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Nonlinear Dynamical Analysis in Models of Mathematical Biology

濟州大學校 大學院

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Nonlinear Dynamical Analysis in Models of Mathematical Biology

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CONTENTS

<Abstract>

Nonlinear Dynamical Analysis In Models of Mathematical Biology

Mathematical biology is a fast growing subject and is very exciting modern application of mathematics. Mathematical biology consists of Modeling and Analysis. This thesis is dealing with the interactions of the vegetable, prey, and predator population. As a first step we set the Vegetable-Prey-Predator models, and then analyzed them. Analysis started by investigating the stability and unstability conditions of the dynamical systems. Moreover, through the concrete examples we discovered the critical change in dynamics of Vegetable-Prey-Predator models such as extinction of the prey and predator population. That is to say, as vegetable grow unboundedly, prey and predator population also grow for some times. But if predator population is too large, then prey deduce extremely toward extinction. To prevent the extinction effectively we should regulate predator population artificially when it is too large.

I. Preliminaries

Definition 1.1. **[14]** $(\Re_+, \Re^n)$

(1) $\Re_+$ is a set of all positive real numbers.

(2) For each positive integer n, $\Re^n$ be the set of all ordered n-tuples

$$\mathbf{x} = (x_1, x_2, \cdots, x_n),$$

where $x_1, \cdots, x_n$ are real numbers, called the coordinates of $\mathbf{x}$. The elements of $\Re^n$ are called points, or vectors, especially when $n > 1$.

Definition 1.2. **[5]** (Mathematical model)

A mathematical model is an equation, or a set of equations, whose solution depends on time. The equation that defines the mathematical model can also be called the **state equation**.

Definition 1.3. **[5]** (Linear and nonlinear models)

A mathematical model is **linear** if the state equation can be written in the form $\mathcal{L}u = f$, where $\mathcal{L}$ is a linear operator that satisfies the following condition

$$\mathcal{L}(u_1 + u_2) = \mathcal{L}(u_1) + \mathcal{L}(u_2)\,,$$

$$\mathcal{L}(\lambda u) = \lambda \mathcal{L}(u).$$

A mathematical model is **nonlinear** if it is not linear.

A mathematical model is **semilinear** if the state equation can be written as $\mathcal{L}u + \varepsilon\mathcal{N}_\varepsilon u = f$, where $\mathcal{L}$ is a linear operator, ε is a parameter, and the operator $\mathcal{N}_\varepsilon$ may depend on ε.

Definition 1.4. [7] (Of class C^r)

Let f be defined and continuous on an open set $\mathcal{D} \subset \Re^n$. Then f is said to be **of class** C^r in $\mathcal{D}$ if all the partial derivatives of f of order up to and including r exist and are continuous everywhere in $\mathcal{D}$.

Definition 1.5. [12] (A dynamical system)

A dynamical system or **flow** is a C^1 map

$$\phi : \Re \times \mathcal{D} \to \mathcal{D},$$

where $\mathcal{D}$ is an open set in $\Re^n$ and writing $\phi(t, \mathbf{x}) = \phi_t(\mathbf{x})$ the map $\phi_t : \mathcal{D} \to \mathcal{D}$ satisfies

(1) $\phi_0 : \mathcal{D} \to \mathcal{D}$ is the identity,

(2) The composition $\phi_t \circ \phi_s = \phi_{t+s}$ holds for each t, s in $\Re$.

Definition 1.6. [16] (A vector field)

A tangent vector v_p to the Euclidean space $\Re^n$ consists of a pair of points $v, p \in \Re^n$.

v is called the vector part and p is called the point of application of v_p (see Figure 1.1).

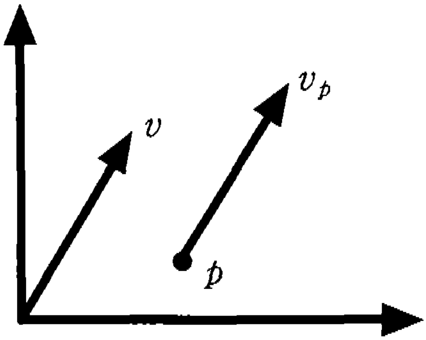


Figure 1.1.

Let $p \in \Re^n$. **The tangent space of** $\Re^n$ **at** p is the set

$$\Re_p{}^n = \{v_p | v \in \Re^n\}.$$

A vector field V on an open subset U of $\Re^n$ is a function that assigns to each $p \in U$ a tangent vector $v_p \in \Re_p{}^n$.

Definition 1.7. [12] (An equilibrium point)
Let $\phi : \Re \times \mathcal{D} \to \mathcal{D}$ be a dynamical system. A point $\mathbf{x} \in \mathcal{D}$ is called **an equilibrium point** of ϕ **(or a steady states, fixed point)** if $\phi(t, \mathbf{x}) = \mathbf{x}$ for all t.

Definition 1.8. [1] (Characteristic polynomial and eigenvalues)
Given an $n \times n$ matrix $A = [a_{ij}]$. The polynomial

$$P(\lambda) = \det(A - \lambda E_n)$$

is called **a characteristic polynomial** of A.(Here E_n denotes the $n \times n$ identity matrix and λ is a scalar). The root of $P(\lambda)$ are called **the eigenvalues** of A.

Definition 1.9. [4] For

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{1.1}$$

(1) We call the equilibrium point **a node** when both eigenvalues of A in (1.1) are real and both of them simultaneously greater (**unstable node, source**) or smaller (**stable node, sink**) than zero (see Figure 1.2(a) and (b), respectively).

(2) We call the equilibrium point **a saddle** when one eigenvalue, say λ_1, is smaller than zero and the other λ_2 is larger than zero (see Figure 1.2(c)).

(3) We call the equilibrium point **a focus** when both eigenvalues are complex (see Figure 1.2(d)(e)).

(4) We call the equilibrium point **a Jordan node** when $\lambda_1 = -\lambda_2$ (see Figure 1.2(f)).

(5) We call the equilibrium point **a centre** when $\lambda_1 = -\lambda_2$ and the real part of both eigenvalues is zero (see Figure 1.2(g)).

(6) We call the equilibrium point **a singular node** when $\lambda_1 = 0$ and $\lambda_2 \neq 0$ (see Figure 1.2(h)).

(7) We call the equilibrium point **a nilpotent** when A is not identically zero but has two zero eigenvalues (see Figure 1.2(i)).

(8) We call the equilibrium point **a bicritical node** when it has an associated normal form

$$\begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix},$$

which occurs if $a_{12} = a_{21} = 0$ and $a_{11} = a_{22}$ in equation (1.1) (see Figure 1.2(j)).

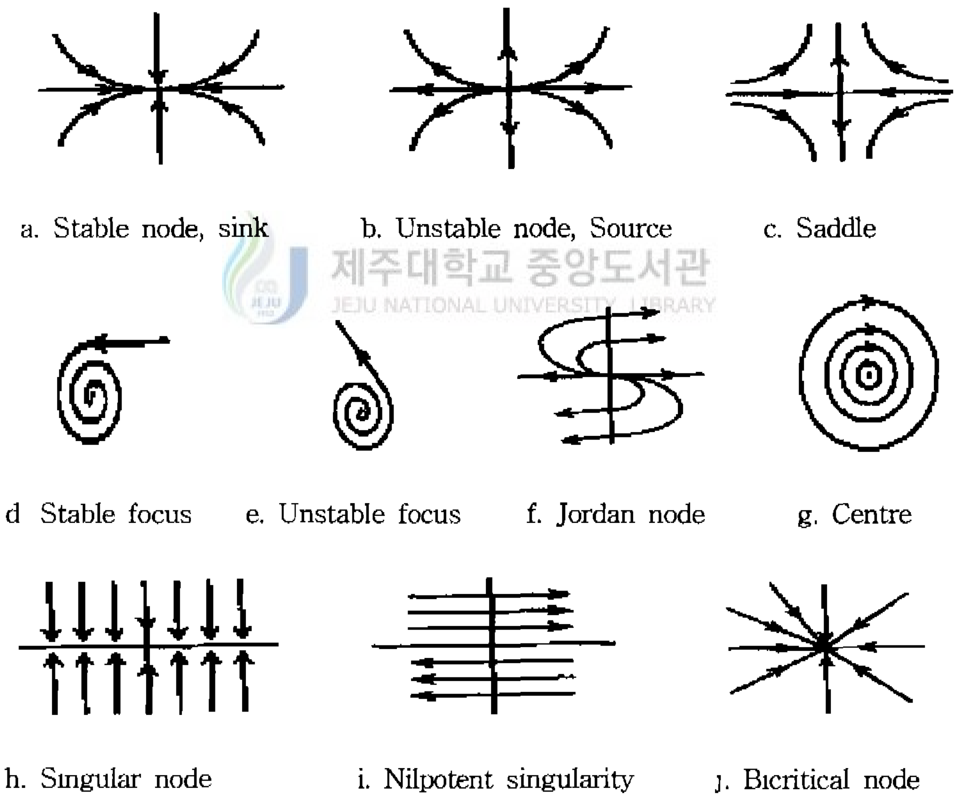


Figure 1.2. bi-dimensional singularities.

Definition 1.10. [4] (An attracting set)

Let $\phi : \Re \times \mathcal{D} \to \mathcal{D}$ be a dynamical system. We call a set, $A \subset \mathcal{D}$, **an attracting set** if it is a closed invariant set and in addition there is a neighborhood $\mathcal{U}(A)$ such that for every $x \in \mathcal{U}(A)$

$$\phi(t, \mathbf{x}) \in \mathcal{U}(A)\,,$$

for positive times

$$\lim_{t\to\infty} \phi(t, \mathbf{x}) \to A\,.$$

Definition 1.11. [4] (Orbits, Periodic orbits)

Let $\phi : \Re \times \mathcal{D} \to \mathcal{D}$ be a dynamical system. Fixed $\mathbf{x} \in \mathcal{D}$, we call the set

$$\mathcal{O}(x) = \{y \in \mathcal{D} \subseteq \Re^n : y = \phi(t, \mathbf{x}) \text{for some}\, t \in \Re\}\,.$$

an orbit through the point $\mathbf{x}$. The orbit is said to be **periodic** if there exist t_0 such that

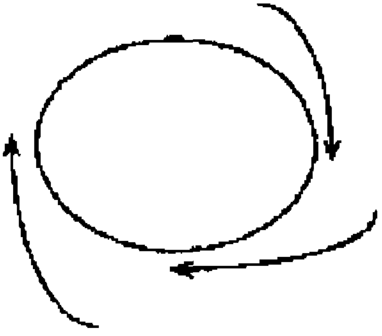
$$\phi(t + t_0, \mathbf{x}) = \phi(t, \mathbf{x}) \quad \text{for all } t \text{ (see Figure 1.3}(a))\,.$$

A periodic orbit is called the **limit cycle**.

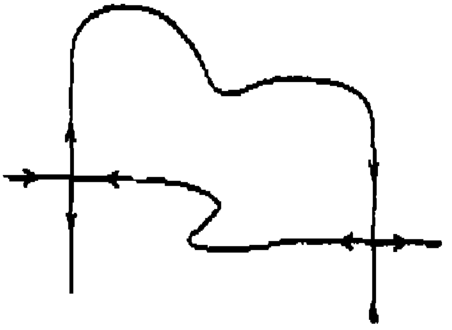
Definition 1.12. [4] (Heteroclinic and Homoclinic orbits)

The orbit $\mathcal{O}(\mathbf{x})$ starting in a nonstable equilibrium point and finishing in another equilibrium point is called a **heteroclinic orbit**.

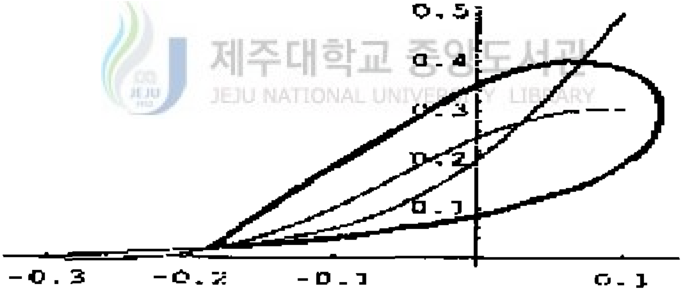
Similarly, **a homoclinic orbit** starts and finishes in the same nonstable equilibrium point (see Figure 1.3(b), (c)).



(a) Periodic Orbit



(b) Heteroclinic Orbit



(c) Homoclinic Orbit

Figure 1.3.

Definition 1.13. [5] (Bifurcation)
Let $\phi_\lambda : \Re \times \mathcal{D} \to \mathcal{D}$ be a dynamical system, and let $\mathbf{x}_0$ be an equilibrium of ϕ_{λ_0}.

We say that an equilibrium of ϕ_λ bifurcates from $\mathbf{x}_0$ if there is a continuous curve $\mathbf{x}(\lambda)$ of equilibriums of ϕ_λ such that

$$\lim_{\lambda\to\lambda_0} \mathbf{x}(\lambda) = \mathbf{x}_0\,.$$

The common value $(\lambda_0, \mathbf{x}_0)$ is called **a bifurcation point**.

Definition 1.14. [4]

Let $\phi : \Re \times \mathcal{D} \to \mathcal{D}$ be a dynamical system. An equilibrium point $\mathbf{x}_0$ of ϕ is said to be **stable** if for every neighborhood

$$\mathcal{U}_\varepsilon(\mathbf{x}_0) = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < \varepsilon\}\,,$$

there exists a neighborhood

$$\mathcal{U}_\delta(\mathbf{x}_0) = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < \delta\}\,,$$

such that for every $\mathbf{x} \in \mathcal{U}_\varepsilon(\mathbf{x}_0)$ and every positive time t, the orbit $\phi(t,\mathbf{x})$ is in $\mathcal{U}_\delta(\mathbf{x})$. That is, $\quad \mathbf{x}(t) \in \mathcal{U}_\delta(\mathbf{x}_0) \quad$ for all $t \geq 0$.

It is **an asymptotically stable** if

$$\lim_{t\to\infty} \phi(t,\mathbf{x}) = \mathbf{x}_0\,.$$

Definition 1.15. (Linearized systems)

Let

$$\mathbf{x}' = \frac{dx}{dt} = \phi(t,\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1.2}$$

be a nonlinear system, and let $\mathbf{x}_1$ be an equilibrium point.

We define the linearized systems of (1.2) at $\mathbf{x}_1$ by

$$\mathbf{x}' = \nabla_{\mathbf{x}}\phi(t, \mathbf{x}_1)\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 .$$

Example for linearized systems

$$(1.3) \qquad \begin{cases} \dfrac{du}{dt} = u(1-u) - \dfrac{auv}{u+d} = f(u,v)\,, \\ \dfrac{dv}{dt} = bv\left(1 - \dfrac{v}{u}\right) = g(u,v)\,. \end{cases}$$

Linearizing about the steady state, (u^*, v^*), by writing

$$x(t) = u(t) - u^*, \quad y(t) = v(t) - v^*\,,$$

then (1.3) becomes to

$$\begin{cases} \dfrac{dx}{dt} = xf_u(u^*, v^*) + yf_v(u^*, v^*)\,, \\ \dfrac{dy}{dt} = xg_u(u^*, v^*) + yg_v(u^*, v^*)\,. \end{cases}$$

Theorem 1.16. [4] (Linear stability)

An equilibrium point $\mathbf{x}_0$ is **(asymptotically) stable** if all the eigenvalues of the linearized system at the point $\mathbf{x}_0$ have negative real parts.

Definition 1.17. [1]

An n-th order polynomial $\mathcal{P}(\lambda)$ with real coefficients is called **stable** if all zeros of $\mathcal{P}(\lambda)$ have negative parts. The stable polynomial is also called **a Hurwitz polynomial**. It is called **unstable** if at least one of the zeros of $\mathcal{P}(\lambda)$ has a positive real part. It is called **critical** if $\mathcal{P}(\lambda)$ is neither stable nor unstable.

Theorem 1.18. [1]

$$\mathcal{P}(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0\,.$$

For $\mathcal{P}(\lambda)$ to be Hurwitz polynomial, it is necessary that

$$a_{n-1} > 0, \cdots, a_1 > 0, a_0 > 0\,.$$

Theorem 1.19. [8] (Maximum and minimum of two variable function) Let f be of class C^2 in a neighborhood of the critical point p_0, and let

$$\triangle = (f_{12}(p_0))^2 - f_{11}(p_0)f_{22}(p_0)\,.$$

Then, if $\triangle > 0$, p_0 is **a saddle point** for f. If $\triangle < 0$, p_0 is **an extremal** for f, and is **a maximum** if

$$f_{11}(p_0) < 0\,,$$

and a **minimum** if

$$f_{11}(p_0) > 0\,.$$

Remark [2] (Logistic population growth)

Verhulst in 1836 proposed that a self-limiting process should operate when a population becomes too large. He suggested

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}),$$

where r and K are positive constants. This is called **logistic growth** in a population. In this model the per capita birth rate is

$$r(1 - \frac{N}{K}),$$

that is, dependent on N. The constant K is **the carrying capacity** of the environment, which is usually determined by the available sustaining resources.

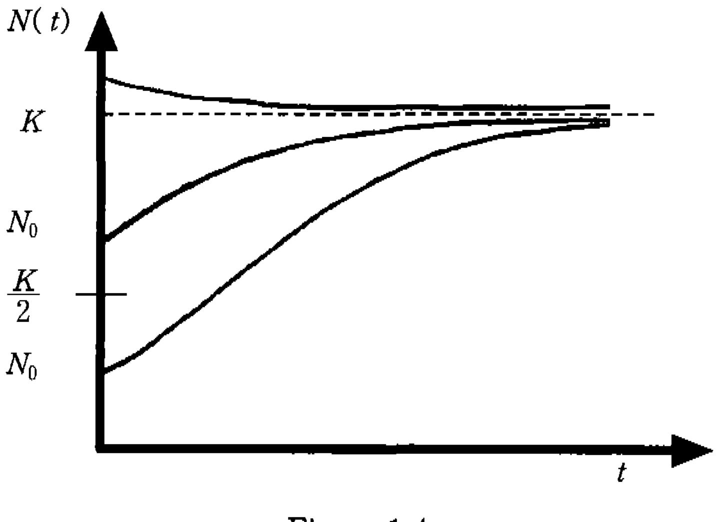


Figure 1.4.

2. the Prey-Predator Model

First, we will consider systems involving two species, prey and predator. Then we will expand to three species, vegetable, prey, and predator.

2.1 Lotka - Volterra Systems

Volterra in 1926 first proposed a simple model for the predation of one species by another to explain the oscillatory levels of certain fish catches in the Adriatic. If $N(t)$ is the prey population and $P(t)$ is that of predator at time t then Volterra's model is

$$(2.1) \qquad \begin{cases} \dfrac{dN}{dt} = N(a - bP)\,, \\ \dfrac{dP}{dt} = P(cN - d)\,, \end{cases}$$

where a, b, c and d are positive constants.

The assumptions in the model are :

(1) The prey in the absence of any predation grow unboundedly ; aN.

(2) The effect of the predation is to reduce the prey's per capita growth rate by a term proportional to the prey and predator populations ; $-bNP$.

(3) In the absence of any prey for sustenance the predator's death rate results in exponential decay ; $-dP$.

(4) The prey's contribution to the predator's growth rate is ; cNP.

Step I : Nondimensionalization

Nondimensionalization reduces the number of parameters by grouping them in a meaningful way. It is extremely useful to write the system in nondimensional form. Although there is no unique way of doing this it is often a good idea to relate the variables to some key relevant parameter.

As a first step in analyzing the Lotka-Volterra model (2.1) we nondimensionalize the system by writing

$$u(\tau) = \frac{cN(t)}{d}, v(\tau) = \frac{bP(t)}{a}, \tau = \frac{d}{a},$$

and it becomes to

(2.2)
$$\begin{cases} \dfrac{du}{d\tau} = u(1-v), \\ \dfrac{dv}{d\tau} = \alpha v(u-1), \end{cases}$$

(2.3)
$$\Rightarrow \quad \frac{dv}{du} = \alpha \frac{v(u-1)}{u(1-v)},$$

which has singular points at $u = v = 0$ and $u = v = 1$.

We can integrate (2.3) exactly to get the phase trajectories.

(2.4)
$$\alpha u + v - \ln u^{\alpha} v = H,$$

where $H > H_m$ is a constant.

The Theorem 1.19 implies that $H_m = 1 + \alpha$ is the minimum of H over all (u, v) and it occurs at $u = v = 1$. For a given $H > 1 + \alpha$, the trajectories in the phase plane are closed as illustrated in Figure 2.1.

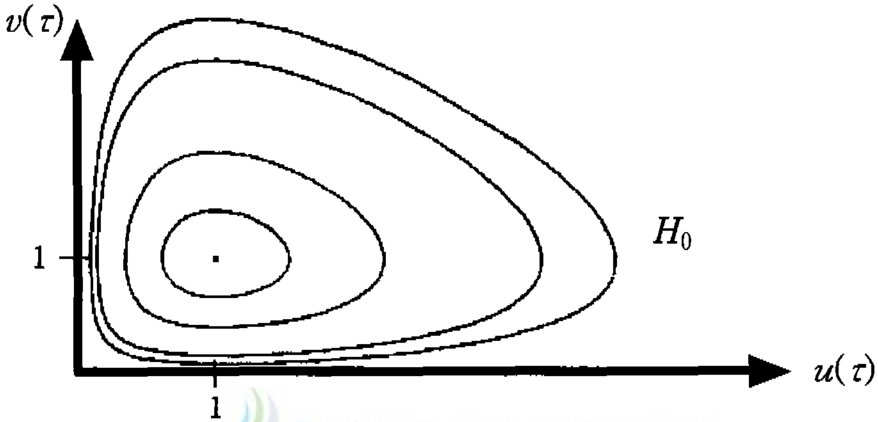


Figure 2.1. closed phase plane trajectories, from (2.4) with various H.

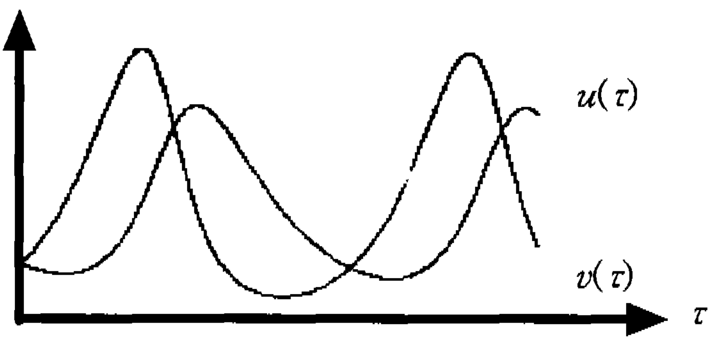


Figure 2.2. periodic solutions for the prey $u(\tau)$ and $v(\tau)$

for the Lotka-Volterra system (2.2).

A closed trajectory in the u, v plane implies periodic solutions in τ for u and v in (2.2). The initial conditions, $u(0)$ and $v(0)$, determine the constant H in (2.4) and hence the phase trajectory in Figure 2.1. Typical periodic solutions $u(\tau)$ and $v(\tau)$ are illustrated in Figure 2.2. From (2.2), we can see immediately that u has a turning point when $v = 1$ and v has one when $u = 1$.

A major inadequacy of the Lotka-Volterra model is clear from Figure 2.1, the solutions are structurally stable. Suppose, for example, $u(0)$ and $v(0)$ are such that u and v for τ are on the trajectory H_0 which passes close to the u and v axes. Then any small perturbation will move the solution onto another trajectory which does not lie everywhere close to the original one H_0. Thus a small perturbation can have a very marked effect, at the very least on the amplitude of the oscillation.

One of the unrealistic assumptions in the Lotka-Volterra models, (2.1) and (2.2), is that the prey growth is unbounded in the absence of predation. To be more realistic the growth rate should depend on both the prey and predator densities.

As a reasonable first step we might expect the prey to satisfy a logistic growth, say, in the absence of any predators, or have some similar growth dynamics which has some maximum carrying capacity. The predation term, which is the functional response of the predator to change in the prey density, generally shows some saturation effect. Instead of bNP, we take $PNR(N)$ where $NR(N)$ saturate for large N.

That is, $$\lim_{N\to\infty} NR(N) = A < \infty .$$

2.2 Analysis of a Prey-Predator Model with Limit Cycle Periodic Behaviour : Parameter Domains of Stability

Let $N(t)$ is the prey population and $P(t)$ is that of the predator at time t. Consider another prey-predator model

$$(2.5) \qquad \begin{cases} \dfrac{dN}{dt} = N\left[r\left(1-\dfrac{N}{K}\right) - \dfrac{kP}{N+D}\right], \\ \dfrac{dP}{dt} = P\left[s\left(1-\dfrac{hP}{N}\right)\right], \end{cases}$$

where r, K, k, D, s and h are positive constants.

k : the maximum number of eating by predator per capita,

$\dfrac{k}{N+D}$: per capita eating rates of predator,

$\dfrac{kN}{N+D}$: the number of eating per capita,

K : carrying capacity, r : growth rates of prey,

s : growth rates of predator.

Step I : Nondimensionalization

Let us write

$$u(\tau) = \frac{N(t)}{K},\ v(\tau) = \frac{hP(t)}{K},\ \tau = rt\,,$$

$$a = \frac{k}{h}r,\ b = \frac{s}{r},\ d = \frac{D}{K}\,,$$

then (2.5) becomes to

$$\text{(2.6)} \qquad \begin{cases} \dfrac{du}{d\tau} = u(1-u) - \dfrac{auv}{u+d} = f(u,v)\,, \\ \dfrac{dv}{d\tau} = bv\left(1 - \dfrac{v}{u}\right) = g(u,v)\,, \end{cases}$$

which has only 3 dimensional parameters a, b and d.

Step II : Steady states

The equilibrium or steady states populations u^*, v^* are solutions of

$$\frac{du}{d\tau} = f(u^*, v^*) = 0, \quad \frac{dv}{d\tau} = g(u^*, v^*) = 0\,.$$

$$u^* = \frac{(1-a-d) + \sqrt{(1-a-d)^2 + 4d}}{2}, \quad v^* = u^*\,.$$

We are interested in the stability condition of the steady states. For the linear analysis write

$$u(\tau) = x(\tau) - u^*, \quad v(\tau) = y(\tau) + v^*\,,$$

$$\begin{pmatrix} \dfrac{dx}{d\tau} \\ \dfrac{dy}{d\tau} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} \dfrac{df}{du} & \dfrac{df}{dv} \\ \dfrac{dg}{du} & \dfrac{dg}{dv} \end{pmatrix},$$

$$(u^*, v^*) = \begin{pmatrix} u^*\left[\dfrac{au^*}{(u^*+d)^2} - 1\right] & -\dfrac{au}{u^*+d} \\ b & -b \end{pmatrix}.$$

For stability we require $Re(\lambda) < 0$ and so the necessary and sufficient conditions for linear stability are, from the last equation,

$$\det A = bu^*\left(1 + \frac{a}{u^*+d} - \frac{au^*}{(u^*+d)^2}\right) > 0,$$

$$\mathrm{tr}A < 0 \leftrightarrow u^*\left[\frac{au^*}{(u^*+d)^2} - 1\right] < b,$$

$$b > \left[a - \sqrt{(1-a-d)^2 + 4d}\right] \cdot \left[\frac{1+a+d-\sqrt{(1-a-d)^2+4d}}{2a}\right].$$

3. Modeling of Vegetable-Prey-Predator

3.1 Vegetable-Prey-Predator Model I

Let $V(t)$ is the vegetable, $P_1(t)$ is the prey, and $P_2(t)$ is the predator. Consider a Vegetable-Prey-Predator Model with the assumption that the vegetable's growth is unbounded in the absence of prey.

$$(3.1) \qquad \begin{cases} \dfrac{dV}{dt} = a - bVP_1, \\ \dfrac{dP_1}{dt} = P_1(cV - dP_2), \\ \dfrac{dP_2}{dt} = P_2(eP_1 - s), \end{cases}$$

where a, b, c, d, e and s are all positive constants.

a : growth rates of vegetable,

b : per capita eating rates of prey,

c : growth rates of prey,

d : per capita eating rates of predator,

e : growth rates of predator,

s : death rates of predator.

3.2 Vegetable-Prey-Predator Model II

In (3.1), Model I, we did not consider the vegetable's deduction happened by nature' destruction. Therefore, we can consider a Model containing that concept.

(3.2)
$$\begin{cases} \dfrac{dV}{dt} = a - bVP_1 + \varepsilon V\,, \\ \dfrac{dP_1}{dt} = P_1(cV - dP_2)\,, \\ \dfrac{dP_2}{dt} = P_2(eP_1 - s)\,, \end{cases}$$

where a, b, c, d, e and s are all positive constants having the same meaning in (3.1) and ε indicates the deduction rates of vegetable happened by nature's destruction.

3.3 Vegetable-Prey-Predator Model III

Next, we can consider another model having some maximum carrying capacity in the vegetable's growth, a logistic growth.

$$
(3.3) \qquad \begin{cases} \dfrac{du}{dt} = au(1-u) - buv\,, \\ \dfrac{dv}{dt} = v(cu - dw)\,, \\ \dfrac{dw}{dt} = w(ev - s)\,. \end{cases}
$$

In (3.3), we suppose that the carrying capacity equals one, $K = 1$.

3.4 Vegetable-Prey-Predator Model IV

$$
(3.4) \qquad \begin{cases} \dfrac{du}{dt} = au(1-u) - buv\,, \\ \dfrac{dv}{dt} = v(cu - dw - \delta)\,, \\ \dfrac{dw}{dt} = w(ev - s)\,. \end{cases}
$$

In (3.4), δ indicates the deduction rates of prey happened by the diseases or nature's disasters.

4. Analysis of a Vegetable-Prey-Predator Model

4.1 Analysis of a Vegetable-Prey-Predator Model I :

$$
(3.1) \qquad \begin{cases} \dfrac{dV}{dt} = a - bVP_1\,, \\ \dfrac{dP_1}{dt} = P_1(cV - dP_2)\,, \\ \dfrac{dP_2}{dt} = P_2(eP_1 - s)\,. \end{cases}
$$

Step I : Nondimensionalization

Put : $cV = u$, $bP_1 = v$, $dP_2 = w$, $ac = \alpha$, $\frac{e}{b} = \beta$, $s = \gamma$. (3.1) becomes to

$$
(4.1) \qquad \begin{cases} \dfrac{du}{dt} = \alpha - uv = f(u,v,w)\,, \\ \dfrac{dv}{dt} = v(u-w) = g(u,v,w)\,, \\ \dfrac{dw}{dt} = w(\beta v - \gamma) = h(u,v,w)\,. \end{cases}
$$

Step II : Steady States

$$
f = g = h = 0,\ (u^*, v^*, w^*) = \left(\frac{\alpha\beta}{\gamma}, \frac{\gamma}{\beta}, \frac{\alpha\beta}{\gamma}\right)\,.
$$

Step III : Community matrix

$$
\begin{pmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{pmatrix} \left(\frac{\alpha\beta}{\gamma}, \frac{\gamma}{\beta}, \frac{\alpha\beta}{\gamma}\right) = \begin{pmatrix} -\dfrac{\gamma}{\beta} & -\dfrac{\alpha\beta}{\gamma} & 0 \\ \dfrac{\gamma}{\beta} & 0 & -\dfrac{\gamma}{\beta} \\ 0 & \dfrac{\alpha\beta^2}{\gamma} & 0 \end{pmatrix},
$$

$$
\begin{aligned}
P(\lambda) &= |\lambda I - A|\,, \\
&= \begin{pmatrix} \lambda + \dfrac{\gamma}{\beta} & \dfrac{\alpha\beta}{\gamma} & 0 \\ -\dfrac{\gamma}{\beta} & \lambda & \dfrac{\gamma}{\beta} \\ 0 & -\dfrac{\alpha\beta^2}{\gamma} & \lambda \end{pmatrix}, \\
&= \lambda^3 + \frac{\gamma}{\beta}\lambda^2 + \alpha(\beta + 1) + \alpha\gamma\,.
\end{aligned}
$$

Let $\lambda_1, \lambda_2, \lambda_3$ are solutions of $P(\lambda) = 0$.

Sum of three solutions ; $\lambda_1 + \lambda_2 + \lambda_3 = -\dfrac{\gamma}{\beta}$.

Multiple of three solutions ; $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = -\alpha\gamma$.

From the elementary calculus we know if they are all real values, then they are all negative, and steady states is stable. And if they are one real and two complex values, then

$$\lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + 2\mathrm{Re}(\lambda_2) = -\frac{\gamma}{\beta},$$

$$\mathrm{Re}(\lambda_2) = -\frac{(\gamma + \beta\lambda_1)}{2\beta}.$$

Therefore,

$$\mathrm{Re}(\lambda_2) < 0 \leftrightarrow \gamma + \beta\lambda_1 > 0 \leftrightarrow \lambda_1 > -\frac{\gamma}{\beta}.$$

To show : $\lambda_1 > -\dfrac{\gamma}{\beta}$.

(1) $P \in C^1,\ P(0) = \alpha\gamma > 0$.

(2) $$P\left(-\frac{\gamma}{\beta}\right) = \lambda^3 + \frac{\gamma}{\beta}\lambda^2 + \alpha(\beta+1)\lambda + \alpha\gamma,$$

$$= \left(-\frac{\gamma}{\beta}\right)^3 + \frac{\gamma}{\beta}\left(-\frac{\gamma}{\beta}\right)^2 + \alpha(\beta+1)\left(-\frac{\gamma}{\beta}\right) = \alpha\gamma,$$

$$= -\frac{\alpha\gamma}{\beta} < 0.$$

From (1) and (2),

$$\exists \lambda_0 \in \left(-\frac{\gamma}{\beta}, 0\right) \text{ with } P(\lambda_0) = 0 \,.$$

But $P(\lambda) = 0$ has an unique real solution λ_1, and $\lambda_0 = \lambda_1$. Therefore, we can conclude that steady states $\left(\frac{\alpha\beta}{\gamma}, \frac{\gamma}{\beta}, \frac{\alpha\beta}{\gamma}\right)$ is always stable.

4.2 Analysis of a Vegetable-Prey-Predator Model II :

$$(3.2) \qquad \begin{cases} \dfrac{dV}{dt} = a - bVP_1 + \varepsilon V \,, \\ \dfrac{dP_1}{dt} = P_1(cV - dP_1) \,, \\ \dfrac{dP_2}{dt} = P_2(eP_1 - s) \,. \end{cases}$$

Step I : Nondimensionalization

$$\text{Put : } cV = u,\ bP_1 = v,\ dP_2 = w,\ ac = \alpha,\ \frac{e}{b} = \beta,\ s = \gamma \,.$$

Then, (3.2) becomes to

$$(4.2) \qquad \begin{cases} \dfrac{du}{dt} = \alpha - uv + \varepsilon u = f(u,v,w) \,, \\ \dfrac{dv}{dt} = v(u - w) = g(u,v,w) \,, \\ \dfrac{dw}{dt} = w(\beta v - \gamma) = h(u,v,w) \,. \end{cases}$$

Step II : Steady States

$$\left(-\frac{\alpha}{\varepsilon},0,0\right), \quad \left(\frac{\alpha\beta}{\gamma-\beta\varepsilon},\frac{\gamma}{\beta},\frac{\alpha\beta}{\gamma-\beta\varepsilon}\right).$$

Step III : Community Matrix

$$A = \begin{pmatrix} -v+\varepsilon & -u & 0 \\ v & u-w & -v \\ 0 & \beta w & \beta v-\gamma \end{pmatrix},$$

$$A_{21} = A\left(-\frac{\alpha}{\varepsilon},0,0\right) = \begin{pmatrix} \varepsilon & \frac{\alpha}{\varepsilon} & 0 \\ 0 & -\frac{\alpha}{\varepsilon} & 0 \\ 0 & 0 & -\gamma \end{pmatrix},$$

$$\det(\lambda I - A_{21}) = \begin{vmatrix} \lambda-\varepsilon & -\frac{\alpha}{\varepsilon} & 0 \\ 0 & \lambda+\frac{\alpha}{\varepsilon} & 0 \\ 0 & 0 & \lambda+\gamma \end{vmatrix},$$

$$= (\lambda-\varepsilon)\left(\lambda+\frac{\alpha}{\varepsilon}\right)(\lambda+\gamma).$$

Eigenvalues of A_{21} are ε, $-\frac{\alpha}{\varepsilon}$, $-\gamma$.

The Theorem 1.16 implies that steady states $\left(-\frac{\alpha}{\varepsilon},0,0\right)$ is unstable for all $\varepsilon \in \Re$.

$$A_{22} = A\left(\frac{\alpha\beta}{\gamma-\beta\varepsilon},\frac{\gamma}{\beta},\frac{\alpha\beta}{\gamma-\beta\varepsilon}\right) = \begin{pmatrix} -\frac{\gamma}{\beta}+\varepsilon & \frac{\alpha\beta}{\gamma-\beta\varepsilon} & 0 \\ \frac{\gamma}{\beta} & 0 & -\frac{\gamma}{\beta} \\ 0 & \frac{\alpha\beta^2}{\gamma-\beta\varepsilon} & 0 \end{pmatrix},$$

$$\det(\lambda I - A_{22}) = \lambda^3 + \frac{\gamma - \beta\varepsilon}{\beta}\lambda^2 + \frac{\alpha\gamma}{\gamma - \beta\varepsilon}(1+\beta)\lambda + \alpha\gamma = P(\lambda)\,.$$

If $\varepsilon > \dfrac{\gamma}{\beta}$ then, we know that the steady states $A\left(\dfrac{\alpha\beta}{\gamma-\beta\varepsilon}, \dfrac{\gamma}{\beta}, \dfrac{\alpha\beta}{\gamma-\beta\varepsilon}\right)$ is unstable by the Theorem 1.18.

When $\varepsilon < \dfrac{\gamma}{\beta}$, $P'(\lambda) = 0$ has two solutions with negative real parts.

Therefore, If $P(\lambda) = 0$ has three solutions, then they are all negative.

If $P(\lambda) = 0$ has one negative solution, say $\lambda_1 < 0$, and two complex solutions λ_2, λ_3 , then

$$\begin{aligned}\lambda_1 + \lambda_2 + \lambda_3 &= \lambda_1 + 2\mathrm{Re}(\lambda_2)\,,\\ &= -\frac{\gamma - \beta\varepsilon}{\beta}\,,\\ &= -\frac{\gamma}{\beta} + \varepsilon\,.\end{aligned}$$

$$\begin{aligned}2\mathrm{Re}(\lambda_2) &= -\frac{\gamma}{\beta} + \varepsilon - \lambda_1 < 0\,,\\ &\leftrightarrow \varepsilon - \frac{\gamma}{\beta} < \lambda_1 < 0\,.\end{aligned}$$

$$P\left(\varepsilon - \frac{\gamma}{\beta}\right) = -\alpha\gamma(1+\beta) + \alpha\gamma = -\alpha\beta\gamma < 0\,.$$

Therefore, $\left(\dfrac{\alpha\beta}{\gamma-\beta\varepsilon}, \dfrac{\gamma}{\beta}, \dfrac{\alpha\beta}{\gamma-\beta\varepsilon}\right)$ is stable.

4.3 Analysis of a Vegetable-Prey-Predator Model III :

$$(3.3)\qquad \begin{cases} \dfrac{du}{dt} = au(1-u) - buv\,, \\ \dfrac{dv}{dt} = v(cu - dw)\,, \\ \dfrac{dw}{dt} = w(ev - s)\,. \end{cases}$$

step I : Steady States

$$(1,0,0), (0,v^*,0), \left[1-\frac{bs}{ae}, \frac{s}{e}, \frac{c}{d}\left(1-\frac{bs}{ae}\right)\right].$$

step II : Community Matrix

$$A = \begin{pmatrix} a-2au-bv & -bu & 0 \\ cv & cu-dw & -dv \\ 0 & ew & ev-s \end{pmatrix},$$

$$A_{31} = A(1,0,0) = \begin{pmatrix} -a & -b & 0 \\ 0 & c & 0 \\ 0 & 0 & -s \end{pmatrix},$$

$$A_{32} = A(0,v^*,0) = \begin{pmatrix} a-bv^* & 0 & 0 \\ cv^* & 0 & -dv^* \\ 0 & 0 & ev^*-s \end{pmatrix},$$

$$A_{33} = A\left[1-\frac{bs}{ae}, \frac{s}{e}, \frac{c}{d}\left(1-\frac{bs}{ae}\right)\right],$$

$$= \begin{pmatrix} -a\left(1-\dfrac{bs}{ae}\right) & -b\dfrac{bs}{ae} & 0 \\ \dfrac{cs}{e} & 0 & -\dfrac{ds}{e} \\ 0 & \dfrac{ce}{d}\left(1-\dfrac{bs}{ae}\right) & 0 \end{pmatrix}$$

Step III : Characteristic Polynomials

$$P_{31}(\lambda) = \det(\lambda I - A_{31}) = (\lambda + a)(\lambda - c)(\lambda + s)\,,$$

$$P_{32}(\lambda) = \det(\lambda I - A_{32}) = \lambda(\lambda - a + bv^*)(\lambda - ev^* + s)\,,$$

$$P_{33}(\lambda) = \det(\lambda I - A_{33})\,,$$

$$= \lambda^3 + a\left(1 - \frac{bs}{ae}\right)\lambda^2 + cs\left(1 + \frac{b}{e}\right)\left(1 - \frac{bs}{ae}\right)\lambda + acs\left(1 - \frac{bs}{ae}\right)^2.$$

Step IV : Eigenvalues

$$P_{31}(\lambda) = 0 \quad \Rightarrow \lambda_{11} = -a,\ \lambda_{12} = c,\ \lambda_{13} = -s\,,$$

$$P_{32}(\lambda) = 0 \quad \Rightarrow \lambda_{21} = 0,\ \lambda_{22} = a - bv^*,\ \lambda_{23} = ev^* - s\,.$$

Step V : Stability and Unstability of the steady states

Using Theorem 1.16 and Theorem 1.8 we can investigate the stability and unstability.

(1) $(1, 0, 0)$: unstable.

(2) $(0, v^*, 0)$:

$$a - bv^* < 0 \,\text{and}\, ev^* - s < 0 \Rightarrow s > ev^* > e\frac{a}{b} = \frac{ea}{b}\,.$$

Therefore, if $s < \dfrac{ea}{b}$, then $(0, v^*, 0)$ is unstable.

If $s > \dfrac{ea}{b}$, then

$\dfrac{a}{b} < v^* < \dfrac{s}{e}$: $(0, v^*, 0)$ is stable.

Otherwise : $(0, v^*, 0)$ is unstable.

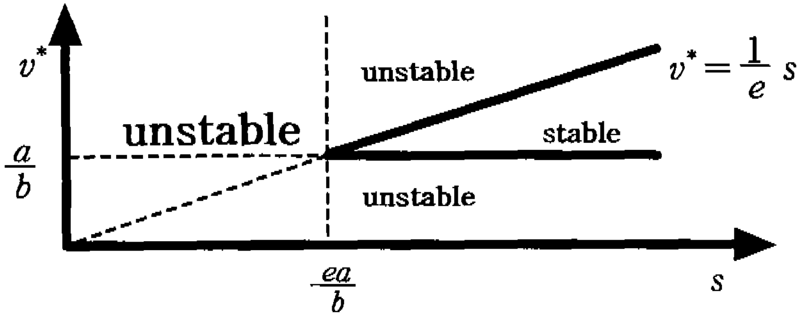


Figure 4.1. stable and unstable area of $(0, v^*, 0)$.

(3) $\left[1 - \frac{bs}{ae}, \frac{s}{e}, \frac{c}{d}\left(1 - \frac{bs}{ae}\right)\right]$:

$$P_{33}(\lambda) = \lambda^3 + a\left(1 - \frac{bs}{ae}\right)\lambda^2 + cs\left(1 + \frac{b}{e}\right)\left(1 - \frac{bs}{ae}\right)\lambda + acs\left(1 - \frac{bs}{ae}\right)^2$$

Case I ; When $1 - \frac{bs}{ae} < 0 \left(s > \frac{ae}{b}\right)$,

$\left[1 - \frac{bs}{ae}, \frac{s}{e}, \frac{c}{d}\left(1 - \frac{bs}{ae}\right)\right]$ is unstable by Theorem 1.16.

Case II ; When $1 - \frac{bs}{ae} = 0 \left(s = \frac{ae}{b}\right)$,

All eigenvalues are zero, and so

$$\left[1 - \frac{bs}{ae}, \frac{s}{e}, \frac{c}{d}\left(1 - \frac{bs}{ae}\right)\right] \text{is critical}.$$

4.4 Analysis of a Vegetable-Prey-Predator Model IV :

$$
(3.4) \qquad \begin{cases} \dfrac{du}{dt} = au(1-u) - buv\,, \\ \dfrac{dv}{dt} = v(cu - dw - \delta)\,, \\ \dfrac{dw}{dt} = w(ev - s)\,. \end{cases}
$$

Step I : Steady States

$$
(0,0,0), (1,0,0), \left(0, \frac{s}{e}, -\frac{\delta}{d}\right), \left[\frac{\delta}{c}, \frac{a}{b}\left(1-\frac{\delta}{c}\right), 0\right],
$$

$$
\left[1-\frac{bs}{ae}, \frac{s}{e}, \frac{c}{d}\left(1-\frac{bs}{ae}\right) - \frac{\delta}{d}\right].
$$

Step II : Community Matrix

$$
A = \begin{pmatrix} a - 2au - bv & -bu & 0 \\ cv & cu - dw - \delta & -dv \\ 0 & ew & ev - s \end{pmatrix},
$$

$$
A_{41} = A(0,0,0) = \begin{pmatrix} a & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & -s \end{pmatrix},
$$

$$
A_{42} = A(1,0,0) = \begin{pmatrix} -a & -b & 0 \\ 0 & c-\delta & 0 \\ 0 & 0 & -s \end{pmatrix},
$$

Case III ; When $1-\frac{bs}{ae}>0\ \left(s<\frac{ae}{b}\right)$,

$$P_{33}(\lambda)=\lambda^3+a\left(1-\frac{bs}{ae}\right)\lambda^2+cs\left(1+\frac{b}{e}\right)\left(1-\frac{bs}{ae}\right)\lambda+acs\left(1-\frac{bs}{ae}\right)^2.$$

$P'_{33}(\lambda)$ has two solutions with negative real parts.

If $P_{33}(\lambda)$ has one negative real, say $\lambda_{31}<0$, and two complex solutions $\lambda_{32},\lambda_{33}$, then

$$\begin{aligned}\lambda_{31}+\lambda_{32}+\lambda_{33}&=\lambda_{31}+2\mathrm{Re}\lambda_{32}\,,\\ &=-a\left(1-\frac{bs}{ae}\right).\end{aligned}$$

$$\begin{aligned}2\mathrm{Re}\lambda_{32}&=-a\left(1-\frac{bs}{ae}\right)-\lambda_{31}\,,\\ &\leftrightarrow -a\left(1-\frac{bs}{ae}\right)<\lambda_{31}<0.\end{aligned}$$

$$P_{33}\left[-a\left(1-\frac{bs}{ae}\right)\right]=-acs\frac{b}{e}\left(1-\frac{bs}{ae}\right)^2<0.$$

Therefore, there is $\lambda_{31}\in\left(-a\left(1-\frac{bs}{ae}\right),0\right)$ with $P_{33}(\lambda)=0$.

Hence, $\left[1-\frac{bs}{ae},\frac{s}{e},\frac{c}{d}\left(1-\frac{bs}{ae}\right)\right]$ is stable.

$$A_{43} = A\left(0, \frac{s}{e}, -\frac{\delta}{d}\right) = \begin{pmatrix} a\left(1-\frac{bs}{ae}\right) & 0 & 0 \\ \frac{cs}{e} & 0 & -\frac{ds}{e} \\ 0 & -\frac{e\delta}{d} & 0 \end{pmatrix},$$

$$A_{44} = A\left[\frac{\delta}{c}, \frac{a}{b}\left(1-\frac{\delta}{c}\right), 0\right],$$

$$= \begin{pmatrix} -\frac{a\delta}{c} & -\frac{b\delta}{c} & 0 \\ \frac{ac}{b}\left(1-\frac{\delta}{c}\right) & 0 & -\frac{ad}{b}\left(1-\frac{\delta}{c}\right) \\ 0 & -\frac{e\delta}{d} & 0 \end{pmatrix},$$

$$A_{45} = A\left[1-\frac{bs}{ae}, \frac{s}{e}, \frac{c}{d}\left(1-\frac{bs}{ae}\right)-\frac{\delta}{d}\right],$$

$$= \begin{pmatrix} -a\left(1-\frac{bs}{ae}\right) & \left(1-b\frac{bs}{ae}\right) & 0 \\ \frac{cs}{e} & 0 & -\frac{ds}{e} \\ 0 & \frac{e}{d}\left[c\left(1-\frac{bs}{ae}\right)-\delta\right] & 0 \end{pmatrix}$$

Step III : Characteristic Polynomials

$$P_{41}(\lambda) = \det(\lambda I - A_{41}) = (\lambda - a)(\lambda + \delta)(\lambda + s),$$

$$P_{42}(\lambda) = \det(\lambda I - A_{42}) = (\lambda + a)(\lambda - c + \delta)(\lambda + s),$$

$$P_{43}(\lambda) = \det(\lambda I - A_{43}) = \left(\lambda - a + \frac{bs}{e}\right)(\lambda^2 - s\delta),$$

$$P_{44}(\lambda) = \det(\lambda I - A_{44}),$$

$$= \left[\lambda + s - \frac{ae}{b}\left(1 - \frac{\delta}{c}\right)\right] \cdot \left[\lambda^2 + \frac{a\delta}{c}\lambda + a\delta\left(1 - \frac{\delta}{c}\right)\right],$$

$$P_{45}(\lambda) = \det(\lambda I - A_{45}),$$

$$= \lambda^3 + a\left(1 - \frac{bs}{ae}\right)\lambda^2 + cs\left(1 + \frac{b}{e}\right)\left(1 - \frac{bs}{ae}\right)\lambda$$

$$+ \frac{s}{e} \cdot \left[bc\left(1 - \frac{bs}{ae}\right) + ec\left(1 - \frac{bs}{ae}\right) - e\delta\right]\lambda$$

$$+ as\left(1 - \frac{bs}{ae}\right) \cdot \left[c\left(1 - \frac{bs}{ae}\right) - \delta\right].$$

Step IV : Eigenvalues

$P_{41}(\lambda) = 0 \quad \Rightarrow \lambda_{11} = a,\ \lambda_{12} = -\delta,\ \lambda_{13} = -s,$

$P_{42}(\lambda) = 0 \quad \Rightarrow \lambda_{21} = -a,\ \lambda_{22} = c - \delta,\ \lambda_{23} = -s,$

$P_{43}(\lambda) = 0 \quad \Rightarrow \lambda_{31} = a - \frac{bs}{e},\ \lambda_{32} = \sqrt{s\delta},\ \lambda_{33} = -\sqrt{s\delta}.$

Step V : Stability and Unstability of the steady states

Using theorem 1.16 and theorem 1.8 we can investigate the stability and unstability.

(1) $(u_1{}^*, v_1{}^*, w_1{}^*) = (0, 0, 0)$: unstable.

(2) $(u_2{}^*, v_2{}^*, w_2{}^*) = (1,0,0)$:

$\delta < c$: unstable, $\delta > c$: stable.

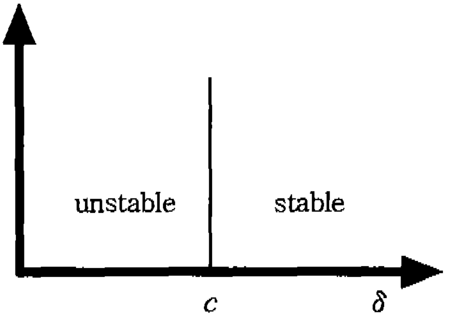


Figure 4.2. stable and unstable area of $(1,0,0)$.

(3) $(u_3{}^*, v_3{}^*, w_3{}^*) = \left(0, \frac{s}{e}, -\frac{\delta}{d}\right)$: always unstable.

(4) $(u_4{}^*, v_4{}^*, w_4{}^*) = \left[\frac{\delta}{c}, \frac{a}{b}\left(1-\frac{\delta}{c}\right), 0\right]$:

$$P_{44}(\lambda) = \left[\lambda + s - \frac{ae}{b}\left(1-\frac{\delta}{c}\right)\right] \cdot \left[\lambda^2 + \frac{a\delta}{c}\lambda + a\delta\left(1-\frac{\delta}{c}\right)\right] = 0.$$

$\left[\lambda^2 + \frac{a\delta}{c}\lambda + a\delta\left(1-\frac{\delta}{c}\right)\right] = 0$ has two solutions with negative real parts

Therefore,

$-s + \frac{ae}{b}\left(1-\frac{\delta}{c}\right) > 0$: $(u_4{}^*, v_4{}^*, w_4{}^*)$ is unstable.

$$-s+\frac{ae}{b}\left(1-\frac{\delta}{c}\right)<0 : ({u_4}^*, {v_4}^*, {w_4}^*) \text{ is stable.}$$

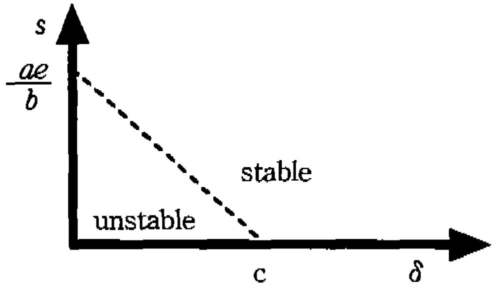


Figure 4.3. stable and unstable area of $({u_4}^*, {v_4}^*, {w_4}^*)$.

(5) $({u_5}^*, {v_5}^*, {w_5}^*) = \left[1-\frac{bs}{ae}, \frac{s}{e}, \frac{c}{d}\left(1-\frac{bs}{ae}\right)-\frac{\delta}{d}\right]$:

$$P_{45}(\lambda) = \lambda^3 + a\left(1-\frac{bs}{ae}\right)\lambda^2 + cs\left(1+\frac{b}{e}\right)\left(1-\frac{bs}{ae}\right)\lambda + \frac{s}{e}\cdot$$

$$\left[bc\left(1-\frac{bs}{ae}\right)+ec\left(1-\frac{bs}{ae}\right)-e\delta\right]\lambda + as\left(1-\frac{bs}{ae}\right)\left[c\left(1-\frac{bs}{ae}\right)-\delta\right]$$

Case I ; When $1-\frac{bs}{ae}<0 \left(s>\frac{ae}{b}\right)$,

$({u_5}^*, {v_5}^*, {w_5}^*)$ is unstable by Theorem 1.18.

Case II ; When $1-\frac{bs}{ae}=0 \left(s=\frac{ae}{b}\right)$,

$$\begin{aligned} P_{45}(\lambda) &= \lambda^3 - 3\delta\lambda\,, \\ &= \lambda(\lambda^2 - s\delta)\,, \\ &= \lambda(\lambda-\sqrt{s\delta})(\lambda+\sqrt{s\delta})\,. \end{aligned}$$

$$\lambda_{41} = 0,\ \lambda_{42} = \sqrt{s\delta},\ \lambda_{43} = \sqrt{s\delta}\,.$$

Case III ; When $1 - \frac{bs}{ae} > 0 \left(s < \frac{ae}{b}\right)$,

If $c\left(1 - \frac{bs}{ae}\right) - \delta < 0$, i.e $s > \frac{ae}{b}\left(1 - \frac{\delta}{c}\right)$, then

Multiple of three eigenvalues ;

$$\lambda_{51} \cdot \lambda_{52} \cdot \lambda_{53} = -as\left(1 - \frac{bs}{ae}\right)\left[c\left(1 - \frac{bs}{ae}\right) - \delta\right] > 0\,.$$

Therefore $(u_5{}^*, v_5{}^*, w_5{}^*)$ is unstable by Theorem 1.16.

If $s < \frac{ae^2}{(b+1)bc}\left(\frac{bc}{e} + c - \delta\right)$ then

$$\begin{aligned} & cs\left(1 + \frac{b}{e}\right)\left(1 - \frac{bs}{ae}\right)\lambda + \frac{s}{e}\left[bc\left(1 - \frac{bs}{ae}\right) + ec\left(1 - \frac{bs}{ae}\right) - e\delta\right], \\ &= -\frac{bc}{ae^2(b+1)}s\left[s - \frac{ae^2}{(b+1)bc}\left(\frac{bc}{e} + c - \delta\right)\right], \\ & < 0\,. \end{aligned}$$

Therefore $(u_5{}^*, v_5{}^*, w_5{}^*)$ is unstable by Theorem 1.16.

Otherwise :

If $P_{45}(\lambda) = 0$ has three real solutions, then they are negative. If not, $P_{45}(\lambda) = 0$ has one negative real, say $\lambda_{51} < 0$, and two complex solutions, $\lambda_{52}, \lambda_{53}$, then

$$\lambda_{51} + \lambda_{52} + \lambda_{53} = \lambda_{51} + 2\mathrm{Re}(\lambda_{52}) = -a\left(1 - \frac{bs}{ae}\right),$$

$$2\mathrm{Re}(\lambda_{52}) = \lambda_{51} - -a\left(1 - \frac{bs}{ae}\right) < 0\,,$$

$$\leftrightarrow \lambda_{51} > -a\left(1 - \frac{bs}{ae}\right).$$

To show : $\lambda_{51} > -a\left(1 - \dfrac{bs}{ae}\right)$.

$$\because \qquad P_{45}\left(-a\left(1 - \frac{bs}{ae}\right)\right) = -\frac{abc}{e}s\left(1 - \frac{bs}{ae}\right) < 0\,.$$

Therefore, $(u_5{}^*, v_5{}^*, w_5{}^*)$ is stable.

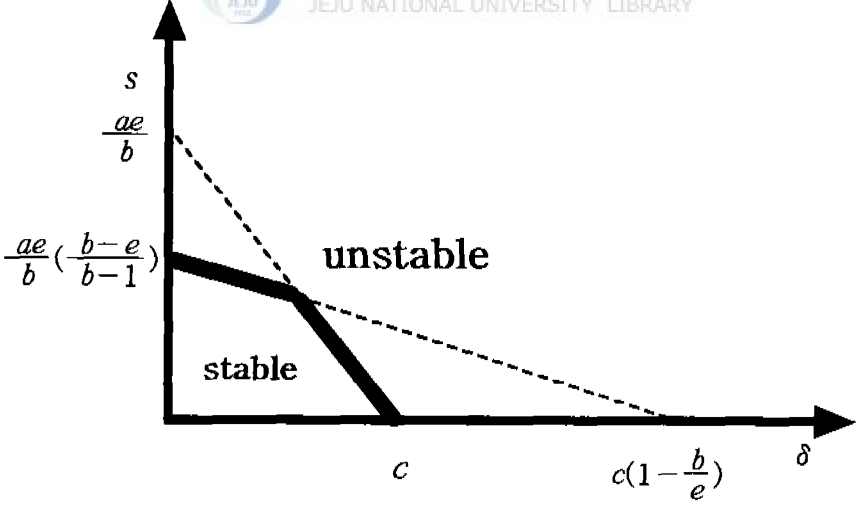


Figure 4.4. stable and unstable area of $(u_5{}^*, v_5{}^*, w_5{}^*)$.

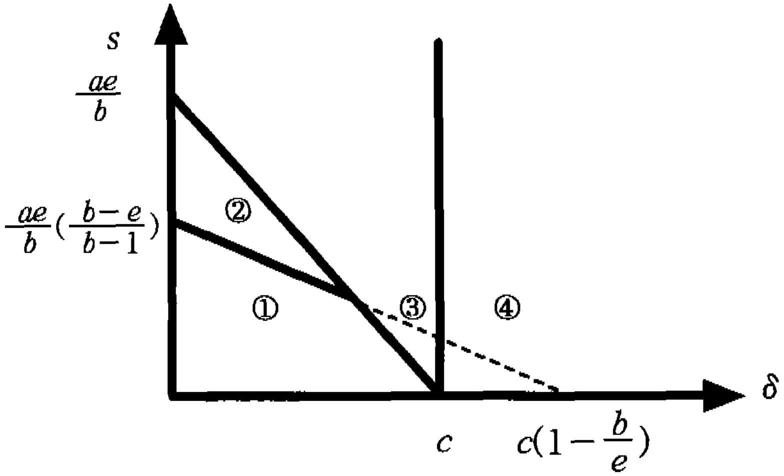


Figure 4.5. unstable area of Model IV.

Area (1) : $(u_2{}^*, v_2{}^*, w_2{}^*)$, $(u_4{}^*, v_4{}^*, w_4{}^*)$ is unstable.

Area (2) : $(u_2{}^*, v_2{}^*, w_2{}^*)$, $(u_4{}^*, v_4{}^*, w_4{}^*)$, $(u_5{}^*, v_5{}^*, w_5{}^*)$ is unstable.

Area (3) : $(u_2{}^*, v_2{}^*, w_2{}^*)$, $(u_5{}^*, v_5{}^*, w_5{}^*)$ is unstable.

Area (4) : $(u_5{}^*, v_5{}^*, w_5{}^*)$ is unstable.

5. Examples

Example 5.1 (Vegetable-Prey-Predator Model II)

(3.2)
$$\begin{cases} \dfrac{dV}{dt} = 0.5 - VP_1 - 0.189V\,, \\ \dfrac{dP_1}{dt} = P_1(V - P_2)\,, \\ \dfrac{dP_2}{dt} = P_2(1.1P_1 - 0.01)\,, \end{cases}$$

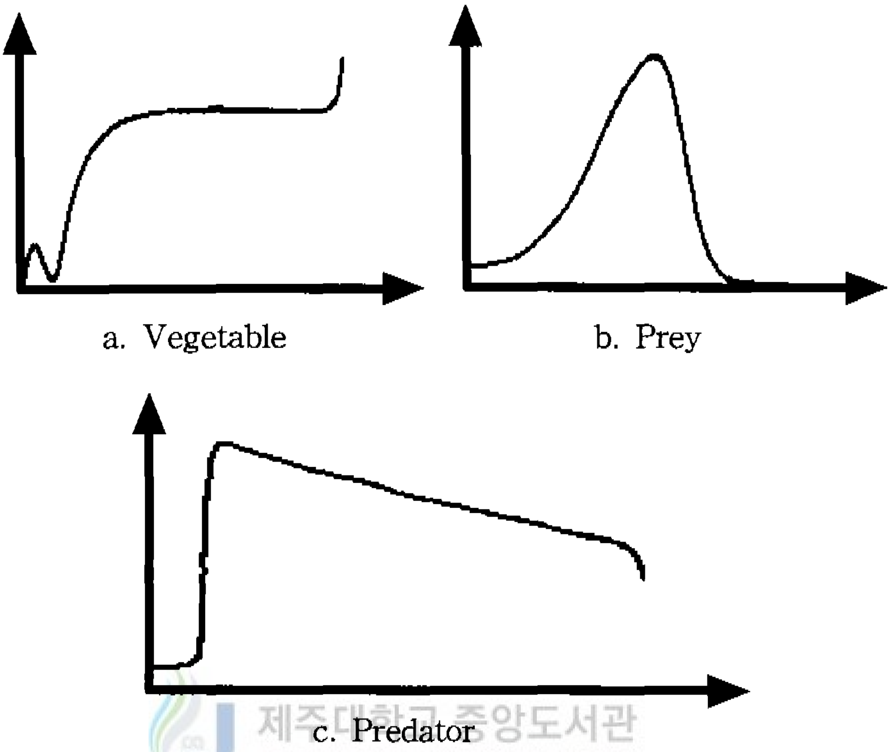


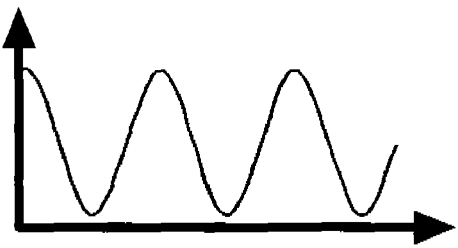
Figure 5.1.

Remark 5.1

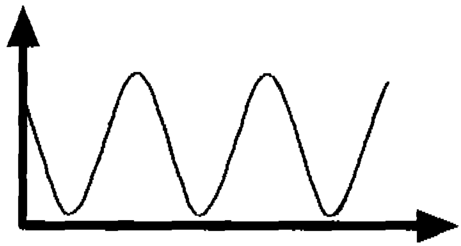
As vegetable population grow unboundedly, prey and predator population also grow for some times. But if predator population is too large, then prey deduce extremely toward extinction. And so, finally predator population also deduce toward extinction. Therefore, to prevent extinction effectively we need to regulate predator population artificially when it is too large. It is a way to permit the hunting temporarily (see Figure 5.1.).

Example 5.2 (Vegetable-Prey-Predator Model IV)

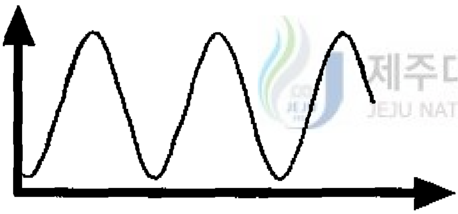
$$
(3.4) \qquad \begin{cases} \dfrac{du}{dt} = 0.1u(1-u) - 0.5uv\,, \\ \dfrac{dv}{dt} = v(0.7u - 0.5w - 0.3)\,, \\ \dfrac{dw}{dt} = w(0.2v - 0.01)\,. \end{cases}
$$



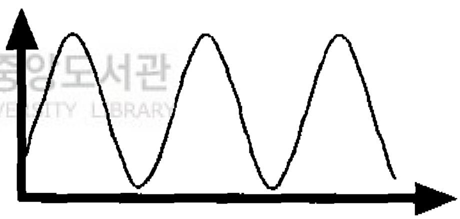
(a). $u(t)$: $t=(80000, 80510)$



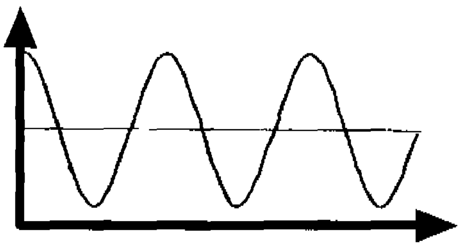
(b). $u(t)$: $t=(89000, 89510)$



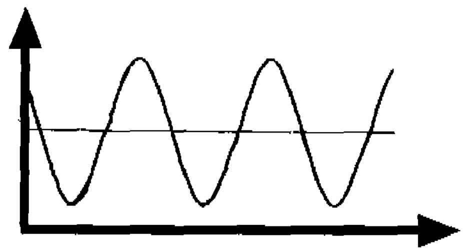
(c). $v(t)$: $t=(80000, 80510)$



(d). $v(t)$: $t=(89000, 89510)$



(e). $w(t)$: $t=(80000, 80510)$



(f). $w(t)$: $t=(89000, 89510)$

Figure 5.2.

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<국문 초록>

비선형 생태학 모델의 동력학적 분석

수리 생물학은 빠르게 성장하고 있는 주제이며, 매우 흥미있는 현대 응용 수학의 한 분야이다. 수리생물학은 모델링 작업과 그 모델의 분석 작업으로 이루어진다. 본 논문은 식물과 초식 동물 그리고 육식 동물 사이의 상호 관계를 다루고 있다. 첫 단계로서 우리는 V-P-P 모델을 설정하고, 그 다음으로 그 모델을 분석해 보았다. 분석 작업은 시스템의 안정성과 불안정성의 조건을 조사함으로서 출발하였다. 더욱이, 구체적인 예제를 통하여 우리는 V-P-P 모델의 동력학에서 초식 동물과 육식 동물의 멸종과 같은 중요한 변화를 발견하였다.

식물이 무한정으로 번식해감에 따라 초식 동물과 육식 동물의 수도 얼마 동안은 증가한다. 하지만, 육식 동물의 수가 너무 커지면 초식 동물은 멸종을 향해 극도로 감소한다. 초식 동물과 육식 동물의 멸종을 막기 위해서 우리는 그 수가 너무 커지기 전에 육식 동물의 수를 인위적인 방법으로 조정하여야만 한다.

감사의 글

어느덧 2년의 석사과정이 시간이 훌쩍 지나버린 것 같습니다. 입학할 때만 하더라도 멀게만 보였던 2년이라는 시간이 부지불식간에 흘러가 버렸습니다. 끝마쳤다는 기쁨 과 함께 그 기간 동안 좀더 열심히 보낼 수 있었을 텐데 하는 아쉬움도 남습니다. 힘들었던 시기였지만 여러분들의 도움으로 무사히 마칠 수 있었습니다. 직접 찾아 뵙고 고마움을 전하는 것이 도리인줄 알면서도 이렇게 지면을 통해 몇 분께 인사드립니다. 우선 부족한 저를 잘 이끌어주시고 여러모로 많은 도움을 주셨던 고봉수 교수님께 진심으로 감사드립니다. 또한 내가 힘들 때마다 옆에서 격려와 용기를 심어 주신 수학과 모든 교수님들께 감사드립니다. 논문 을 준비하는 과정에서 수학교육과 함량규 선생님이 제게 무엇보다 많은 도움을 주셨습니다. 저로 인해 많은 불편함이 있었음에도 불구하고 불평하기는커녕 늘 도움을 주셨던 기억이 지금도 생생합니다. 특히 힘들 때마다 기댈 수 있는 가족이 늘 함께 있어 제게는 무엇보다는 큰 힘이 되었습니다. 앞으로 이 모든 분들의 사랑과 기대에 어긋남이 없도록 열심히 생활하겠습니다.

1999년 12월