m, n) = \begin{cases} 1 & \textit{if } \min(m,n) = 1, \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. Case 1. it is clear that $\phi_{Z^+}(A) = \phi_{R^+}(A) = 1$ or 0. Thus the first case holds.

Case 2. From Example 3.2.6, there exists a 3×3 matrix such that $\phi_{Z^+}(A) = 3$ but $\phi_{R^+}(A) = 2$. Hence $\Phi(\mathbf{Z}^+, \mathbf{R}^+, m, n) \leq 2$. To show that the second case holds, it suffices to show that for $\phi_{R^+}(A) = 1$, $\phi_{Z^+}(A) = 1$ for all m, n. If this is proved then for $\phi_{Z^+}(A) = 2$, $\phi_{R^+}(A) = 2$, since $\phi_{R^+}(A) \leq \phi_{Z^+}(A) = 2$ and $\phi_{R^+}(A) \neq 1$. Then the proof would end. Therefore put $A = \mathbf{u}\mathbf{v}^{\mathbf{t}}$ where $\mathbf{u}, \mathbf{v}$ are vectors with entries in $\mathbf{R}^+$, i.e., $\phi_{R^+}(A) = 1$. Let u_i be a nonzero entry in a vector $\mathbf{u}$, and note that (u_j/u_i) in lowest terms as (p_j/q_j) (i.e., p_j/q_j is a simple fraction). and let d be the least common multiple of the q_j's. For any k and j, we have $a_{ik} = u_j v_k = (p_j/q_j)u_i v_k \in \mathbf{Z}^+$, and we see that q_j divides $u_j v_k$. Consequently, d divides $u_i v_k$ (for any k). Thus $\mathbf{b}^t = (1/d)(u_i \mathbf{v})$ is a vector over $\mathbf{Z}^+$, as is $\mathbf{a} = (d/u_i)\mathbf{u}$ over $\mathbf{Z}^+$, since we notes that $u_j/u_i = p_i/q_j$ and that $\mathbf{a} = (d/u_i)\mathbf{u} = [\frac{d}{u_i}u_j] = d[\frac{p_j}{q_j}]$ and d be the least common multiple of q_j's. Further, $\mathbf{a}\mathbf{b}^t = (\frac{d}{u_i}\mathbf{u})(\frac{u_i}{d}\mathbf{v}) = \mathbf{u}\mathbf{v}^t = A$, so that $\phi_{Z^+}(A) = 1$. Suppose that $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$; if $\phi_R(A) = 1$, then each column of A is a multiple of the first nonzero column of A. Consequently, each column of A is a nonnegative multiple of that column, and hence $\phi_{R^+}(A) = 1$ as well. Thus we have $\Phi(\mathbf{R}^+, \mathbf{R}, m.n) \geq 1$. □

Theorem 3.2.8. *Let* $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$. *If* $\phi_R(A) = 2$ *then* $\phi_{R^+} = (A) = 2$.

Proof. We will show this result using Theorem 4.2.8 in chapter IV. That is, we begin with the assumption that for $\chi_R(A) = 2$, $\chi_{R^+}(A) = 2$. If $\phi_R(A) = 2$, then $\chi_R(A)$, since the column and the factor rank are the the same on field. Further, in general it holds that $\phi_{R^+}(A) \leq \chi_{R^+}(A)$. Thus $\phi_{R^+}(A) \leq 2$. Also $\phi_{R^+}(A) \geq \phi_R(A) = 2$, since $\mathbf{R}^+$ is a subsemiring of $\mathbf{R}$. Therefore

$\phi_{R^+}(A) = 2$. □

Example 3.2.9. Let $M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Then a straight forward calculation shows that $\phi_R(M) = 3$, since $\chi_R(M) = 3$. By Corollary 3.1.3 $\phi_{B_1}(M) \leq \phi_{R^+}(M)$, and note $\phi_{B_1}(M) = 4$. Thus we have $\phi_{R^+}(M) = 4$.

Using Theorem 3.2.8, Example 3.2.9 and Proposition 2.5, the following corollary can be proved.

Corollary 3.2.10. *We have that*

$$\Phi(\mathbf{R}^+, \mathbf{R}, m, n) = \begin{cases} 1 & \textit{if } \min(m,n) = 1, \\ 2 & \textit{if } \min(m,n) = 2, \\ 3 & \textit{otherwise.} \end{cases}$$

IV. The Comparisons of Column Rank

4.1. Column Rank Inequalities

Suppose that $\mathbb{K}$ and $\mathbb{L}$ are semirings and that $\xi : \mathbb{K} \to \mathbb{L}$ is a semiring homomorphism. The map $\Xi : \mathcal{M}_{m,n}(\mathbb{K}) \to \mathcal{M}_{m,n}(\mathbb{L})$ as in chapter III was defined. Note that Ξ is also an homomorphism.

Theorem 4.1.1. *Let $\mathbb{K}$ and $\mathbb{L}$ be semirings and $\xi : \mathbb{K} \to \mathbb{L}$ be a semiring homomorphism. Then $\chi_K(A) \geq \chi_L(A)$ for every matrix $A \in \mathcal{M}_{m,n}(\mathbb{K})$.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ be column vectors in $\mathcal{M}_{m,1}(\mathbb{K})$ which span the column space of A. Since ξ is a homomorphism, $\{\Xi(\mathbf{x}_1), \Xi(\mathbf{x}_2), \cdots, \Xi(\mathbf{x}_k)\}$ spans the column space of $\Xi(A)$, since for $\mathbf{y} \in < A >$ where $< A >=$ the column space of$A$, $\mathbf{y} = r_1\mathbf{x}_1 + r_2\mathbf{x}_2 + \cdots + r_k\mathbf{x}_k$ for $r_i \in \mathbb{K}$. Then $\Xi(\mathbf{y}) = \xi(r_1)\Xi(\mathbf{x}_1) + \cdots \xi(r_k)\Xi(\mathbf{x}_k)$. Hence $\chi_K(A) \geq \chi_L(\Xi(A))$. □

Corollary 4.1.2. *If $\mathbb{K}$ is a subsemiring of $\mathbb{L}$, then $\chi_K(A) \geq \chi_L(A)$. In paricular,*

(1) *if $j \geq k$, then $\chi_{B_k} \geq \chi_{B_j}$,*

(2) *$\chi_{Z^+} \geq \chi_Z$, $\chi_Z \geq \chi_P$ and $\chi_P \geq \chi_R$ for any subring, $\mathbf{P}$, of the reals with identity and*

(3) *$\chi_{Z^+} \geq \chi_{P^+}$ and $\chi_{P^+} \geq \chi_{R^+}$ for any subring with identity, $\mathbf{P}^+$, of $\mathbf{R}^+$.*

Corollary 4.1.3. *Let $\mathbb{S}$ be an anti-negative semiring with multiplicative identity 1. Then $\chi_{B_1}(\bar{A}) \leq \chi_S(A)$ for all matrices $A \in \mathcal{M}_{m,n}(\mathbb{S})$. In particular, if A is any $(0,1)$ matrix, then $\chi_{B_1}(A) \leq \chi_S(A)$.*

Proof. The mapping $\xi : \mathbb{S} \to \mathbf{B}_1$ defined by

$$\xi(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a \neq 0 \end{cases}$$

is a semiring homomorphism. The proof is similar to Corollary 3.1.3. □

Corollary 4.1.4. *Let a and b be intgers with $a, b \geq 2$. Suppose thet A is a matrix with entries in $\{0, 1, 2, \cdots, a-1\}$. Then $\chi_{Z_a}(A) \leq \chi_{Z_{ab}}(A) \leq \chi_Z(A)$.*

Proof. The mapping $\eta : \mathbf{Z} \to \mathbf{Z}_{ab}$ and $\zeta : \mathbf{Z}_{ab} \to \mathbf{Z}_a$ of a ring into a factor ring are homomorphisms. □

4.2. The Case of Equality in Column Rank

Theorem 4.2.1. *Suppose that $\mathbf{C}_1$ and $\mathbf{C}_2$ are chain semirings and that $\mathbf{C}_1$ is a subsemiring of $\mathbf{C}_2$. If $A \in \mathcal{M}_{m,n}(\mathbf{C}_1)$, then $\chi_{C_1}(A) = \chi_{C_2}(A)$.*

Proof. The proof is similar to the proof of Theorem 3.2.1. □

Theorem 4.2.2. *Suppose that $j \geq k$, so that $\mathbf{B}_j \subset \mathbf{B}_k$. If $A \in \mathcal{M}_{m,n}(\mathbf{B}_j)$, then $\chi_{B_j}(A) = \chi_{B_k}(A)$.*

Proof. Since $\mathbf{B}_j$ and $\mathbf{B}_k$ are chain semirings which $\mathbf{B}_j$ is a subsemiring of $\mathbf{B}_k$, from above theorem, $\chi_{B_j}(A) = \chi_{B_k}(A)$. □

Corollary 4.2.3. *For any (0,1) matrix A, any chain semiring $\mathbf{C}$ which contains 0 and 1, and any integer k with $k \geq 1$, we have $\chi_{B_1}(A) = \chi_{B_k}(A) = \chi_C(A) = \chi_F(A)$.*

Suppose that $\mathbb{K}$ is a subsemiring of $\mathbb{L}$ (or that there exists a semiring homomorphism from $\mathbb{K}$ into $\mathbb{L}$).

Let $X(\mathbb{K}, \mathbb{L}, m, n)$ denote the maximum integer j such that there exists a matrix in $\mathcal{M}_{m,n}(\mathbb{K})$ with $\chi_K(A) \leq j$ we have $\chi_K(A) = \chi_L(A)$. Note that $X \geq 0$, since for any semiring $\mathbb{S}$, $\chi_S(A) = 0$ if and only if A is the zero matrix. Using this notation, we have established the following equalities $\chi_{B_j} = \chi_{B_k}$ for any $j \leq k$, and $\chi_{C_1} = \chi_{C_2}$ for any chain semirings $\mathbf{C}_1$ and $\mathbf{C}_2$ with $\mathbf{C}_1 \subset \mathbf{C}_2$. Further, from the results in the preceding section, we have shown that for any matrix $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$,

$$\chi_{B_1}(\bar{A}) \leq \chi_{R^+}(A) \tag{4.1}$$

and that

$$\chi_R \leq \chi_{R^+}, \tag{4.2}$$

$$\chi_{R^+} \leq \chi_{Z^+}. \tag{4.3}$$

We will show that equality does not hold in general for any of (4.1)–(4.3). Our approach will be to investigate the values of $X(\mathbb{K}, \mathbb{L}, m, n)$ for appropriate semirings $\mathbb{K}$ and $\mathbb{L}$.

Example 4.2.4. Let $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. The Boolean column space of M is spanned by the vectors $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, so that $\chi_{B_1}(M) = 2$, by definition of column rank, while $\chi_{B_1}(M) = 3$. Consequenty, M provides us with an example to show that strict inequality can hold in part of (1).

The theorem below follows from from Example 4.2.4 and Proposition 2.5.

Theorem 4.2.5. *Suppose that* $A \in \mathcal{M}_{m,n}(\mathbf{B}_1)$*. If* $\min(m,n) \leq 2$ *then* $\chi_{B_1}(A) = \chi_{R^+}(A)$*. On the other hand, if* $m, n \geq 3$*, there is a matrix* $M \in \mathcal{M}_{m,n}(\mathbf{B}_1)$ *such that* $\chi_{B_1}(M) < \chi_{R^+}(M)$*.*

Example 4.2.6. Let $N = [2 \quad 3]$. The second column of N is $\frac{3}{2}$ times the first, so $\chi_{R^+}(A) = 2$. However, no integer multiple of the first column equals the second, and visa versa, so we see that $\chi_{Z^+}(N) = 2 > \chi_{R^+}(N)$.

The Example above will help us establish the following.

Theorem 4.2.7. *We have that* $X(\mathbf{Z}^+, \mathbf{R}^+, m, n) = 1$*.*

Proof. We observe that any nonzero matrix has at least column rank 1. From Example 4.2.6 and Proposition 2.1, we always have a matrix $N \in \mathcal{M}_{m,n}(\mathbf{Z}^+)$ with $\chi_{R^+}(N) = 1$ and $\chi_{Z^+}(N) = 2$, whenever $n \geq 2$. Therefore $X(\mathbf{Z}^+, \mathbf{R}^+, m, n) \leq 1$, since for any $A \in \mathcal{M}_{m,n}(\mathbf{Z}^+)$ with $\chi_{Z^+}(A) = 1$, $\chi_{R^+}(A) = 1$. □

Suppose that $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$; $\chi_R(A) = 1$ implies that each column of A is a multiple of the first nonzero column of A. Consequently, each column of A is a nonnegative multiple of that column, and hence $\chi_{R^+}(A) = 1$ as well. Thus $X(\mathbf{R}^+, \mathbf{R}, m, n) \geq 1$.

Theorem 4.2.8. *Let* $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$*. If* $\chi_R(A) = 2$*, then* $\chi_{R^+}(A) = 2$*.*

Proof. Suppose that $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$. We will show that $\chi_R(A) = 2$ implies $\chi_{R^+}(A) = 2$ by using induction on n. Certainly the statement holds for $n = 2$. So suppose that it holds for some $n \geq 2$. If $A \in \mathcal{M}_{m,n+1}(\mathbf{R}^+)$ and $\chi_R(A) = 2$ and let the columns of A be $\mathbf{c_1}, \mathbf{c_2}, \cdots, \mathbf{c_{n+1}}$, and let $B = [\mathbf{c}_1 \mid \mathbf{c}_2 \mid \cdots \mid \mathbf{c}_n]$. If $\chi_R(B) = 1$, then certainly any column of A

can be written as a linear combination over $\mathbf{R}^+$ of $\mathbf{c_{n+1}}$ and a nonzero column of B, so that $\chi_{R^+}(A) = 2$. If $\chi_R(B) = 2$, then $\chi_{R^+}(B) = 2$ by the induction step. Hence three are columns $\mathbf{c_i}$ and $\mathbf{c_j}$ of A such that each of $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$ can be written as a linear combination over $\mathbf{R}^+$ of $\mathbf{c_i}$ and $\mathbf{c_j}$. Further, since $\chi_R(A) = 2$, we have $x\mathbf{c}_{n+1} + y\mathbf{c}_i + z\mathbf{c}_j = 0$ for some $x, y, z \in \mathbf{R}$. The $\mathbf{c}$'s have nonnegative entries, so one of x, y and z is positive while another is negative. It follows that that there are numbers α and β in $\mathbf{R}^+$ such that one of the following equality holds $\mathbf{c}_{n+1} = \alpha\mathbf{c}_i + \beta\mathbf{c}_j; \mathbf{c}_i = \alpha\mathbf{c}_{n+1} + \beta\mathbf{c}_j; \mathbf{c}_j = \alpha c_{n+1} + \beta\mathbf{c}_i$. In the first case, $\mathbf{c}_i$ and $\mathbf{c}_j$ span the column space of A over $\mathbf{R}^+$. In the second case, $\mathbf{c}_{n+1}$ and $\mathbf{c}_j$ span that space, and in the third case, $\mathbf{c_{n+1}}$ and $\mathbf{c_i}$ span the space, which completes the induction. □

Example 4.2.9. Let $M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Then $\chi_R(M) = 3$ since $\mathbf{a}_1 + \mathbf{a}_3 - \mathbf{a}_2 = \mathbf{a}_4$, where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{a}_4$ are columns of A by turns. But $\chi_{R^+}(M) = 4$. Therefore $\chi_R(M) < \chi_{R^+}(A)$.

Using Theorem 4.2.8, Example 4.2.9 and Proposition 2.5, the following corollary can be proved.

Corollary 4.2.10. *We have that*

$$X(\mathbf{R}^+, \mathbf{R}, m, n) = \begin{cases} 1 & \text{if } \min(m,n) = 1, \\ 2 & \text{if } \min(m,n) = 2, \\ 3 & \text{otherwise.} \end{cases}$$

V. The Comparisons of Maximal Column Rank

5.1. Maximal Column Rank Inequalities

Suppose that $\mathbb{K}, \mathbb{L}$ are semirings and that $\xi : \mathbb{K} \to \mathbb{L}$ is a semiring homomorphisms. We identify an $m \times n$ matrix $A = [a_{ij}]$ whose entries lie in $\mathbb{K}$, with the $m \times n$ matrix $\Xi(A)$ whose (i, j)th entry equals to $\xi(a_{ij})$. Thus $\Xi : \mathcal{M}_{m,n}(\mathbb{K}) \to \mathcal{M}_{m,n}(\mathbb{L})$ and $A \in \mathcal{M}_{m,n}(\mathbb{K})$ and so $\Xi(A) \in \mathcal{M}_{m,n}(\mathbb{L})$. A homomorphism does not increase the maximal column rank.

Theorem 5.1.1. *Let* $\mathbb{K}$ *and* $\mathbb{L}$ *be semirings and* $\xi : \mathbb{K} \to \mathbb{L}$ *be a semiring homomophism. Then* $\psi_k(A) \geq \psi_L(A)$ *for every matrix* $A \in \mathcal{M}_{m,n}(\mathbb{K})$.

Proof. Let $\psi_k(A) = p$ and let $A = [\mathbf{a}_1|\mathbf{a}_2|\cdots|\mathbf{a}_n]$, where for each $i = 1, 2, \cdots n$, $\mathbf{a}_i$ is a column vector of A. Then there exists the maximal number p of linearly independent column vectors of A. Therefore we assume that the first p columns $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_p$ are the largest linearly independent columns of A, without loss of generality.

Put $\Xi(A) = [\Xi(\mathbf{a}_1) \mid \Xi(\mathbf{a}_2) \mid \cdots \mid \Xi(\mathbf{a}_n)]$, where each $\Xi(\mathbf{a}_i)$ is represented as the image of $\mathbf{a}_i$ by ξ. We will show that the largest number of linearly independent columns of $\Xi(A)$ is at most p. To show this, let us choose any number i with $i > p$. Then for $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_p, \mathbf{a}_i$, there exists at least one $\mathbf{a}_k$ among $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_p, \mathbf{a}_i$ such that $\mathbf{a}_k = r_1\mathbf{a}_1 + \cdots + r_{k-1}\mathbf{a}_{k-1} + r_{k+1}\mathbf{a}_{k+1} + \cdots + r_p\mathbf{a}_p + r_i\mathbf{a}_i$, where $r_j \in \mathbb{K}$, by the maximality of p.

Since ξ is a semiring homomorphisms, so is Ξ. Thus

$$\begin{aligned}\Xi(\mathbf{a}_k) &= \Xi(r_1\mathbf{a}_1 + \cdots + r_{k-1}\mathbf{a}_{k-1} + r_{k+1}\mathbf{a}_{k+1} + \cdots + r_p\mathbf{a}_p + r_i\mathbf{a}_i) \\ &= \Xi(r_1\mathbf{a}_1) + \cdots + \Xi(r_{k-1}\mathbf{a}_{k-1}) + \Xi(r_{k+1}\mathbf{a}_{k+1}) \\ &+ \cdots + \Xi(r_p\mathbf{a}_p) + \Xi(r_i\mathbf{a}_i) \\ &= \xi(r_1)\Xi(\mathbf{a}_1) + \cdots + \xi(r_{k-1})\Xi(\mathbf{a}_{k-1}) + \xi(r_{k+1})\Xi(\mathbf{a}_{k+1}) \\ &+ \cdots + \xi(r_p)\Xi(\mathbf{a}_p) + \xi(r_i)\Xi(\mathbf{a}_i).\end{aligned}$$

Therefore since $\xi(r_k) \in \mathbb{L}$, for any i with $i > p$, $\{\Xi(\mathbf{a}_1), \cdots, \Xi(\mathbf{a}_{k-1}), \Xi(\mathbf{a}_{k+1}), \Xi(\mathbf{a}_i)\}$ is the linearly dependent set over $\mathbb{L}$ and so this set has at most p linearly independent set. Continue to consider another number j with $j > p$ and $j \neq i$. Then $\{\Xi(\mathbf{a}_1), \cdots, \Xi(\mathbf{a}_p), \Xi(\mathbf{a}_i), \Xi(\mathbf{a}_j)\}$ is also linearly dependent set similarly over $\mathbb{L}$ and has at most p linearly independent set. Consequently, continuing in this manner, for all $1, 2, \cdots, n$, $\{\Xi(\mathbf{a}_1), \Xi(\mathbf{a}_2), \cdots, \Xi(\mathbf{a}_n)\}$ has at most p linearly independent set. The largest number of linearly independent columns of $\Xi(A)$ is not larger than p. Therefore we have that $\psi_K(A) \geq \psi_L(A)$. □

If $\mathbb{K}$ is a subsemiring of $\mathbb{L}$, then the canonical injection of $\mathbb{K}$ into $\mathbb{L}$ is a homomorphism, and hence by Theorem 5.1.1, $\psi_K(A) \geq \psi_L(\Xi(A))$ for each matrix $A \in \mathcal{M}_{m,n}(\mathbb{K})$. In this case, we abbreviate the above to $\psi_k \geq \psi_L$.

Corollary 5.1.2. *If $\mathbb{K}$ is a subsemiring of $\mathbb{L}$, then $\psi_K \geq \psi_L$.*

In particular,

(1) *if $j \geq k$ then $\psi_{B_k} \geq \psi_{B_j}$,*

(2) *$\psi_{Z^+} \geq \psi_Z, \psi_Z \geq \psi_P, \psi_P \geq \psi_R$ for any subring, $\mathbf{P}$, of the reals with identity and*

(3) *$\psi_{Z^+} \geq \psi_{P^+}$ for any subseming with identity, $\mathbf{P}^+$, of $\mathbf{R}$.*

Let the entries of the matrix A be in a semiring $\mathbb{S}$. We define the *Pattern* of A to be the $m \times n$ matrix $\bar{A} = [\bar{a}_{ij}]$ where $\bar{a}_{ij} = 0$ if $a_{ij} = 0$, $\bar{a}_{ij}$ equals to the multiplicative identity of $\mathbb{S}$, otherwise.

Corollary 5.1.3. *Let $\mathbb{S}$ be an anti-negative semiring with multiplicative identity 1. Then $\psi_{B_1}(\bar{A}) \leq \psi_S(A)$ for all matrces $\in \mathcal{M}_{m,n}(\mathbb{S})$. In particular, if A is any (0,1) marrix, then $\psi_{B_1}(A) \leq \psi_S(A)$.*

Proof. The mapping $\xi : \mathbb{S} \to \mathbf{B}_1$ defined by $\xi(a) = 0$ if $a = 0$ and $\xi(a) = 1$, otherwise, is a semiring homomorphism. Therefore from Theorem 5.1.1, it is complete. □

Example 5.1.4. Let $\mathbb{S} = \mathbf{R}^+$ and let $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$. Then $\Xi(A) = \bar{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and therefore $\psi_{R^+}(A) = 2$ while $\psi_{B_1}(A) = 1$. Hence $\psi_{R^+}(A) > \psi_{B_1}(\bar{A})$.

Corollary 5.1.5. *Let A and B be integers with a, b and $a, b \geq 2$. Suppose A is a matrix with entries in $\{0, 1, 2, \cdots a-1\}$. Then $\psi_{Z_a}(A) \leq \psi_{Z_{ab}}(A) \leq \psi_Z(A)$.*

Proof. Consider the canonical mappings $\eta : \mathbf{Z} \to \mathbf{Z}_{ab}$ and $\zeta : \mathbf{Z}_{ab} \to \mathbf{Z}_a$ and, of ring into a factoring are homomorphisms. The corollary follows from Theorem 5.1.1. □

5.2. The Case of Equality in Maximal Column Rank

Theorem 5.2.1. *Suppose that $\mathbf{C}_1$ and $\mathbf{C}_2$ are chain semirings and that $\mathbf{C}_1$ is a subsemiring of $\mathbf{C}_2$. If $A \in \mathcal{M}_{m,n}(\mathbf{C}_1)$, then $\psi_{C_1}(A) = \psi_{C_2}(A)$.*

Proof. Since $\mathbf{C}_1 \subset \mathbf{C}_2$, we have $\psi_{C_1}(A) \geq \psi_{C_2}(A)$. Let $\mathbf{C}(A)$ be the chain semiring consisting of 0, 1 and the entries in A. Then $\mathbf{C}(A) \subset \mathbf{C}_1$. Using the process of the proof of Theorem 3.2.1, it is clear that $\psi_{C_1}(A) = \psi_{C(A)}(A) = \psi_{C_2}(A)$. □

Theorem 5.2.2. *Suppose that $j \leq k$, so that $\mathbf{B}_j \subset \mathbf{B}_k$. If $A \in \mathcal{M}_{m,n}(\mathbf{B}_j)$, then $\psi_{B_j}(A) = \psi_{B_k}(A)$.*

Proof. Since $\mathbf{B}_j \subset \mathbf{B}_k$, $\psi_{B_j} \geq \psi_{B_k}$. Let $\mathbf{C}(A)$ be the chain semiring generated by the entries of A, 0 (empty set) and 1(universal set). Then $\mathbf{C}(A) \subset \mathbf{B}_j$. Thus $\psi_{C(A)}(A) \geq \psi_{B_j}(A)$. By the analagous manner with Theorem 5.2.1, $\psi_{B_j}(A) = \psi_{B_k}(A)$. □

Corollary 5.2.3. *For any (0,1) matrix A, any chain semiring $\mathbf{C}$ which contains 0 and 1, any nteger with $k \geq 1$, we have $\psi_{B_1}(A) = \psi_{B_k}(A) = \psi_C(A) = \psi_F(A)$.*

Suppose that $\mathbb{K}$ is a subsemiring of $\mathbb{L}$. Let $\Psi(\mathbb{K}, \mathbb{L}, m, n)$ denote the maximum integer j such that there exists a matrix in $\mathcal{M}_{m,n}(\mathbb{K})$ with maximum column rank j and for every $A \in \mathcal{M}_{m,n}(\mathbb{K})$ with $\psi_K(A) \leq j$, we have $\psi_K(A) = \psi_L(A)$. Note that $\Psi \geq 0$, since for any semiring S, $\psi_S(A) = 0$ if and only if A is the zero matrix. Using this notation, we have established the following equality ; $\psi_{B_j} = \psi_{B_k}$ for any $j \leq k$. and $\psi_{C_1} = \psi_{c_2}$ for any semirings $\mathbf{C}_1$ and $\mathbf{C}_2$ with $\mathbf{C}_1 \subset \mathbf{C}_2$. Further, from the results in the preceding section, we have shown that for any matrix $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$,

$$\psi_{B_1} \leq \psi_{R+} \tag{5.1}$$

and that

$$\psi_R \leq \psi_{R^+}, \tag{5.2}$$

$$\psi_{R^+} \leq \psi_{Z^+}. \tag{5.3}$$

We will show that equality does not hold in general for any of (5.1)–(5.3). Our approach will be to investigate the values of Ψ for appropriate anti-negative semirings $\mathbb{K}$ and $\mathbb{L}$. First we will look at (5.1).

Example 5.2.4. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Then $\psi_{B_1}(A) = 2$ since the second and the third column generate the first. Thus the largest number of linearly independent columns is 2, while $\psi_{R^+}(A) = 3$ since three columns are linearly independent over $\mathbf{R}^+$. Thus $\psi_{B_1}(A) < \phi_{R^+}(A)$.

The following Theorem is easily established from Example 5.2.4 and Proposition 2.5.

Theorem 5.2.5. *Suppose that* $A \in \mathcal{M}_{m,n}(\mathbf{B}_1)$. *If* $\min(m, n) \leq 2$, *then* $\psi_{B_1}(A) = \psi_{R^+}(A)$. *Otherwise, there is a matrix* $A \in \mathcal{M}_{m,n}(\mathbf{B}_1)$ *such that* $\psi_{B_1}(A) < \psi_{R^+}(A)$.

Proof. When either m or n is 1, it is clear. Suppose that $n = 2$ and let $A = [\mathbf{a}_1|\mathbf{a}_2]$ be a $m \times 2$ matrix where $\mathbf{a}_1$ and $\mathbf{a}_2$ are the first and second columns of A, respectively. If $\psi_{B_1}(A) = 1$, then since the entries of A is in Boolean algebra $\mathbf{B}_1$, $\mathbf{a}_1 = \mathbf{a}_2$ and both nonzero columns. Hence $\psi_{R^+}(A) = 1$. If $\psi_{B_1}(A) = 2$, then there exists at least one positive integer j such that either $a_{j1} = 0$ and $a_{j2} = 1$ or, $a_{j1} = 1$ and $a_{j2} = 0$. Then clearly $\psi_{R^+}(A) = 2$. Consider the converse. Now note that $\psi_{B_1}(A) \leq \psi_{R^+}(A)$ since there exists a semiring homomorphism ξ of $\mathbf{R}^+$ into $\mathbf{B}_1$, by Corollary 5.1.2. If $\psi_{R^+}(A) = 1$ then $\psi_{B_1}(A) = 1$ since $\psi_{B_1}(A) \leq \psi_{R^+}(A) = 1$ and $\psi_{B_1}(A) \neq 0$. Similarly if $\psi_{R^+}(A) = 2$ then $\psi_{B_1}(A) \leq \psi_{R^+}(A) = 2$. Then clearly $\psi_{B_1}(A) = 2$ (if $\psi_{B_1}(A) = 1$ then $\psi_{R^+}(A) = 1$, contradiction to the assumption that $\psi_{R^+}(A) = 2$).

On the other hand, suppose that $m = 2$ and let $A = [\mathbf{a_1}|\mathbf{a_2}|\cdots|\mathbf{a_n}]$ be a $2 \times n$ matrix, where each $\mathbf{a_i}$ is a column vector of A. Since each entry of A is either 0 or 1, the columns of A are the only forms of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus the maximul column rank of A on $\mathbf{R}^+$ is at most 2. Therefore, if $\psi_{B_1}(A) = 2$, then A has at least two columns among three columns $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then clearly $\psi_{R^+}(A)$ is also 2. Also if $\psi_{B_1}(A) = 1$ then $\psi_{R^+}(A) = 1$ as the preceding proof.

Conversely, if $\psi_{R^+}(A) = 1$, then $\psi_{B_1}(A) = 1$ since $\psi_{B_1}(A) \leq \psi_{R^+}(A) = 1$ and $\psi_{B_1}(A) \neq 0$. Also if $\psi_{R^+}(A) = 2$, then $\psi_{B_1}(A) = 2$ since $\psi_{B_1}(A) \leq \psi_{R^+}(A) = 2$ and $\psi_{B_1}(A)$ is nether 0 nor 1. Therefore the first case holds. Now for $\min(m,n) \geq 3$, from Example 3.2.4 and Proposition 2.5, there exists M in $\mathcal{M}_{m,n}(\mathbf{B}_1)$ such that $\psi_{B_1}(M) < \psi_{R^+}(A)$. □

Therefore the following corollary is true.

Corollary 5.2.6. *If* $A \in \mathcal{M}_{m,n}(\mathbf{B}_1)$, *then*

$$\Psi(\mathbf{R}^+, \mathbf{B}_1, m, n) = \begin{cases} 1 & \text{if } \min(m,n) = 1, \\ 2 & \text{if } \min(m,n) = 2, \\ 3 & \text{if } m, n \geq 3. \end{cases}$$

Proof. Case 1 and Case 2 hold immediatly from Theorem 5.2.5.

Case 3. Let $\min(m,n) \geq 3$. If $A \in \mathcal{M}_{m,n}(\mathbf{B}_1)$ with $\psi_{R^+}(A) = 3$, then $\psi_{R^+}(A) = \psi_{B_1}(A) = 3$ since $\psi_{B_1}(A) \leq \psi_{R^+}(A) = 3$ but $\psi_{B_1}(A) \neq 0, 1$ and 2 from Theorem 5.4. By definition of Ψ, Case 3 holds. □

The following example enables us to have the comparison of the maximal column rank between *Nonnegative integer* and *Nonnegative real.*

Example 5.2.7. Let $A = [3,\ 4]$. The second column of A is $\frac{3}{2}$ times the first, thus $\psi_{R^+0}(A) = 1$. However, no integer multiple of the first column equals to the second, and vise versa, so we see that $\psi_{Z^+}(A) = 2 > \psi_{R^+}(A)$.

Thus the below theorem is easily proved from Example 5.2.7 and Proposition 2.5.

Theorem 5.2.8. *We have that* $\Psi(\mathbf{Z}^+,\ \mathbf{R}^+,\ m,\ n) = 1$.

Proof. From Example 5.2.7 and Proposition 2.5, we always have a matrix $A \in \mathcal{M}_{m,n}(\mathbf{Z}^+)$ with $\psi_{R^+}(A) = 1$ and $\psi_{Z^+}(A) = 2$ whenever $n \geq 2$. □

Suppose that $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$; if $\psi_R(A) = 1$, then each column of A is a multiple of the first nonzero column of A. Consequently, each column of A is a nonnegaive multiple of that column, and hence $\psi_{R^+}(A) = 1$ as well. Thus we have $\Psi(\mathbf{R}^+,\ \mathbf{R},\ m,\ n) \geq 1$.

The next result establishes the more.

Theorem 5.2.9. *Let* $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$. *If* $\psi_R(A) = 2$, *then* $\psi_{R^+}(A) = 2$.

Proof. Suppose that $\psi_R(A) = 2$. Then the largest number of linearly independent columns is 2 over $\mathbf{R}$. That is, for any three columns of A, $\mathbf{a}_i$, $\mathbf{a}_j$ and $\mathbf{a}_k$, $x\mathbf{a}_i + y\mathbf{a}_j + z\mathbf{a}_k = 0$ for some x, y, and z in $\mathbf{R}$. Since $\mathbf{a}_i$, $\mathbf{a}_j$, and $\mathbf{a}_k$ are over $\mathbf{R}^+$, all x, y and z are not the same signs. Let x be positive and let the others be negative. Then $\mathbf{a}_i = (-\frac{y}{x})\mathbf{a}_j + (-\frac{z}{x})\mathbf{a}_k$. Since $-\frac{y}{x}$ and $-\frac{z}{x}$ is positive, $\mathbf{a}_i$, $\mathbf{a}_j$ and $\mathbf{a}_k$ are linearly dependent over $\mathbf{R}^+$. In other case, it holds similarly. Thus three arbitrary columns in A are linearly dependent. Now extend to four columns in A, say $\mathbf{a}_i$, $\mathbf{a}_j$, $\mathbf{a}_k$ and $\mathbf{a}_l$. In the above manner, for $\mathbf{a}_i$, $\mathbf{a}_j$ and $\mathbf{a}_k$, put $\mathbf{a}_i = r_1\mathbf{a}_j + r_2\mathbf{a}_k$ for some $r_1, r_2 \in \mathbf{R}^+$ and for $\mathbf{a}_j$, $\mathbf{a}_k$ and $\mathbf{a}_l$,$\mathbf{a}_j = r_3\mathbf{a}_k + r_4\mathbf{a}_l$ for some $r_3, r_4 \in \mathbf{R}^+$. Then $\mathbf{a}_i = r_1(r_3\mathbf{a}_k + r_4\mathbf{a}_l) + r_2\mathbf{a}_k = (r_1r_3 + r_2)\mathbf{a}_k + r_1r_4\mathbf{a}_l$. Thus these four columns $\mathbf{a}_i$, $\mathbf{a}_j$, $\mathbf{a}_k$ and $\mathbf{a}_l$ are generated by two columns $\mathbf{a}_k$ and $\mathbf{a}_l$ over $\mathbf{R}^+$. In other case, it holds clearly. Continuing in this process, for the columns of A more than four, they have two generating columns over $\mathbf{R}^+$. Thus we can obtain that $\psi_{R^+}(A) = 2$. □

Now we shall see that there exists for $\psi_R(A) \geq 3$ the case when does not hold the above theorem.

Example 5.2.10. Let $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Then $\psi_R = 3$ since $\mathbf{a_1} + \mathbf{a_3} - \mathbf{a_2} = \mathbf{a_4}$, where $\mathbf{a_1}, \mathbf{a_2}$, $\mathbf{a_3}$ and $\mathbf{a_4}$ are the columns of A by turns. From Corollary 5.1.2, $4 = \psi_{B_1}(A) \leq \psi_{R^+}(A)$. Hence $\psi_R(A) < \psi_{R^+}(A)$.

Lemma 5.2.11. *Let* $\mathbb{F}$ *be a field. If* $\min(m, n) = k$ *and* $A \in \mathcal{M}_{m,n}(\mathbb{F})$, *then* $\psi_F(A) \leq k$.

Proof. In a field, all kinds of ranks are the same. That is, column, row, factor and maximal column rank are all the same. Thus $\psi_F(A)$ is equal to or smallar than k. □

Using Theorem 5.2.9, Lemma 5.2.11, Proposition 2.5 and Example 5.2.10, the following corollary holds.

Corollary 5.2.12. *We have that*

$$\Psi(\mathbf{R}^+, \mathbf{R}, m, n) = \begin{cases} 1 & \text{if } \min(m,n) = 1, \\ 2 & \text{if } \min(m,n) = 2, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Case 1. If $\min(m,n) = 1$, then $\psi_R(A)$ has the one of 0 and 1 by Lemma 5.2.11. Further, $\psi_R(A) = 1$ if and only if $\psi_{R^+}(A) = 1$, and $\psi_R(A) = 0$ if and only if $\psi_{R^+}(A) = 0$. The first case holds.

Case 2. Suppose that $\min(m,n) = 2$. By Lemma 5.2.11, $\psi_R(A) \leq 2$. We must show that for all $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$ with $\psi_{R^+}(A) \leq 2, \psi_R(A) = \psi_{R^+}(A)$. Since from case 1 for all $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$ with $\psi_{R^+}(A) \leq 1, \psi_R(A) = \psi_{R^+}(A)$, it is suffices to show that for any A with $\psi_{R^+}(A) = 2$, $\psi_R(A) = 2$. Put $\psi_{R^+}(A) = 2$. Since $\psi_R(A) \leq \psi_{R^+}(A)$, $\psi_R(A) \leq 2$. But $\psi_R(A)$ is neither 0 nor 1. Thus $\psi_R(A) = 2$ and so the case holds.

Case 3. From case 1 and case 2, it suffices to show that for any $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$ with $\psi_{R^+}(A) = 3, \psi_R(A) = \psi_{R^+}(A)$. Put $\psi_{R^+}(A) = 3$ and then $\psi_R(A) \leq 3$ since $\psi_R(A) \leq \psi_{R^+}(A)$. But none of 0,1 and 2 can be the value of $\psi_R(A)$. For $\psi_R(A)$ is 0 and 1 if and only if $\psi_{R^+}(A)$ is 0 and 1, respectively. And if $\psi_R(A) = 2$ then $\psi_{R^+}(A) = 2$, by Theorem 5.2.9. Thus $\psi_R(A) \neq o, 1$, and 2. Thus $\psi_R(A) = 3$. Therefore for any $A \in \mathcal{M}_{m,n}(\mathbf{R}^+)$ with $\psi_{R^+}(A) \leq 3$, we have $\psi_R(A) = \psi_{R^+}(A)$. Further Example 5.2.10 and Proposition 2.5 imply that there exists M such that $\psi_{R^+}(M) > \psi_R(M)$. Hence $\Psi(\mathbf{R}^+, \mathbf{R}, m, n) = 3$. □

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< Abstract >

On Comparison of Maximal Column Rank of Anti-Negative Matrices

For a given matrix A over a semiring $\mathbb{S}$, the maximal column rank of A, $\psi_S(A)$, is the largest number of linearly independent columns of A over the given semiring. The maximal column rank is the same as the column rank over the field, but over general semiring two concepts have different mean each other.

In this thesis, we studied the maximal column rank of a matrix A when A could be considered as a matrix over two related semirings. For two semiring of which one is a subsemiring of the other, the maximal colum rank of the smaller semiring is equal to or larger than that of the larger semiring. Also, we obtain the maximum value of maximal column rank such that two maximal column ranks coincide each other. As a result, over the chain semirings or the general Boolean algebras, two maximal column rank are always the same, while over anti-negative real and real, the maximum value is at most 3.

감사의 글

논문을 준비하면서 한편의 정성된 논문이 완성되기까지 많은 분들의 지도와 조언 때로는 세심한 격려와 날카로운 질책까지도 얼마나 소중한 것이었는가 알게 되었습니다. 이시간 제가 대학원생활을 보내고 논문을 쓰기까지 많은 도움을 주시고 이끌어 주신 분들께 감사를 드립니다.

먼저 저를 가장 잘 아시고 보이지 않는 곳에서 인도자가 되시는 하나님께 영광을 돌려 드립니다. 그리고 논문의 방향과 힌트를 주시고 아낌없는 관심과 지도, 희생을 보여주신 송석준 교수님께 감사를 드리겠습니다. 대학원 과정동안 깊이 있고 폭넓은 강의를 해 주신 현진오 교수님, 양영오 교수님, 고봉수 교수님, 고윤희 교수님, 정승달 교수님께 감사를 드립니다. 논문을 섬세히 검토해 주시고 심사를 해주신 현진오 교수님과 양성호 교수님께 감사를 드립니다. 논문의 교정작업을 세밀하게 도와주신 윤용식 교수님께 감사를 드립니다.

그동안 함께 공부하면서 든든한 친구들 같은 대학원동기님들과 후배님들에게도 감사를 드립니다. 여러 믿음의 형제자매님들과 선배님들의 기도과 사랑에 감사를 드리며 멀리 인도에 있는 고민철 선배님께 깊이 감사를 드립니다. 마지막으로 말없이 지켜보시며 희생하시고 격려해주신 부모님과 큰언니와 형부, 멀리 있는 작은언니 그리고 동생들에게 감사를 드리며 이 기쁨을 함께 하겠습니다.

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