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Lower bounds for the eigenvalues of the basic Dirac operator

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Lower bounds for the eigenvalues of the basic Dirac operator

Soon Chan Kim

(Supervised by professor Seoung Dal Jung)

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Department of Mathematics

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CHEJU NATIONAL UNIVERSITY

Basic Dirac 연산자의 고유치의 하한

指導教授 鄭 承 達

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<**Abstract**>

Lower bounds for the eigenvalues of the basic Dirac operator

In this thesis, we review some basic properties of the transvese spin structure and the basic Dirac operator D_b. And we give some estimates of the eigenvalues of the basic Dirac operator by using the transversally conformal change of metric. Moreover, we give a sharp estimate of the basic Dirac operator by using the new connection. We also study the limiting cases of each estimates. In fact, at all of cases, the limiting foliations are minimal and transversally Einsteinian with a constant positive transversal scalar curvature.

1 Introduction

In 1963, A. Lichnerowicz([22]) showed that on a Riemannian spin manifold the square of the Dirac operator D is given by

$$D^2 = \nabla^*\nabla + \frac{1}{4}\,\sigma,$$

where $\nabla^*\nabla$ is the positive spinor Laplacian and σ the scalar curvature. In 1980, Th. Friedrich([6]) proved that any eigenvalue λ of the Dirac operator satisfies the inequality

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M \sigma \tag{1.1}$$

on a compact Riemannian spin manifold (M^n, g_M) with positive scalar curvature σ. The inequality (1.1) has been improved to many cases by many authors([8, 9, 10, 15, 14, 11, 18, 19, 20]). In particular, in 2001, S. D. Jung([10]) proved the lower bound for the eigenvalues λ of the basic Dirac operator D_b on a foliated Riemannian manifold with a transverse spin structure, which is introduced by J. Brüning and F. W. Kamber([2]). Namely, let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ of codimension $q > 1$ and a bundle-like metric g_M such that the mean curvature form κ is basic harmonic. Then the eigenvalue λ of the basic Dirac operator D_b satisfies the inequality

$$\lambda^2 \geq \frac{q}{4(q-1)}\ \inf_M(\sigma^\nabla + |\kappa|^2), \tag{1.2}$$

where σ^∇ is the transversal scalar curvature of $\mathcal{F}$. In the limiting case, the foliation $\mathcal{F}$ is minimal, transversally Einsteinian with constant transversal scalar curvature σ^∇. In 2004, S. D. Jung et al.([15]) improved the above inequality (1.2) by the first eigenvalue of the basic Yamabe operator Y_b, which is defined by

$$Y_b = 4\frac{q-1}{q-2}\ \Delta_B + \sigma^\nabla, \tag{1.3}$$

where Δ_B is a basic Laplacian acting on basic functions. In fact, any eigenvalue λ of the basic Dirac operator D_b satisfies the inequality

$$\lambda^2 \geq \frac{q}{4(q-1)}\,(\mu_1 + \inf_M |\kappa|^2), \tag{1.4}$$

where μ_1 is the first eigenvalue of the basic Yamabe operator.

In this thesis, we give another estimates for the square of the eigenvalue λ of the basic Dirac operator D_b.

This artcle is organized as followings. In Chapter 2, we review the known facts on the foliated Riemannian manifold. In Chapter 3, we study some basic properties of the transverse spin structure and the basic Dirac operator D_b. In Chapter 4, we give the new proof of the estimate (1.4) with the conformal change of the transversal twistor operator. In Chapter 5, we give a sharper estimate than (1.2) with a modified connection. Namely, any eigenvalue λ of the basic Dirac operator D_b satisfies

$$\lambda^2 \geq \frac{q}{4(q-1)}\,\inf_M \Big(\sigma^2 + \frac{q+1}{q}\,|\kappa|^2\Big). \tag{1.5}$$

In Chapter 6, we apply some techniques and concerning conformal change of the Riemannian metric to get a sharper estimate than (1.4). Namely,

$$\lambda^2 \geq \frac{q}{4(q-1)}\,\Big(\mu_1 + \frac{q+1}{q}\,\inf_M |\kappa|^2\Big), \tag{1.6}$$

where μ_1 is the first eigenvalue of the basic Yamabe operator.

2 The geometry of foliations

2.1 Definition

Definition 2.1 *A family $\mathcal{F} \equiv \{L_\alpha\}_{\alpha\in A}$ of connected subsets of a manifold M^{p+q} is called a p-dimensional(or codimension q)* **foliation** *if*

(1) $\cup_\alpha L_\alpha = M$,

(2) $\alpha \neq \beta \Longrightarrow L_\alpha \cap L_\beta = \varnothing$,

(3) *for any point $p \in M$ there exists a C^r-chart(local coordinate system) (φ_p, U_p), such that $p \in U_p$ and if $U_p \cap L_\alpha \neq \varnothing$, then $\varphi_p(U_p \cap L_\alpha) = A_c \cap \varphi(U_p)$, where*

$$A_c = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \mid y = \text{constant}\}.$$

Here $(\varphi_\alpha, U_\alpha)$ is called a distinguished(or foliated) chart.

Roughly speaking, a foliation corresponds to a decomposition of a manifold into a union of connected submanifolds of dimension p called *leaves*.
Remark. From (3), we know that on $U_i \cap U_j \neq \varnothing$, the coordinate change $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ has the form

$$\varphi_j \circ \varphi_i^{-1}(x, y) = (\varphi_{ij}(x, y), \gamma_{ij}(y)), \tag{2.1}$$

where $\gamma_{ij} : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism. Let $pr : \mathbb{R}^{p+q} \to \mathbb{R}^q$ be the projection and N be a q-dimensional manifold with a chart (V, ψ_V). Then $\psi_V^{-1} \circ pr \circ \varphi_U : U \to V \subset N$ is a submersion and the leaves of $\mathcal{F}$ in U are given as the fibers of a submersion $f = \psi_V^{-1} \circ pr \circ \varphi_U : U \to V$ onto an open subset V of a model manifold N.

2.2 Riemannian foliation

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a foliation $\mathcal{F}$ of codimension q. Let L be a tangent bundle of a foliation $\mathcal{F}$ and then L is the integrable subbundle of TM. i.e.,

$$X,\ Y \in \Gamma(U, L) \Longrightarrow [X, Y] \in \Gamma(U, L).$$

The normal bundle Q of $\mathcal{F}$ on M is the quotient bundle $Q = TM/L$. Then the metric g_M defines a splitting σ in the exact sequence of vector bundles

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0,$$

with $\sigma : Q \to L^{\perp}$ isomorphism. Thus $g_M = g_L \oplus g_{L^{\perp}}$ induces a metric g_Q on Q. With $g_Q = \sigma^* g_{L^{\perp}}$, the splitting map $\sigma : (Q, g_Q) \to (L^{\perp}, g_{L^{\perp}})$ is a metric isomorphism.

Definition 2.2 *A key fact to study of transversal geometry is the existence of the* **Bott connection** $\mathring{\nabla}$ *in* Q *defined by*

$$\mathring{\nabla}_X s = \pi[X, Y_s] \quad \text{for } X \in \Gamma L,\ s \in \Gamma Q, \tag{2.2}$$

where $Y_s \in \Gamma TM$ *is any vector field projecting to* s *under* $\pi : TM \to Q$.

The right hand side in (2.2) is independent of the choice of Y_s. Trivially $\mathring{R}(X, Y) = 0$ for $X,\ Y \in \Gamma L$.

Definition 2.3 *A foliation is* **Riemannian** *if there exists a metric* g_Q *on* Q *satisfying* $\mathring{\nabla}_X g_Q = 0$ *(or* $\theta(X) g_Q = 0$*) for any* $X \in \Gamma L$*. i.e.,*

$$X g_Q(s, t) = g_Q(\mathring{\nabla}_X s, t) + g_Q(s, \mathring{\nabla}_X t) \quad \text{for } X \in \Gamma L,\ s, t \in \Gamma Q.$$

Definition 2.4 *A Riemannian metric* g_M *on* M *is* **bundle-like** *with respect to* $\mathcal{F}$ *if the fiber metric* g_Q *induced on* Q *turns the foliation into a Riemannian foliation.*

A Riemannian foliation admits a bundle-like metric. In fact, we choose any fiber metric g_L on L, a splitting $\sigma : Q \to L^\perp$ and set g_M equal to the orthogonal sum $g_L + g_Q$ on $TM \cong L \oplus \sigma Q$.

For a distinguished coordinate system (x_α, y_α) in U_α, $\{\partial/\partial x_j\}(j = 1, \cdots, p)$ is a basis of L and $\{\omega_i = dx_i + A_i^a dy_a\}(a = 1, \cdots, q)$ forms a basis of L^*. So $\{\omega_1, \cdots, \omega_p, dy_1, \cdots, dy_q\}$ forms a basis of the cotangent bundle T^*M. Then g_M has local expression

$$g_M = \sum g_{ij}(x, y)\omega_i \otimes \omega_j + \sum g_{ab}(x, y)dy_a \otimes dy_b. \tag{2.3}$$

In particular, if g_M is a bundle-like metric, then g_M is of the form

$$g_M = \sum g_{ij}(x, y)\omega_i \otimes \omega_j + \sum g_{ab}(y)dy_a \otimes dy_b. \tag{2.4}$$

Definition 2.5 *In each distinguished coordinate chart $(U_\alpha, (x_\alpha, y_\alpha))$, a frame field $\{X_1, \cdots, X_p, X_{p+1}, \cdots, X_{p+q}\}$ is an* **adapted frame field** *to the foliation $\mathcal{F}$ if $\{X_1, \cdots, X_p\}$ and $\{\pi(X_{p+1}), \cdots, \pi(X_{p+q})\}$ are basis of ΓL and ΓQ, respectively.*

Theorem 2.6 ([27]) *The followings are equivalent.*

(1) *The metric g_M is a bundle-like metric*

(2) *There exists an orthonormal adapted frame $\{X_i, X_a\}$ such that $\overset{\circ}{\nabla}_X \pi(X_a) = 0$ for any $X \in \Gamma L$.*

(3) *There exists an orthonormal adapted frame $\{X_i, X_a\}$ such that*

$$g_M(\nabla_{X_a} X_i, X_b) + g_M(\nabla_{X_b} X_i, X_a) = 0.$$

(4) *All geodesics orthogonal to a leaf at one point are orthogonal to each leaf at every point.*

Definition 2.7 *The* **transverse Levi-Civita connection** *on* $Q = TM/L$ *is defined by*

$$\nabla_X s = \begin{cases} \overset{\circ}{\nabla}_X s = \pi[X, Y_s] & \text{for } X \in \Gamma L, \\ \pi(\nabla^M_X Y_s) & \text{for } X \in \Gamma L^\perp, \end{cases} \tag{2.5}$$

where $s \in \Gamma Q$ *and* $Y_s \in \Gamma L^\perp$ *corresponding to* s *under the canonical isomorphism* $L^\perp \cong Q$.

The transverse Levi-Civita connection ∇ is metrical and torsion free. That is, $\nabla_X g_Q = 0$ for all $X \in \Gamma TM$ and for all $Y, Z \in \Gamma TM$

$$T_\nabla(Y, Z) = \nabla_Y \pi(Z) - \nabla_Z \pi(Y) - \pi[Y, Z] = 0.$$

Definition 2.8 *The transversal* **sectional curvature** K^∇, **Ricci curvature** ρ^∇ *and* **scalar curvature** σ^∇ *are defined by*

$$K^\nabla(s,t) = \frac{g_Q(R^\nabla(s,t)t, s)}{g_Q(s,s)g_Q(t,t) - g_Q(s,t)^2}, \quad \forall s, t \in \Gamma Q$$
$$\rho^\nabla(s) = \sum_a R^\nabla(s, E_a)E_a, \quad \sigma^\nabla = g_Q(\rho^\nabla(E_a), E_a),$$

where $\{E_a\}$ *is a local orthonormal basic frame of* Q *and* R^∇ *is the curvature tensor for* ∇, *which is defined by*

$$R^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for } X, Y \in TM.$$

Definition 2.9 $\mathcal{F}$ *is said to be transversally* **Einsteinian** *if the model space* N *is Einsteinian, i.e.,* $\rho^\nabla = \frac{1}{q}\sigma^\nabla I$, *where* σ^∇ *is constant.*

2.3 Transversal divergence theorem

The *second fundamental form* $\alpha : \Gamma L \times \Gamma L \to \Gamma Q$ of $\mathcal{F}$ is given by

$$\alpha(X, Y) = \pi(\nabla^M_X Y) \quad \text{for } X, Y \in \Gamma L. \tag{2.6}$$

It is trivial that α is Q-valued, bilinear and symmetric.

Definition 2.10 *The* **mean curvature vector field** τ *of* $\mathcal{F}$ *is then defined by*

$$\tau = \sum_i \alpha(E_i, E_i) = \sum_i \pi(\nabla^M_{E_i} E_i), \tag{2.7}$$

where $\{E_i\}_{i=1,\cdots,p}$ *is an orthonormal basis of* L. *The dual form* κ, *the* **mean curvature form** *of* $\mathcal{F}$, *is then given by*

$$\kappa(X) = g_Q(\tau, X) \quad \text{for } X \in \Gamma Q \tag{2.8}$$

The foliation $\mathcal{F}$ *is said to be* **minimal** (*or* **harmonic**) *if* $\kappa = 0$. *The foliation* $\mathcal{F}$ *is* **totally geodesic** *if and only if* $\alpha = 0$.

Theorem 2.11 (Tautness theorem [1,23]) *Let* $(M, g_M, \mathcal{F})$ *be a closed, oriented Riemannian manifold with a Riemannian foliation* $\mathcal{F}$ *of codimension* $q \geq 2$ *and a bundle-like metric* g_M. *If the transversal Ricci operator* ρ^∇ *is positive definite, then* $\mathcal{F}$ *is taut, i.e., there exists a bundle-like metric* $\tilde{g}_M$ *for which all leaves are minimal submanifolds.*

Let $V(\mathcal{F})$ be the space of all vector fields Y on M satisfying $[Y, Z] \in \Gamma L$ for all $Z \in \Gamma L$. An element of $V(\mathcal{F})$ is called an *infinitesimal automorphism* of $\mathcal{F}$. Let

$$\bar{V}(\mathcal{F}) = \{\bar{Y} = \pi(Y) | Y \in V(\mathcal{F})\}. \tag{2.9}$$

It is trivial that an element s of $\bar{V}(\mathcal{F})$ satisfies $\nabla_X s = 0$ for all $X \in \Gamma L$([28]). Hence we have [25]

$$\bar{V}(\mathcal{F}) \cong \Omega^1_B(\mathcal{F}). \tag{2.10}$$

Theorem 2.12 (Transversal divergence theorem [30]) *Let* $(M, g_M, \mathcal{F})$ *be a closed, oriented, connected Riemannian manifold with a transversally orientable foliation* $\mathcal{F}$ *and a bundle-like metric* g_M *with respect to*

$\mathcal{F}$. Then for any vector field $X \in V(\mathcal{F})$

$$\int_M div_\nabla(\bar{X}) = \int_M g_Q(\bar{X}, \tau), \tag{2.11}$$

where $div_\nabla(\bar{X})$ denotes the transverse divergence of $\bar{X}$ with respect to the transverse Levi-Civita connection.

Corollary 2.13 *If $\mathcal{F}$ is minimal, then we have that for any $X \in V(\mathcal{F})$,*

$$\int_M div_\nabla(\bar{X}) = 0. \tag{2.12}$$

2.4 Basic De-Rham Cohomology

Definition 2.14 *A differential form $\omega \in \Omega^r(M)$ is* **basic***, if*

$$i(X)\omega = 0,\ \theta(X)\omega = 0 \text{ for } X \in \Gamma L. \tag{2.13}$$

In a distinguished chart $(x_1, \ldots, x_p; y_1, \ldots, y_q)$ of $\mathcal{F}$, a basic form w is expressed by

$$\omega = \sum_{a_1 < \cdots < a_r} \omega_{a_1 \cdots a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r},$$

where the functions $\omega_{a_1 \cdots a_r}$ are independent of x, i.e. $\frac{\partial}{\partial x_i}\omega_{a_1 \cdots a_r} = 0$.

Let $\Omega^r_B(\mathcal{F})$ be the set of all basic r-forms on M. The exterior derivative d preserves basic forms, since $\theta(X)d\omega = d\theta(X)\omega = 0,\ i(X)d\omega = \theta(X)\omega - di(X)\omega = 0$ for a basic form ω. Hence $\Omega^r_B(\mathcal{F})$ constitutes a subcomplex

$$d : \Omega^r_B(\mathcal{F}) \to \Omega^{r+1}_B(\mathcal{F})$$

of the De Rham complex $\Omega^*(M)$ and the restriction $d_B = d|\Omega^*_B(\mathcal{F})$ is well defined. Its cohomology

$$H_B(\mathcal{F}) = H(\Omega^*_B(\mathcal{F}), d_B)$$

is the *basic cohomology* of $\mathcal{F}$. It plays the role of the De Rham cohomology of the leaf space $M/\mathcal{F}$ of the foliation. Let δ_B be the formal adjoint operator of d_B. Then we have the following proposition ([1, 10]).

Proposition 2.15 *On a Riemannian foliation $\mathcal{F}$, we have*

$$d_B = \sum_a \theta_a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B), \tag{2.14}$$

where κ_B is the basic component of κ, $\{E_a\}$ is a local orthonormal basic frame in Q and $\{\theta_a\}$ its g_Q-dual basic 1-form.

The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega_B^1(\mathcal{F})$. We already know that κ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric ([28]).

Definition 2.16 *The* **basic Laplacian** *acting on $\Omega_B^*(\mathcal{F})$ is defined by*

$$\Delta_B = d_B\delta_B + \delta_B d_B. \tag{2.15}$$

The following theorem is proved in the same way as the corresponding usual result in De Rham-Hodge Theory.

Theorem 2.17 ([28]) *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed oriented manifold (M, g_M). Assume g_M to be bundle-like metric with $\kappa \in \Omega_B^1(\mathcal{F})$. Then there is a decomposition into mutually orthogonal subspaces*

$$\Omega_B^r(\mathcal{F}) = im d_B \oplus im \delta_B \oplus \mathcal{H}_B^r(\mathcal{F})$$

with finite dimensional $\mathcal{H}_B^r(\mathcal{F}) = \{\omega \in \Omega_B^r(\mathcal{F}) | \Delta_B\omega = 0\}$. Moreover,

$$H_B^r(\mathcal{F}) \cong \mathcal{H}_B^r(\mathcal{F}).$$

If $\mathcal{F}$ is the foliation by points of M, the basic Laplacian is the ordinary Laplacian. In the more general case, the basic Laplacian and its spectrum provide information about the transverse geometry of $(M, \mathcal{F})$([26]).

2.5 Curvatures of transversally conformal metrics

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar{g}_Q = e^{2u} g_Q$. Let $\bar{\nabla}$ be the metric and torsion free connection corresponding to $\bar{g}_Q$. Then we have the following proposition.

Proposition 2.18 ([15]) *On a Riemannian foliation, we have that for $X, Y \in \Gamma TM$,*

$$\bar{\nabla}_X \pi(Y) = \nabla_X \pi(Y) + X(u)\pi(Y) + Y(u)\pi(X) - g_Q(\pi(X), \pi(Y)) d_B u, \tag{2.16}$$

where $d_B u := grad_\nabla(u) = \sum_a E_a(u) E_a$ is a transversal gradient of u.

Proof. Since $\bar{\nabla}$ is the metric and torsion free connection with respect to $\bar{g}_Q$ on Q, we have

$$\begin{aligned} 2\bar{g}_Q(\bar{\nabla}_X s, t) =& X\bar{g}_Q(s,t) + Y_s \bar{g}_Q(\pi(X), t) - Z_t \bar{g}_Q(\pi(X), s) \\ &+ \bar{g}_Q(\pi[X, Y_s], t) + \bar{g}_Q(\pi[Z_t, X], s) - \bar{g}_Q(\pi[Y_s, Z_t], \pi(X)), \end{aligned}$$

where $\pi(Y_s) = s$ and $\pi(Z_t) = t$. From this formula, the proof is completed. □

Proposition 2.19 *On a Riemannian foliation, the curvature tensor associated with $\bar{g}_Q$ is given by*

$$\begin{aligned} R^{\bar{\nabla}}(X,Y)s =& R^{\nabla}(X,Y)s - g_Q(\pi(Y), s)\nabla_X d_B u + g_Q(\pi(X), s)\nabla_Y d_B u \\ &+ \{Y(u)s(u) - g_Q(\pi(Y), s)|d_B u|^2 - g_Q(\nabla_Y d_B u, s)\}\pi(X) \\ &- \{X(u)s(u) - g_Q(\pi(X), s)|d_B u|^2 - g_Q(\nabla_X d_B u, s)\}\pi(Y) \\ &+ \{X(u) g_Q(\pi(Y), s) - Y(u) g_Q(\pi(X), s)\} d_B u \end{aligned}$$

for any $X, Y \in TM$ and $s \in \Gamma Q$.

Proof. By a direct calculation, from (2.16), we have

$$\bar\nabla_X\bar\nabla_Y s$$
$$= \nabla_X\nabla_Y s + X(u)\nabla_Y s + Y(u)\nabla_X s + s(u)\nabla_X\pi(Y) - g_Q(\pi(Y), s)\nabla_X d_B u$$
$$+ \{g_Q(\nabla_Y s, d_B u) + Y(u)s(u) + s(u)Y(u) - g_Q(\pi(Y), s)|d_B u|^2\}\pi(X)$$
$$+ \{g_Q(\nabla_X s, d_B u) + g_Q(s, \nabla_X d_B u) + X(u)s(u)\}\pi(Y)$$
$$+ \{g_Q(\nabla_X\pi(Y), d_B u) + g_Q(\pi(Y), \nabla_X d_B u) + X(u)Y(u)\}s$$
$$+ \{-g_Q(\pi(X), \nabla_Y s) - Y(u)g_Q(\pi(X), s) - s(u)g_Q(\pi(X), \pi(Y))$$
$$- g_Q(\nabla_X\pi(Y), s) - g_Q(\pi(Y), \nabla_X s)\}d_B u.$$

Hence the proof is completed. □

Lemma 2.20 *On a Riemannian foliation, the mean curvature form $\kappa_{\bar g}$ associated with $\bar g_Q = e^{2u}g_Q$ satisfies $\kappa_{\bar g} = e^{-2u}\kappa$. Moreover, $v_{\bar g} = e^{qu}v_g$ for volume element v_g of g_Q.*

Proof. From (2.7), (2.8), we have

$$g_M(\tau, X) = g_M(\sum_i \nabla^M_{E_i}E_i, X), \quad \forall X \in \Gamma Q,$$

where $\{E_i\}_{i=1,\cdots,p}$ is an orthonormal basis of L. Let $\bar g_M = g_L + \bar g_Q$ be a transversally conformal metric of g_M. Then $\bar E_i = E_i$ $(i = 1, \cdots, p)$. Hence we have that, for any $X \in Q$,

$$\begin{aligned}\bar g_M(\tau_{\bar g}, X) =& \bar g_M(\sum_i \bar\nabla^M_{\bar E_i}\bar E_i, X) = \bar g_M(\sum_i \bar\nabla^M_{E_i}E_i, X)\\ =&\frac{1}{2}\sum_i\{E_i\bar g_M(E_i, X) + E_i\bar g_M(E_i, X) - X\bar g_M(E_i, E_i)\\ &+ \bar g_M([E_i, E_i], X) + \bar g_M([X, E_i], E_i) - \bar g_M([E_i, X], E_i)\}\\ =& g_M(\sum_i \nabla^M_{E_i}E_i, X) = g_M(\tau, X).\end{aligned}$$

In the last equality of the above equation, we used the fact that $g_M(X,Y) = 0$ for $X \in \Gamma L$, $Y \in \Gamma Q$ and $g_L = \bar{g}_L$. Hence we have that, for any $X \in Q$,

$$e^{2u} g_Q(\tau_{\bar{g}}, X) = \bar{g}_M(\tau_{\bar{g}}, X) = g_M(\tau, X) = g_Q(\tau, X),$$

which implies $\tau_{\bar{g}} = e^{-2u}\tau$ and so $\kappa_{\bar{g}} = e^{-2u}\kappa$. On the other hand, the volume form $v_{\bar{g}}$ of $\bar{g}_Q$ is given by

$$v_{\bar{g}} = \bar{\theta}_1 \wedge \cdots \wedge \bar{\theta}_q.$$

Since $\bar{\theta}_a = e^u \theta_a$, the last statement follows. □

Lemma 2.21 *On a Riemannian foliation, the basic Laplacian $\bar{\Delta}_B$ associated with $\bar{g}_Q = e^{2u} g_Q$ satisfies*

$$\bar{\Delta}_B f = e^{-2u}\{\Delta_B f - (q-2) g_Q(d_B f, d_B u)\} \tag{2.17}$$

for any basic function f.

Proof. From (2.14) and (2.15), we have

$$\begin{aligned}
\bar{\Delta}_B f :=& \bar{\delta}_B \bar{d}_B f \\
=& -\sum_{a,b} i(\bar{E}_a)\bar{\nabla}_{\bar{E}_a}(\bar{\theta}_b \wedge \bar{\nabla}_{\bar{E}_b} f) + i(\kappa_{\bar{g}})\bar{d}_B f \\
=& -\sum_{a,b} \bar{E}_b(f)\bar{g}_Q(\bar{E}_a, \bar{\nabla}_{\bar{E}_a}\bar{E}_b) - \sum_a \bar{E}_a\bar{E}_a(f) + \kappa_{\bar{g}}(f),
\end{aligned}$$

where $\{\bar{E}_a\}$ is an orthonormal basic frame associated to $\bar{g}_Q$ and $\{\bar{\theta}_a\}$ its $\bar{g}_Q$-dual 1-form. Note that from (2.16)

$$\bar{\nabla}_{E_a}\bar{E}_b = \bar{E}_b(u)E_a - e^{-u}\delta_{ab} d_B u. \tag{2.18}$$

Hence we have

$$\begin{aligned}
\bar{\Delta}_B f =& -\sum_{a,b} \bar{E}_b(f)\bar{E}_b(u) + \sum_{a,b} \bar{E}_b(f)\delta_{ab}\bar{E}_a(u) \\
&+ \sum_a \bar{E}_a(u)\bar{E}_a(f) - e^{-2u}\sum_a E_a E_a(f) + e^{-2u}\kappa(f) \\
=& e^{-2u}\{\Delta_B f - (q-2)E_a(u)E_a(f)\},
\end{aligned}$$

which proves (2.17). □

From (2.17), we have the following corollary.

Corollary 2.22 *For any transversally conformal change* $\bar{g}_Q = e^{2u} g_Q = h^{\frac{4}{q-2}} g_Q$ $(q \geq 3)$ *on* $\mathcal{F}$*, we have*

$$e^{2u}\bar{\Delta}_B(h^{-1}f) = h^{-1}\Delta_B f - fh^{-2}\Delta_B h. \tag{2.19}$$

Proof. From (2.17), we have

$$e^{2u}\bar{\Delta}_B(h^{-1}f) = \Delta_B(h^{-1}f) - (q-2)g_Q(d_B u, d_B(h^{-1}f)).$$

On the other hand, a direct calculation gives

$$\Delta_B(h^{-1}f) = -fh^{-2}\Delta_B h + h^{-1}\Delta_B f - 2fh^{-3}|d_B h|^2 + 2h^{-2}g_Q(d_B h, d_B f).$$

Since $u = \frac{2}{q-2}\ln h$, we have

$$g_Q(d_B u, d_B(h^{-1}f)) = -\frac{2}{q-2}\, fh^{-3}|d_B h|^2 + \frac{2}{q-2}\, h^{-2}g_Q(d_B h, d_B f).$$

Hence we completed the proof. □

The transversal Ricci curvature $\rho^{\bar{\nabla}}$ of $\bar{g}_Q = e^{2u}g_Q$ and the transversal scalar curvature $\sigma^{\bar{\nabla}}$ of $\bar{g}_Q$ are related to the transversal Ricci curvature ρ^{∇} of g_Q and the transversal scalar curvature σ^{∇} of g_Q by the following lemma.

Lemma 2.23 ([15]) *On a Riemannian foliation* $\mathcal{F}$*, we have that for any* $X \in Q$*,*

$$\begin{aligned} e^{2u}\rho^{\bar{\nabla}}(X) =& \rho^{\nabla}(X) + (2-q)\nabla_X d_B u + (2-q)|d_B u|^2 X \\ &+ (q-2)X(u)d_B u + \{\Delta_B u - \kappa(u)\}X, \end{aligned} \tag{2.20}$$

$$e^{2u}\sigma^{\bar{\nabla}} = \sigma^{\nabla} + (q-1)(2-q)|d_B u|^2 + 2(q-1)\{\Delta_B u - \kappa(u)\}. \tag{2.21}$$

Corollary 2.24 *On a Riemannian foliation $\mathcal{F}$, the scalar curvature $\sigma^{\bar\nabla}$ associated with $\bar g_Q = e^{2u}g_Q = h^{\frac{4}{q-2}}g_Q$ is simplied as*

$$h^{\frac{q+2}{q-2}}\sigma^{\bar\nabla} = 4\frac{q-1}{q-2}\ \{\Delta_B h - \kappa(h)\} + \sigma^{\nabla}h. \tag{2.22}$$

Proof. Since $u = \frac{2}{q-2}\ \ln h$ for $q \geq 3$, we have $d_B u = \frac{2}{q-2}\ h^{-1}d_B h$. Hence we have

$$\Delta_B u = \frac{2}{q-2}\ h^{-2}|d_B h|^2 + \frac{2}{q-2}\ h^{-1}\Delta_B h. \tag{2.23}$$

From (2.21), we have

$$\sigma^{\bar\nabla} = h^{\frac{-4}{q-2}}\sigma^{\nabla} + 4\ \frac{q-1}{q-2}\ h^{-\frac{q+2}{q-2}}\Delta_B h - 4\ \frac{q-1}{q-2}\ h^{-\frac{q+2}{q-2}}\kappa(h),$$

which implies (2.22). □

Now we define the generalized *basic Yamabe operator* $\tilde Y_b$ by

$$\tilde Y_b = 4\ \frac{q-1}{q-2}\ (\Delta_B - \nabla_\kappa) + \sigma^{\nabla}. \tag{2.24}$$

Lemma 2.25 *On a Riemannian foliation $\mathcal{F}$ of codimension $q \geq 3$, the generalized basic Yamabe operator of the transversally conformal metric satisfies the following equation: For $\bar g_Q = h^{\frac{4}{q-2}}g_Q$,*

$$\bar{\tilde Y}_b(h^{-1}f) = h^{\frac{-q-2}{q-2}}\tilde Y_b f. \tag{2.25}$$

Proof. From (2.19), (2.22) and (2.24), we have

$$\begin{aligned}\bar{\tilde Y}_b(h^{-1}f) =& 4\ \frac{q-1}{q-2}\ \{\bar\Delta_B(h^{-1}f) - \bar\nabla_\kappa(h^{-1}f) + \sigma^{\bar\nabla}(h^{-1}f)\\ =& 4\frac{q-1}{q-2}\ h^{\frac{-q-2}{q-2}}\{\Delta_B f - \kappa(f)\} + h^{\frac{-q-2}{q-2}}\sigma^{\nabla}f,\end{aligned}$$

which implies (2.25). □

Definition 2.26 *For any vectors* $X, Y \in TM$ *and* $s \in \Gamma Q$*, the* **transversal Weyl conformal curvature tensor** W^∇ *is defined by*

$$
\begin{aligned}
&W^\nabla(X,Y)s \\
&= R^\nabla(X,Y)s \\
&+ \frac{1}{q-2}\{g_Q(\rho^\nabla(\pi(X)),s)\pi(Y) - g_Q(\rho^\nabla(\pi(Y)),s)\pi(X) \\
&\qquad + g_Q(\pi(X),s)\rho^\nabla(\pi(Y)) - g_Q(\pi(Y),s)\rho^\nabla(\pi(X))\} \\
&- \frac{\sigma^\nabla}{(q-1)(q-2)}\{g_Q(\pi(X),s)\pi(Y) - g_Q(\pi(Y),s)\pi(X)\}.
\end{aligned} \tag{2.26}
$$

By a direct calculation, the transversal Weyl conformal curvature tensor W^∇ vanishes identically for $q = 3$, where $q = codim\mathcal{F}$. Moreover, we have the following theorem ([13]).

Theorem 2.27 *Let* $(M, g_M, \mathcal{F})$ *be a Riemannian manifold with a foliation* $\mathcal{F}$ *and a bundle-like metric* g_M *with respect to* $\mathcal{F}$*. Then the transversal Weyl conformal curvature tensor is invariant under any transversally conformal change of* g_M*.*

Proof. By a long calculation with Proposition 2.19 and Lemma 2.23, we have that $W^{\bar\nabla} = W^\nabla$. □

3 Transversal Dirac operators

3.1 Clifford algebras

Definition 3.1 *Let V be a vector space over a field $K = \{\mathbb{R}, \mathbb{C}\}$ of dimension n and g a non-degenerate bilinear form on V. The* **Clifford algebra** $Cl(V, g)$ *associated to g on V is the algebra over K generated by V with the relation*

$$v \cdot w + w \cdot v = -2g(v, w) \tag{3.1}$$

for v, $w \in V$. The product "$\cdot$" is called the Clifford multiplication.

Remark. (1) If $(E_1, E_2, \cdots, E_n)$ is a g-orthonormal basis of V, then

$$\{E_{i_1} \cdot E_{i_2} \cdot ... \cdot E_{i_k} | 1 \leq i_1 < i_2 < \cdots < i_k \leq n,\ 0 \leq k \leq n\}$$

is a basis of $Cl(V, g)$, and so $\dim Cl(V, g) = 2^n$.

(2) There is a canonical isomorphism of vector spaces between the exterior algebra and the Clifford algebra of (V, g) which is given by :

$$\wedge^* V \overset{\cong}{\rightarrow} Cl(V, g) \quad \text{as} \quad E_{i_1} \wedge ... \wedge E_{i_k} \mapsto E_{i_1} \cdot ... \cdot E_{i_k}.$$

This isomorphism does not depend on the choice of the basis. Let us denote $Cl_n = Cl(\mathbb{R}^n, <, >)$. Then we have the following proposition.

Proposition 3.2 ([21]) *For all $v \in \mathbb{R}^n$ and all $\varphi \in Cl_n$, we have*

$$v \cdot \varphi \simeq v \wedge \varphi - i(v)\varphi \quad \textit{and} \quad \varphi \cdot v \simeq (-1)^p (v \wedge \varphi + i(v)\varphi),$$

where $\wedge$ denotes the exterior, $i(v)$ the interior product and $\varphi \in \wedge^p \mathbb{R}^n \subset \wedge^* \mathbb{R}_n \simeq Cl_n$.

Definition 3.3 *The* **Pin group** $Pin(V)$ *is defined by*

$$Pin(V) = \{a \in Cl(V) | a = a_1 \cdots a_k, \|a_i\| = 1\}. \tag{3.2}$$

The **Spin group** *is defined by*

$$Spin(V) = \{a \in Pin(V) | aa^t = 1\}, \tag{3.3}$$

where $a^t = a_k \cdots a_1$ *for any* $a = a_1 \cdots a_k$. *Equivalently,* $Spin(V) = \{e_1 \cdots e_{2k} || e_i| = 1\}$.

Let V be a real vector space. Then $Spin(V)$ is a compact and connected Lie group, and for $\dim V \geq 3$, it is also simply connected. Thus, for $\dim V \geq 3$, $Spin(V)$ is the universal cover of $SO(V)$(for detail, see [21]).

3.2 Basic Dirac operator

Let $(M, g_M, \mathcal{F})$ be a Riamannian manifold with a transversally oriented Riemannian foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Let $SO(q) \to P_{SO} \to M$ be the principal bundle of (oriented) transverse orthonormal framings. Then a *transverse spin structure* is a principal $Spin(q)$-bundle P_{Spin} together with two sheeted covering $\xi : P_{Spin} \to P_{SO}$ such that $\xi(p \cdot q) = \xi(p)\xi_0(g)$ for all $p \in P_{Spin}$, $g \in Spin(q)$, where $\xi_0 : Spin(q) \to SO(q)$ is a covering. In this case, the foliation $\mathcal{F}$ is called a *transverse spin foliation*. We then define the *foliated spinor bundle* $S(\mathcal{F})$ associated to P_{Spin} by

$$S(\mathcal{F}) = P_{Spin} \times_{Spin(q)} S_q,$$

where S_q is the irreducible spinor space associated to Q. The Hermitian metric $< \cdot, \cdot >$ on $S(\mathcal{F})$ induced from g_Q satisfies the following relation:

$$< \varphi, \psi > = < v \cdot \varphi, v \cdot \psi > \tag{3.4}$$

for every $v \in Q$, $g_Q(v,v) = 1$ and φ, $\psi \in S_q$. And the Riemannian connection ∇ on P_{SO} defined by (2.5) can be lifted to one on P_{Spin}, in particular, to one on $S(\mathcal{F})$, which will be denoted by the same letter.

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M such that $\Delta_B \kappa = 0$. The existence of such a metric is assured from [23, 24]. Let $S(\mathcal{F})$ be a foliated spinor bundle on a transverse spin foliation $\mathcal{F}$ and $< \cdot, \cdot >$ a Hermitian metric on $S(\mathcal{F})$. By the Clifford multiplication " $\cdot$ " in the fibers of $S(\mathcal{F})$ for any vector field $X \in Q$ and any spinor field $\Psi \in S(\mathcal{F})$, the Clifford multiplication $X \cdot \Psi \in S(\mathcal{F})$ is well-defined. Then we have

Proposition 3.4 *For any X, $Y \in \Gamma Q$ and $\Phi \in \Gamma S(\mathcal{F})$, the following properties hold:*

$$(X \cdot Y + Y \cdot X)\Phi = -2g_Q(X, Y)\Phi, \tag{3.5}$$

$$< X \cdot \Psi, \Phi > + < \Psi, X \cdot \Phi >= 0. \tag{3.6}$$

Proposition 3.5 ([8,21]) *The spinorial covariant derivative on $S(\mathcal{F})$ is given locally by:*

$$\nabla \Psi_\alpha = \frac{1}{4} \sum_{a,b} g_Q(\nabla E_a, E_b) E_a \cdot E_b \cdot \Psi_\alpha, \tag{3.7}$$

where Ψ_α is an orthonormal basis of S_q. And the curvature transform R^S on $S(\mathcal{F})$ is given as

$$R^S(X,Y)\Phi = \frac{1}{4} \sum_{a,b} g_Q(R^\nabla(X,Y)E_a, E_b) E_a \cdot E_b \cdot \Phi \tag{3.8}$$

for $X, Y \in TM$, where $\{E_a\}$ is an orthonormal basis of the normal bundle Q.

Proposition 3.6 ([21]) *For any* $X \in \Gamma TM$, $Y \in \Gamma Q$, *and* $\Psi, \Phi \in \Gamma S(\mathcal{F})$, *we have*

$$\nabla_X(Y \cdot \Psi) = (\nabla_X Y) \cdot \Psi + Y \cdot (\nabla_X \Psi), \tag{3.9}$$

$$X < \Psi, \Phi > = < \nabla_X \Psi, \Phi > + < \Psi, \nabla_X \Phi > . \tag{3.10}$$

We now define a canonical section $\mathcal{R}^\nabla$ of $Hom(S(\mathcal{F}), S(\mathcal{F}))$ by the formula

$$\mathcal{R}^\nabla(\Psi) = \sum_{a<b} E_a \cdot E_b \cdot R^S(E_a, E_b)\Psi. \tag{3.11}$$

Theorem 3.7 ([10]) *On the foliated spinor bundle* $S(\mathcal{F})$, *we have the following equation*

$$\sum_a E_a \cdot R^S(X, E_a)\Psi = -\frac{1}{2}\, \rho^\nabla(X) \cdot \Psi \tag{3.12}$$

for all $X \in \Gamma Q$.

Corollary 3.8 ([10]) *On* $S(\mathcal{F})$, *we have* $\mathcal{R}^\nabla = \frac{1}{4}\, \sigma^\nabla$.

Definition 3.9 *The* **transversal Dirac operator** D_{tr} *is locally defined by*

$$D_{tr}\Psi = \sum_a E_a \cdot \nabla_{E_a}\Psi - \frac{1}{2}\kappa \cdot \Psi \quad \textit{for } \Psi \in \Gamma S(\mathcal{F}), \tag{3.13}$$

where $\{E_a\}$ *is a local orthonormal basic frame of* Q.

We can easily prove that D_{tr} is formally self-adjoint. i.e.,

$$\int_M < D_{tr}\Psi, \Phi >_{g_Q} = \int_M < \Psi, D_{tr}\Phi >_{g_Q} \tag{3.14}$$

for all Ψ, $\Phi \in \Gamma S(\mathcal{F})$. We define the subspace $\Gamma_B(S(\mathcal{F}))$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$\Gamma_B(S(\mathcal{F})) = \{\Psi \in \Gamma S(\mathcal{F}) \mid \nabla_X \Psi = 0 \quad \text{for } X \in \Gamma L\}.$$

Trivially, we see that D_{tr} leaves $\Gamma_B(S(\mathcal{F}))$ invariant if and only if the foliation $\mathcal{F}$ is isoparametric, i.e., $\kappa \in \Omega^1_B(\mathcal{F})$. Let $D_b = D_{tr}|_{\Gamma_B(S(\mathcal{F}))}$: $\Gamma_B(S(\mathcal{F})) \to \Gamma_B(S(\mathcal{F}))$. This operator D_b is called the *basic Dirac operator* on (smooth) basic sections.

Theorem 3.10 ([7,10]) *On an isoparametric transverse spin foliation $\mathcal{F}$ with $\delta_B\kappa = 0$, the Lichnerowicz type formula is given by*

$$D^2_{tr}\Psi = \nabla^*_{tr}\nabla_{tr}\Psi + \frac{1}{4}K^{\nabla}_{\sigma}\Psi, \tag{3.15}$$

where $K^{\nabla}_{\sigma} = \sigma^{\nabla} + |\kappa|^2$ and

$$\nabla^*_{tr}\nabla_{tr}\Psi = -\sum_a \nabla^2_{E_a,E_a}\Psi + \nabla_\kappa\Psi \tag{3.16}$$

with $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla^M_X Y}$ for all X, $Y \in \Gamma TM$.

The operator $\nabla^*_{tr}\nabla_{tr}$ is non-negative and formally self-adjoint([10]) such that

$$\int_M < \nabla^*_{tr}\nabla_{tr}\Phi, \Psi >_{g_Q} = \int_M < \nabla_{tr}\Phi, \nabla_{tr}\Psi >_{g_Q} \tag{3.17}$$

for all Φ, $\Psi \in \Gamma S(\mathcal{F})$.

3.3 The transversal Dirac operator of transversally conformal metrics

Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar{g}_Q = e^{2u}g_Q$. Let $P_{SO}(\mathcal{F})$ and $\bar{P}_{SO}(\mathcal{F})$ be the principal bundles of g_Q- and $\bar{g}_Q$-orthogonal frames, respectively. Locally, the section $\bar{s}$ of $\bar{P}_{SO}(\mathcal{F})$ corresponding a section $s = (E_1, \cdots, E_q)$ of $P_{SO}(\mathcal{F})$ is $\bar{s} = (\bar{E}_1, \cdots, \bar{E}_q)$, where $\bar{E}_a = e^{-u}E_a(a = 1, \cdots, q)$. This isometry will be denoted by I_u. Thanks to the isomorphism I_u one can define a transverse spin structure $\bar{P}_{spin}(\mathcal{F})$ on $\mathcal{F}$ in such a way that the diagram

$$\begin{array}{ccc} P_{spin}(\mathcal{F}) & \xrightarrow{\tilde{I}_u} & \bar{P}_{spin}(\mathcal{F}) \\ \downarrow & & \downarrow \\ P_{so}(\mathcal{F}) & \xrightarrow{I_u} & \bar{P}_{so}(\mathcal{F}) \end{array}$$

commutes.

Let $\bar{S}(\mathcal{F})$ be the foliated spinor bundles associated with $\bar{P}_{spin}(\mathcal{F})$. For any section Ψ of $S(\mathcal{F})$, we write $\bar{\Psi} \equiv I_u \Psi$. If $<\ ,\ >_{g_Q}$ and $<\ ,\ >_{\bar{g}_Q}$ denote respectively the natural Hermitian metrics on $S(\mathcal{F})$ and $\bar{S}(\mathcal{F})$, then for any Φ, $\Psi \in \Gamma S(\mathcal{F})$

$$< \Phi, \Psi >_{g_Q} = < \bar{\Phi}, \bar{\Psi} >_{\bar{g}_Q}, \tag{3.18}$$

and the Clifford multiplication in $\bar{S}(\mathcal{F})$ is given by

$$\bar{X} \mathbin{\bar{\cdot}} \bar{\Psi} = \overline{X \cdot \Psi} \quad \text{for } X \in \Gamma Q. \tag{3.19}$$

Proposition 3.11 ([15]) *The connections ∇ and $\bar{\nabla}$ acting respectively on the sections of $S(\mathcal{F})$ and $\bar{S}(\mathcal{F})$, are related, for any vector field X and any spinor field Ψ by*

$$\bar{\nabla}_X \bar{\Psi} = \overline{\nabla_X \Psi} - \frac{1}{2}\, \overline{\pi(X) \cdot d_B u \cdot \Psi} - \frac{1}{2}\, g_Q(d_B u, \pi(X)) \bar{\Psi}. \tag{3.20}$$

Let $\bar{D}_{tr}$ be the transversal Dirac operator associated with the metric $\bar{g}_Q = e^{2u} g_Q$ and acting on the sections of the foliated spinor bundles $\bar{S}(\mathcal{F})$. Let $\{E_a\}$ be a local frame of $P_{SO}(\mathcal{F})$ and $\{\bar{E}_a\}$ a local frame of $\bar{P}_{SO}(\mathcal{F})$. Locally, $\bar{D}_{tr}$ is expressed by

$$\bar{D}_{tr} \bar{\Psi} = \sum_a \bar{E}_a \mathbin{\bar{\cdot}} \bar{\nabla}_{\bar{E}_a} \bar{\Psi} - \frac{1}{2}\, \kappa_{\bar{g}} \mathbin{\bar{\cdot}} \bar{\Psi}, \tag{3.21}$$

where $\kappa_{\bar{g}}$ is the mean curvature form associated with $\bar{g}_Q$. Using (3.19), we have that for any Ψ

$$\bar{D}_{tr} \bar{\Psi} = e^{-u} \Big(\overline{D_{tr} \Psi} + \frac{q-1}{2}\, \overline{d_B u \cdot \Psi} \Big). \tag{3.22}$$

Now, for any function f, we have

$$D_{tr}(f\Psi) = d_B f \cdot \Psi + f D_{tr}\Psi. \tag{3.23}$$

Hence we have

$$\bar{D}_{tr}(f\bar{\Psi}) = e^{-u}\overline{d_B f \cdot \Psi} + f\bar{D}_{tr}\bar{\Psi}. \tag{3.24}$$

From (3.22) and (3.24), we have the following proposition.

Proposition 3.12 ([15]) *Let $\mathcal{F}$ be the transverse spin foliation of codimension q. Then the transverse Dirac operators D_{tr} and $\bar{D}_{tr}$ satisfy*

$$\bar{D}_{tr}(e^{-\frac{q-1}{2}u}\bar{\Psi}) = e^{-\frac{q+1}{2}u}\overline{D_{tr}\Psi} \tag{3.25}$$

for any spinor field $\Psi \in S(\mathcal{F})$.

From Proposition 3.12, if $D_{tr}\Psi = 0$, then $\bar{D}_{tr}\bar{\Phi} = 0$, where $\Phi = e^{-\frac{q-1}{2}u}\Psi$, and conversely. So, on the transverse spin foliation $\mathcal{F}$, the dimesion of the space of the foliated harmonic spinors is a transversally conformal invariant.

Theorem 3.13 *On the transverse spin foliation with the basic harmonic mean curvature form κ, we have that on $\bar{S}(\mathcal{F})$*

$$\bar{D}_{tr}^2\bar{\Psi} = \bar{\nabla}_{tr}^*\bar{\nabla}_{tr}\bar{\Psi} + \frac{1}{4}\, K_\sigma^{\bar{\nabla}}\bar{\Psi}, \tag{3.26}$$

where

$$\bar{\nabla}_{tr}^*\bar{\nabla}_{tr}\bar{\Psi} = -\sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\sum_a \bar{\nabla}_{\bar{E}_a}\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \tag{3.27}$$

$$K_\sigma^{\bar{\nabla}} = \sigma^{\bar{\nabla}} + |\bar{\kappa}|^2 + 2(q-2)\kappa_{\bar{g}}(u). \tag{3.28}$$

Proof. Fix $x \in M$ and choose a local orthonormal basic frame $\{E_a\}$ satisfying $(\nabla E_a)_x = 0$ at $x \in M$. Then by definition,

$$
\begin{aligned}
\bar{D}_{tr}^2 \bar{\Psi} =& \sum_{a,b} \bar{E}_b \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_b} \bar{E}_a \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_a} \bar{\Psi} + \sum_{a,b} \bar{E}_b \,\bar{\cdot}\, \bar{E}_a \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_b} \bar{\nabla}_{\bar{E}_a} \bar{\Psi} \\
& - \frac{1}{2} \sum_b \bar{E}_b \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_b} \kappa_{\bar{g}} \,\bar{\cdot}\, \bar{\Psi} - \frac{1}{2} \sum_b \bar{E}_b \,\bar{\cdot}\, \kappa_{\bar{g}} \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_b} \bar{\Psi} \\
& - \frac{1}{2} \sum_a \kappa_{\bar{g}} \,\bar{\cdot}\, \bar{E}_a \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_a} \bar{\Psi} + \frac{1}{4}\, \kappa_{\bar{g}} \,\bar{\cdot}\, \kappa_{\bar{g}} \,\bar{\cdot}\, \bar{\Psi}.
\end{aligned}
$$

From (2.18), we have

$$
\sum_{a,b} \bar{E}_b \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_b} \bar{E}_a \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_a} \bar{\Psi} = -e^{-u} \Big(q \bar{\nabla}_{\overline{d_B u}} \bar{\Psi} + \sum_a \bar{E}_a \,\bar{\cdot}\, \overline{d_B u} \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_a} \bar{\Psi} \Big),
$$

and

$$
\begin{aligned}
\sum_{a,b} \bar{E}_b \,\bar{\cdot}\, \bar{E}_a \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_b} \bar{\nabla}_{\bar{E}_a} \bar{\Psi} = & - \sum_a \bar{\nabla}_{\bar{E}_a} \bar{\nabla}_{\bar{E}_a} \bar{\Psi} + \sum_{a<b} \bar{E}_a \,\bar{\cdot}\, \bar{E}_b \,\bar{\cdot}\, \bar{R}^S(\bar{E}_a, \bar{E}_b) \bar{\Psi} \\
& + \sum_{a<b} \bar{E}_a \,\bar{\cdot}\, \bar{E}_b \,\bar{\cdot}\, \bar{\nabla}_{[\bar{E}_a, \bar{E}_b]} \bar{\Psi}.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\bar{\nabla}_{\bar{E}_b} \kappa_{\bar{g}} =& e^{-u} \Big(- e^{-2u} E_b(u) \kappa + e^{-2u} \nabla_{E_b} \kappa + e^{-2u} \kappa(u) E_b \\
& - e^{-2u} g_Q(E_b, \kappa) d_B u \Big),
\end{aligned}
$$

and so

$$
\begin{aligned}
\sum_b \bar{E}_b \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_b} \kappa_{\bar{g}} \,\bar{\cdot}\, \bar{\Psi} =& e^{-2u} \Big(\sum_b \overline{E_b \cdot \nabla_{E_b} \kappa \cdot \Psi} + (2-q) \kappa(u) \bar{\Psi} \Big) \\
=& - |\bar{\kappa}|^2 \bar{\Psi} - (q-2) \kappa_{\bar{g}}(u) \bar{\Psi},
\end{aligned}
$$

and

$$
-\frac{1}{2} \sum_b \bar{E}_b \,\bar{\cdot}\, \kappa_{\bar{g}} \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_b} \bar{\Psi} - \frac{1}{2} \sum_a \kappa_{\bar{g}} \,\bar{\cdot}\, \bar{E}_a \,\bar{\cdot}\, \bar{\nabla}_{\bar{E}_a} \bar{\Psi} = \bar{\nabla}_{\kappa_{\bar{g}}} \bar{\Psi}.
$$

Hence we have

$$\bar{D}_{tr}^2\bar{\Psi} = -\sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\sum_a \bar{\nabla}_{\bar{E}_a}\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}$$
$$+\sum_{a<b} \bar{E}_a \bar{\cdot} \bar{E}_b \bar{\cdot} \bar{R}^S(\bar{E}_a, \bar{E}_b)\bar{\Psi} + \frac{1}{2}\,(q-2)\kappa_{\bar{g}}(u)\bar{\Psi} + \frac{1}{4}\,|\bar{\kappa}|^2\bar{\Psi}.$$

From (3.11) and Corollary 3.8, the proof is completed. □

Lemma 3.14 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\int_M < \bar{\nabla}_{tr}^*\bar{\nabla}_{tr}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} = \int_M < \bar{\nabla}_{tr}\bar{\Psi}, \bar{\nabla}_{tr}\bar{\Phi} >_{\bar{g}_Q} \tag{3.29}$$

for all $\Phi,\ \Psi \in S(\mathcal{F})$, where $< \bar{\nabla}_{tr}\bar{\Psi}, \bar{\nabla}_{tr}\bar{\Phi} >_{\bar{g}_Q} = \sum_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Phi} >_{\bar{g}_Q}$.

Proof. Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$ for all a. Then, from (3.27), we have

$$\begin{aligned}
&< \bar{\nabla}_{tr}^*\bar{\nabla}_{tr}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} \\
&= -\sum_a < \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} + < \bar{\nabla}_{\sum_a \bar{\nabla}_{\bar{E}_a}\bar{E}_a}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} + < \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} \\
&= -\sum_a \bar{E}_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} + \sum_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Phi} >_{\bar{g}_Q} \\
&\quad + (1-q)e^{-2u} < \bar{\nabla}_{d_B u}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} + < \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} \\
&= -\,div_{\bar{\nabla}}(V) + \sum_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Phi} >_{\bar{g}_Q} + < \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q},
\end{aligned}$$

where $V \in \Gamma Q \otimes \mathbb{C}$ are defined by $\bar{g}_Q(V, Z) = < \bar{\nabla}_Z\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q}$ for all $Z \in \Gamma Q$. The last line is proved as follows: At $x \in M$,

$$\begin{aligned}
div_{\bar{\nabla}}(V) &= \sum_a \bar{g}_Q(\bar{\nabla}_{\bar{E}_a}V, \bar{E}_a) \\
&= \sum_a \bar{E}_a\bar{g}_Q(V, \bar{E}_a) - \bar{g}_Q(V, \bar{\nabla}_{\bar{E}_a}\bar{E}_a) \\
&= \sum_a \bar{E}_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} - (1-q)e^{-2u} < \bar{\nabla}_{d_B u}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q}.
\end{aligned}$$

By the transversal divergence theorem on $\mathcal{F}$([30]), we have

$$\int_M div_{\bar\nabla}(V)v_{\bar g} = \int_M \bar g_Q(\kappa_{\bar g}, V)v_{\bar g} = \int_M < \bar\nabla_{\kappa_{\bar g}}\bar\Psi, \bar\Phi >_{\bar g_Q} v_{\bar g},$$

where $v_{\bar g}$ is the volume form associated to the metric $\bar g_M = g_L + \bar g_Q$. By integrating, we obtain our result. □

Proposition 3.15 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Then for any spinor fields Ψ, $\Phi \in S(\mathcal{F})$*

$$\int_M < \bar D_{tr}\bar\Psi, \bar\Phi >_{\bar g_Q} v_{\bar g} = \int_M < \bar\Psi, \bar D_{tr}\bar\Phi >_{\bar g_Q} v_{\bar g}. \tag{3.30}$$

Proof. For a transversally conformal change $\bar g_Q = e^{2u}g_Q$, we know that $v_{\bar g} = e^{qu}v_g$. Hence we have, from (3.14), (3.18), and (3.22),

$$\begin{aligned}
&\int_M < \bar D_{tr}\bar\Psi, \bar\Phi >_{\bar g_Q} v_{\bar g} \\
&= \int_M < D_{tr}\Psi, e^{(q-1)u}\Phi >_{g_Q} v_g + \frac{q-1}{2}\int_M e^{(q-1)u} < d_Bu\cdot\Psi, \Phi >_{g_Q} v_g \\
&= \int_M < \Psi, D_{tr}(e^{(q-1)u}\Phi) >_{g_Q} v_g - \frac{q-1}{2}\int_M e^{(q-1)u} < \Psi, d_Bu\cdot\Phi >_{g_Q} v_g \\
&= \int_M < \bar\Psi, e^{-u}(\overline{D_{tr}\Phi} + \frac{q-1}{2}\overline{d_Bu\cdot\Phi}) >_{\bar g_Q} v_{\bar g} \\
&= \int_M < \bar\Psi, \bar D_{tr}\bar\Phi >_{\bar g_Q} v_{\bar g}. \quad \square
\end{aligned}$$

4 Eigenvalue estimate with a transversal twistor operator

4.1 Eigenvalue estimate

Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M such that κ is basic-harmonic. The existence of the bundle-like metric g_M for $(M, \mathcal{F})$ such that κ is basic-harmonic is assured from ([21,22]).

Definition 4.1 *For any real number $s \neq 0$, we put $P_{tr}^s \Psi = \sum_a E_a \otimes P_{E_a}^s \Psi$, where*

$$P_X^s \Psi = \nabla_X \Psi + \frac{1}{s}\, \pi(X) \cdot D_{tr} \Psi \tag{4.1}$$

for any $X \in TM$. This operator P_{tr}^s is called the **transversal twistor operator** *of type s on $S(\mathcal{F})$ and the spinor field in $Ker P_{tr}^s$ is called the* **transversal twistor** *of type s.*

By a direct calculation, we have the following([12]).

Lemma 4.2 *For any spinor field Ψ, we have*

$$\int_M |P_{tr}^s \Psi|^2 = \int_M \Big(f(s) |D_{tr}\Psi|^2 - \frac{1}{4}\, K_\sigma^\nabla |\Psi|^2 \Big) - \frac{1}{s} \int_M F(\Psi), \tag{4.2}$$

where $f(s) = \dfrac{q}{s^2} - \dfrac{2}{s} + 1$ and $F(\Psi) = \mathrm{Re} < \kappa \cdot \Psi, D_{tr}\Psi >$.

Since $f(s)$ has a minimum $\dfrac{q-1}{q}$ at $s = q$, we have the following theorem.

Theorem 4.3 ([10,12]) *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M. Assume that $K_\sigma^\nabla \geq 0$. Then any eigenvalue λ of D_b satisfies*

$$\lambda^2 \geq \frac{q}{4(q-1)}\, \inf_M K_\sigma^\nabla. \tag{4.3}$$

Now, we estimate the eigenvalues of the basic Dirac operator with the transversally conformal change $\bar{g}_Q = e^{2u} g_Q$. On $\bar{S}(\mathcal{F})$, we define the transversal twistor operator $\bar{P}^s_{tr} : TM \otimes \bar{S}(\mathcal{F}) \to \bar{S}(\mathcal{F})$ of type s by

$$\bar{P}^s_X \bar{\Psi} = \bar{\nabla}_X \bar{\Psi} + \frac{1}{s}\, \pi(X) \,\bar{\cdot}\, \bar{D}_{tr} \bar{\Psi}. \tag{4.4}$$

Proposition 4.4 *For any spinor field* Ψ*, we have*

$$\bar{P}^s_X \bar{\Psi} = \overline{P^s_X \Psi} + \frac{q-s-1}{2s}\, \overline{\pi(X) \cdot d_B u \cdot \Psi} - \frac{1}{2}\, g_Q(d_B u, \pi(X)) \bar{\Psi}. \tag{4.5}$$

Proof. From (3.20), (3.22), and (4.1), we have that, for any spinor Ψ,

$$\begin{aligned}
\bar{P}^s_X \bar{\Psi} =& \bar{\nabla}_X \bar{\Psi} + \frac{1}{s}\, \pi(X) \,\bar{\cdot}\, \bar{D}_{tr} \bar{\Psi} \\
=& \overline{\nabla_X \Psi} - \frac{1}{2}\, \overline{\pi(X) \cdot d_B u \cdot \Psi} - \frac{1}{2}\, g_Q(d_B u, \pi(X)) \bar{\Psi} \\
&+ \frac{1}{s}\, \pi(X) \,\bar{\cdot}\, e^{-u} \{ \overline{D_{tr} \Psi} + \frac{q-1}{2}\, \overline{d_B u \cdot \Psi} \} \\
=& \overline{\nabla_X \Psi} + \frac{1}{s}\, \overline{\pi(X) \cdot D_{tr} \Psi} \\
&+ \frac{q-s-1}{2s}\, \overline{\pi(X) \cdot d_B u \cdot \Psi} - \frac{1}{2}\, g_Q(d_B u, \pi(X)) \bar{\Psi} \\
=& \overline{P^s_X \Psi} + \frac{q-s-1}{2s}\, \overline{\pi(X) \cdot d_B u \cdot \Psi} - \frac{1}{2}\, g_Q(d_B u, \pi(X)) \bar{\Psi}. \qquad \square
\end{aligned}$$

Theorem 4.5 *For any spinor field* Ψ*, we have*

$$\bar{P}^q_{tr}(e^{\frac{u}{2}} \bar{\Psi}) = e^{-\frac{u}{2}} \overline{P^q_{tr} \Psi}. \tag{4.6}$$

Hence the dimension of the transversal twistor spinor space of type q *is invariant under the conformal change.*

Proof. From (4.5), we have

$$
\begin{aligned}
\bar{P}^s_X(e^{\frac{u}{2}}\bar{\Psi}) =& e^{\frac{u}{2}}\bar{P}^s_X\bar{\Psi} + X(e^{\frac{u}{2}})\bar{\Psi} + \frac{1}{s}\ e^{-u}\pi(X)\ \bar{\cdot}\ \overline{d_B e^{\frac{u}{2}}\cdot\Psi} \\
=& e^{\frac{u}{2}}\overline{P^s_X\Psi} + \frac{q-s-1}{2s}\ e^{\frac{u}{2}}\overline{\pi(X)\cdot d_B u\cdot\Psi} \\
&- \frac{1}{2}\ e^{\frac{u}{2}}g_Q(d_B u,\pi(X))\bar{\Psi} + \frac{1}{2}\ e^{\frac{u}{2}}X(u)\bar{\Psi} \\
&+ \frac{1}{2s}\ e^{\frac{u}{2}}\overline{\pi(X)\cdot d_B u\cdot\Psi} \\
=& e^{\frac{u}{2}}\{\overline{P^s_X\Psi} + \frac{q-s}{2s}\ \overline{\pi(X)\cdot d_B u\cdot\Psi}\}. \qquad \square
\end{aligned}
$$

Lemma 4.6 *For any spinor field Ψ, we have*

$$
|\bar{P}^s_{tr}\bar{\Psi}|^2 = |\bar{\nabla}_{tr}\bar{\Psi}|^2 + \frac{q-2s}{s^2}\ |\bar{D}_{tr}\bar{\Psi}|^2 - \frac{1}{s}\ \bar{F}(\bar{\Psi}), \tag{4.7}
$$

where $\bar{F}(\bar{\Psi}) = \mathrm{Re} < \kappa_{\bar{g}}\ \bar{\cdot}\ \bar{\Psi}, \bar{D}_{tr}\bar{\Psi} >_{\bar{g}_Q}$.

Proof. By direct calculation, we have, from (4.4) and (3.21),

$$
\begin{aligned}
|\bar{P}^s_{tr}\bar{\Psi}|^2 =& \sum_a < \bar{P}^s_{\bar{E}_a}\bar{\Psi}, \bar{P}^s_{\bar{E}_a}\bar{\Psi} >_{\bar{g}_Q} \\
=& \sum_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Psi} >_{\bar{g}_Q} + \frac{1}{s^2}\ \sum_a < \bar{E}_a\bar{\cdot}\ \bar{D}_{tr}\bar{\Psi}, \bar{E}_a\bar{\cdot}\ \bar{D}_{tr}\bar{\Psi} >_{\bar{g}_Q} \\
&- \frac{1}{s}\ \sum_a\{< \bar{E}_a\bar{\cdot}\ \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{D}_{tr}\bar{\Psi} >_{\bar{g}_Q} + < \bar{D}_{tr}\bar{\Psi}, \bar{E}_a\bar{\cdot}\ \bar{\nabla}_{\bar{E}_a}\bar{\Psi} >_{\bar{g}_Q}\} \\
=& |\bar{\nabla}_{tr}\bar{\Psi}|^2 + \frac{q}{s^2}|\bar{D}_{tr}\bar{\Psi}|^2 - \frac{2}{s}|\bar{D}_{tr}\bar{\Psi}|^2 - \frac{1}{s}\bar{F}(\bar{\Psi}) \\
=& |\bar{\nabla}_{tr}\bar{\Psi}|^2 + (\frac{q}{s^2} - \frac{2}{s})|\bar{D}_{tr}\bar{\Psi}|^2 - \frac{1}{s}\bar{F}(\bar{\Psi}). \qquad \square
\end{aligned}
$$

Corollary 4.7 *For any spinor field Ψ, we have*

$$
\int_M |\bar{P}^s_{tr}\bar{\Psi}|^2 = \int_M \Big(f(s)|\bar{D}_{tr}\bar{\Psi}|^2 - \frac{1}{4}\ K^{\bar{\nabla}}_\sigma|\bar{\Psi}|^2\Big) - \frac{1}{s}\int_M \bar{F}(\bar{\Psi}), \tag{4.8}
$$

where $f(s) = \dfrac{q}{s^2} - \dfrac{2}{s} + 1$.

Let $D_b\Phi = \lambda\Phi(\Phi \neq 0)$ and $\Psi = e^{-\frac{q-1}{2}u}\Phi$. By Proposition 3.12, we get

$$\bar{D}_b\bar{\Psi} = \lambda e^{-u}\bar{\Psi}. \tag{4.9}$$

Since $< X \cdot \Psi, \Psi >$ is pure imaginary for any $X \in \Gamma Q$, we have

$$\bar{F}(\bar{\Psi}) = \mathrm{Re} < \kappa_{\bar{g}} \bar{\cdot} \bar{\Psi}, \bar{D}_{tr}\bar{\Psi} >_{\bar{g}_Q} = \lambda e^{-2u}\mathrm{Re} < \kappa \cdot \Psi, \Psi >_{g_Q} = 0.$$

From (4.8), we get

$$\begin{aligned}\int_M |\bar{P}^s_{tr}\bar{\Psi}|^2 &= \int_M e^{-2u}\Big(f(s)\lambda^2 - \frac{1}{4}\, e^{2u}K^{\bar{\nabla}}_\sigma\Big)|\bar{\Psi}|^2 \\ &\leq \int_M e^{-2u}\Big(f(s)\lambda^2 - \frac{1}{4}\inf_M(e^{2u}K^{\bar{\nabla}}_\sigma)\Big)|\bar{\Psi}|^2. \end{aligned} \tag{4.10}$$

Since $f(s)$ has a minimum $\dfrac{q-1}{q}$ at $s = q$, we have that for any basic function u,

$$\lambda^2 \geq \frac{q}{4(q-1)}\inf_M(e^{2u}K^{\bar{\nabla}}_\sigma).$$

Hence we have the following theorem.

Theorem 4.8 (cf. [15]) *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 2$ and bundle-like metric g_M such that $\kappa \in \Omega^1_B(\mathcal{F})$ and $\delta\kappa = 0$. Assume that $K^{\bar{\nabla}}_\sigma \geq 0$ for some transversally conformal metric $\bar{g}_Q = e^{2u}g_Q$. Then we have*

$$\lambda^2 \geq \frac{q}{4(q-1)}\sup_u\inf_M(e^{2u}K^{\bar{\nabla}}_\sigma), \tag{4.11}$$

where $K^{\bar{\nabla}}_\sigma = \sigma^{\bar{\nabla}} + |\bar{\kappa}|^2 + 2(q-2)\kappa_{\bar{g}}(u)$.

Lemma 4.9 *Let $\mathcal{K}_u = \{u \in \Omega^0_B(\mathcal{F}) | \kappa(u) = 0\}$. Assume that we choose $u \in \mathcal{K}_u$ and the positive function h by $u = \dfrac{2}{q-2}\ln h$ for $q \geq 3$, then we have*

$$e^{2u}K^{\bar{\nabla}}_\sigma = \begin{cases} K^{\nabla}_\sigma + 2(q-1)\Delta_B u + (q-1)(2-q)|d_B u|^2, \\ h^{-1}Y_b h + |\kappa|^2 \ (q \geq 3). \end{cases} \tag{4.12}$$

Proof. From the equation (2.21), we have

$$e^{2u}K_\sigma^{\bar\nabla} = \sigma^\nabla + |\kappa|^2 + 2(q-1)\Delta_B u + (q-1)(2-q)|d_B u|^2 - 2\kappa(u). \quad (4.13)$$

From (2.23), we have

$$\Delta_B u = \frac{2}{q-2}\Big(h^{-2}|d_B h|^2 + h^{-1}\Delta_B h\Big). \quad (4.14)$$

Since $u = \frac{2}{q-2}\ln h$ for $q \geq 3$, we have $d_B u = \frac{2}{q-2}h^{-1}d_B h$ and hence

$$|d_B u|^2 = \Big(\frac{2}{q-2}\Big)^2 h^{-2}|d_B h|^2. \quad (4.15)$$

Hence, from (4.13), we have

$$e^{2u}K_\sigma^{\bar\nabla} = \begin{cases} \sigma^\nabla + |\kappa|^2 + \dfrac{4(q-1)}{q-2}\ h^{-1}\Delta_B h - \dfrac{4}{q-2}\ h^{-1}\kappa(h), \\ h^{-1}Y_b h + |\kappa|^2 - \dfrac{4}{q-2}\ h^{-1}\kappa(h)\ (q \geq 3), \end{cases} \quad (4.16)$$

where $Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^\nabla$, which is called a *basic Yamabe operator* of $\mathcal{F}$, and it is trivial that $e^{2u}K_\sigma^{\bar\nabla} = h^{\frac{4}{q-2}}\ K_\sigma^{\bar\nabla}$. Since $u \in \mathcal{K}_u$, $\kappa(u) = 0 = \kappa(h)$, which proves (4.12). □

Corollary 4.10 ($cf.$ [15]) *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and bundle-like metric g_M such that $\kappa \in \Omega_B^1(\mathcal{F})$ and $\delta\kappa = 0$. Assume that $K_\sigma^\nabla \geq 0$. Then*

$$\lambda^2 \geq \begin{cases} \dfrac{q}{4(q-1)} \sup\limits_{u\in\mathcal{K}_u} \inf\limits_M \{K_\sigma^\nabla + 2(q-1)\Delta_B u + (q-1)(2-q)|d_B u|^2\}, \\ \dfrac{q}{4(q-1)} \sup\limits_{h\in\mathcal{K}_u} \inf\limits_M \{h^{-1}Y_b h + |\kappa|^2\}\ (q \geq 3). \end{cases}$$

Assume that the transversal scalar curvature σ^∇ is non-negative. Then the eigenvalue h_1 associated to the first eigenvalue μ_1 of Y_b can be chosen to be positive and then μ_1 is non-negative. Thus

$$h_1^{-1}Y_b h_1 = \mu_1. \quad (4.17)$$

Since $\sup\inf\{h^{-1}Y_b h\} \geq \mu_1$, we have the following corollary.

Corollary 4.11 ($cf.$ [15]) *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric g_M with $\kappa \in \Omega^1_B(\mathcal{F})$ and $\delta\kappa = 0$. If the transversal scalar curvature satisfies $\sigma^\nabla \geq 0$, then any eigenvalue λ of the Dirac operator corresponding to the eigenspinor Ψ satisfies*

$$\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2), \tag{4.18}$$

where μ_1 is the first eigenvalue of the basic Yamabe operator Y_b of $\mathcal{F}$.

4.2 The limiting case

In this section, we study the limiting case of (4.18).

Proposition 4.12 *If M admits a non-zero transversal twistor spinor field Ψ of type s with the transversally conformal metric $\bar{g}_Q = e^{2u}g_Q$, i.e., $\bar{P}^s_X\bar{\Psi} = 0$, then, for any $X \in TM$,*

$$\nabla_X\Psi = -\frac{1}{s}\,\pi(X)\cdot D_{tr}\Psi + \frac{s-q+1}{2s}\,\pi(X)\cdot d_Bu\cdot\Psi + \frac{1}{2}X(u)\Psi. \tag{4.19}$$

Proof. Let $\bar{\Psi} \in Ker\bar{P}^s_{tr}$. From (4.4), we have

$$\bar{\nabla}_X\bar{\Psi} + \frac{1}{s}\pi(X)\,\bar{\cdot}\,\bar{D}_{tr}\bar{\Psi} = 0.$$

Hence from (3.20) and (3.22), we have

$$\overline{\nabla_X\Psi} - \frac{s-q+1}{2s}\,\overline{\pi(X)\cdot d_Bu\cdot\Psi} - \frac{1}{2}X(u)\bar{\Psi} + \frac{1}{s}\,\overline{\pi(X)\cdot D_{tr}\Psi} = 0,$$

which yields (4.19). □

Theorem 4.13 *If $(M, g_M, \mathcal{F})$ admits a non-vanishing trnsversal twistor spinor field of type q, then the foliation $\mathcal{F}$ is minimal.*

Proof. From Theorem 4.5, the space of transversal twistor spinor space of type q is invariant under the transversally conformal metric change. Let $\Phi \in KerP_{tr}^q$. Then $\bar\Psi = e^{\frac{u}{2}}\bar\Phi \in Ker\bar P_{tr}^q$ be the corresponding transversal twistor spinor field. Hence from (4.4), we have

$$\bar\nabla_{\bar E_a}\bar\Psi + \frac{1}{q}\ \bar E_a \bar\cdot \bar D_{tr}\bar\Psi = 0. \tag{4.20}$$

Hence we get

$$\sum_a \bar E_a \bar\cdot \bar\nabla_{\bar E_a}\bar\Psi = -\frac{1}{q}\sum_a \bar E_a \bar\cdot \bar E_a \bar\cdot \bar D_{tr}\bar\Psi$$

or

$$\bar D_{tr}\bar\Psi + \frac{1}{2}\kappa_{\bar g} \bar\cdot \bar\Psi = \bar D_{tr}\bar\Psi.$$

Therefore we have

$$\kappa_{\bar g} \bar\cdot \bar\Psi = 0,$$

and then $\kappa = 0$, i.e., $\mathcal{F}$ is minimal. □

Lemma 4.14 *Let $\bar\Psi \in Ker\bar P_{tr}^q$. Then we have*

$$\bar D_{tr}^2\bar\Psi = \frac{q}{4(q-1)}\ \sigma^{\bar\nabla}\bar\Psi. \tag{4.21}$$

Proof. From (2.18) and (4.20), we have

$$\begin{aligned}
0 &= \sum_a \bar\nabla_{\bar E_a}\bar\nabla_{\bar E_a}\bar\Psi + \frac{1}{q}\sum_a \bar\nabla_{\bar E_a}\{\bar E_a \bar\cdot \bar D_{tr}\bar\Psi\} \\
&= \sum_a \bar\nabla_{\bar E_a}\bar\nabla_{\bar E_a}\bar\Psi + \frac{1}{q}\sum_a (\bar\nabla_{\bar E_a}\bar E_a) \bar\cdot \bar D_{tr}\bar\Psi + \frac{1}{q}\sum_a \bar E_a \bar\cdot \bar\nabla_{\bar E_a}\bar D_{tr}\bar\Psi \\
&= \sum_a \bar\nabla_{\bar E_a}\bar\nabla_{\bar E_a}\bar\Psi + \frac{1}{q}\sum_a e^{-u}\{E_a(u)\bar E_a - \overline{d_B u}\} \bar\cdot \bar D_{tr}\bar\Psi + \frac{1}{q}\bar D_{tr}^2\bar\Psi,
\end{aligned}$$

which implies

$$-\sum_a \bar\nabla_{\bar E_a}\bar\nabla_{\bar E_a}\bar\Psi = \frac{1-q}{q}\ e^{-u}\overline{d_B u} \bar\cdot \bar D_{tr}\bar\Psi + \frac{1}{q}\bar D_{tr}^2\bar\Psi. \tag{4.22}$$

On the other hand, from Theorem 4.13, the foliation is minimal, i.e., $\kappa = 0$. Hence from (2.18), (3.26) and (4.22), we have

$$\bar{D}_{tr}^2\bar{\Psi} = \frac{1}{q}\ \bar{D}_{tr}^2\bar{\Psi} + \frac{1-q}{q}\ e^{-u}\overline{d_B u}\ \bar{\cdot}\ \bar{D}_{tr}\bar{\Psi} + (1-q)e^{-2u}\bar{\nabla}_{d_B u}\bar{\Psi} + \frac{1}{4}\sigma^{\bar{\nabla}}(\bar{\Psi}).$$

From (3.20) and (3.19), we have

$$\begin{aligned}
\bar{\nabla}_{d_B u}\bar{\Psi} =& \overline{\nabla_{d_B u}\Psi} - \frac{1}{2}\ \overline{d_B u \cdot d_B u \cdot \Psi} - \frac{1}{2}\ g_Q(d_B u, d_B u)\bar{\Psi} \\
=& -\frac{1}{q}\ \overline{d_B u}\ \bar{\cdot}\ \overline{D_{tr}\Psi} + \frac{q-1}{2q}\ |d_B u|^2\bar{\Psi} \\
=& -\frac{1}{q}\ \overline{d_B u}\ \bar{\cdot}\ \{e^u\bar{D}_{tr}\bar{\Psi} - \frac{q-1}{2}\ \overline{d_B u\cdot\Psi}\} + \frac{q-1}{2q}\ |d_B u|^2\bar{\Psi} \\
=& -\frac{1}{q}\ e^u\overline{d_B u}\ \bar{\cdot}\ \bar{D}_{tr}\bar{\Psi}.
\end{aligned}$$

Hence we have

$$\bar{D}_{tr}^2\bar{\Psi} = \frac{1}{q}\ \bar{D}_{tr}^2\bar{\Psi} + \frac{1}{4}\ \sigma^{\bar{\nabla}}(\bar{\Psi}),$$

which means (4.21). $\square$

Hence we have the following theorem.

Theorem 4.15 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric g_M such that $\kappa \in \Omega_B^1(\mathcal{F})$ and $\delta\kappa = 0$. Assume that $K_\sigma^\nabla \geq 0$. If there exists an eigenspinor field Φ_1 of the basic Dirac operator D_b for the eigenvalue λ_1 satisfying*

$$\lambda_1^2 = \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2), \tag{4.23}$$

then $\mathcal{F}$ is minimal and transversally Einsteinian with a positive constant transversal scalar curvature σ^∇.

Proof. Let $D_b\Phi_1 = \lambda_1\Phi_1$ with $\lambda_1^2 = \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2)$. From (4.8), we know that $\bar{\Psi}_1 \in Ker\bar{P}_{tr}^q$ with $\Psi_1 = e^{-\frac{q-1}{2}u}\Phi_1$. Since $\bar{D}_b\bar{\Psi}_1 = \lambda_1 e^{-u}\bar{\Psi}_1$, we have

$$\bar{D}_b^2\bar{\Psi}_1 = -\lambda_1 e^{-2u}\overline{d_B u\cdot\Psi_1} + \lambda_1^2 e^{-2u}\bar{\Psi}_1.$$

From (4.21), we have

$$\lambda_1^2\bar{\Psi}_1 = \frac{q}{4(q-1)}\ e^{2u}\sigma^{\bar{\nabla}}\bar{\Psi}_1 + \lambda_1\overline{d_B u\cdot\Psi_1}.$$

Hence we get

$$\lambda_1^2|\bar{\Psi}_1|^2 = \frac{q}{4(q-1)}\ e^{2u}\sigma^{\bar{\nabla}}|\bar{\Psi}_1|^2 \quad \text{and} \quad d_B u\cdot\Psi_1 = 0, \tag{4.24}$$

which means u is constant and $e^{2u}\sigma^{\bar{\nabla}}$ is positive and constant. So Proposition 4.12 implies that for $\bar{\Psi}_1 \in Ker\bar{P}^q_{tr}$, $\Psi_1 = e^{-\frac{q-1}{2}u}\Phi_1$,

$$\nabla_X\Phi_1 = -\frac{\lambda_1}{q}\ X\cdot\Phi_1. \tag{4.25}$$

By direct calculation with (4.25), we have

$$\begin{aligned}\nabla_X\nabla_{E_a}\Phi_1 =& \nabla_X\{-\frac{\lambda_1}{q}\ E_a\cdot\Phi_1\}\\ =& -\frac{\lambda_1}{q}\ (\nabla_X E_a)\cdot\Phi_1 - \frac{\lambda_1}{q}\ E_a\cdot\nabla_X\Phi_1\\ =& -\frac{\lambda_1}{q}\ (\nabla_X E_a)\cdot\Phi_1 + \frac{\lambda_1^2}{q^2}\ E_a\cdot X\cdot\Phi_1\end{aligned}$$

for any $X \in \Gamma Q$. Hence we have

$$R^S(X,E_a)\cdot\Phi_1 = \frac{2\lambda_1^2}{q^2}\ E_a\cdot X\cdot\Phi_1 + \frac{2\lambda_1^2}{q^2}\ g_Q(X,E_a)\Phi_1$$

and so we have

$$\sum_a E_a\cdot R^S(X,E_a)\cdot\Phi_1 = -\frac{2\lambda_1^2}{q^2}\ (q-1)X\cdot\Phi_1. \tag{4.26}$$

On the other hand, from (2.21) and (4.24), we have

$$\lambda_1^2 = \frac{q}{4(q-1)}\ \sigma^{\nabla}. \tag{4.27}$$

Therefore we have, from (3.12), (4.26), and (4.27),

$$\rho^{\nabla}(X) = \frac{\sigma^{\nabla}}{q} X. \tag{4.28}$$

This means that $\mathcal{F}$ is transversally Einsteinian with a positive constant transversal scalar curvature σ^{∇}. □

5 Eigenvalue estimate with a modified connection

5.1 Eigenvalue estimate

Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M such that κ is basic-harmonic.

Now, we introduce a new connection $\overset{f,g}{\nabla}$ on $S(\mathcal{F})$ as the following:

Definition 5.1 *Let f and g be real-valued basic functions on M. For any tangent vector field X and any spinor field Ψ, we define the modified connection $\overset{f,g}{\nabla}$ on $S(\mathcal{F})$ by*

$$\overset{f,g}{\nabla}_X \Psi = \nabla_X \Psi + f\pi(X) \cdot \Psi + g\kappa \cdot \pi(X) \cdot \Psi, \tag{5.1}$$

where $\pi : TM \to Q$.

Lemma 5.2 *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M. Then, for any basic-harmonic 1-form $\omega \in \Omega^1_B(\mathcal{F})$,*

$$D_{tr}(\omega \cdot \Psi) = -\omega \cdot D_{tr}\Psi - 2\nabla_\omega \Psi. \tag{5.2}$$

Proof. For any spinor field Ψ, a simple calculation gives

$$\begin{aligned}
D_{tr}(\omega\cdot\Psi) =& \sum_a E_a\cdot(\nabla_{E_a}\omega)\cdot\Psi + \sum_a E_a\cdot\omega\cdot\nabla_{E_a}\Psi - \frac{1}{2}\,\kappa\cdot\omega\cdot\Psi \\
=& \sum_a \{E_a\wedge\nabla_{E_a}\omega - i(E_a)\nabla_{E_a}\omega\}\Psi - \sum_a \omega\cdot E_a\cdot\nabla_{E_a}\Psi \\
& - 2\sum_a g_Q(E_a,\omega)\nabla_{E_a}\Psi - \frac{1}{2}\,\{-\omega\cdot\kappa\cdot\Psi - 2g_Q(\kappa,\omega)\Psi\} \\
=& (d_B\omega + \delta_B\omega - i(\kappa_B)\omega)\Psi - \sum_a \omega\cdot E_a\cdot\nabla_{E_a}\Psi - 2\nabla_\omega\Psi \\
& + \frac{1}{2}\,\omega\cdot\kappa\cdot\Psi + g_Q(\kappa,\omega)\Psi \\
=& -\omega\cdot D_{tr}\Psi - 2\nabla_\omega\Psi + (d_B\omega + \delta_B\omega)\Psi.
\end{aligned}$$

Since $\omega\in\Omega^1_B(\mathcal{F})$ is a basic-harmonic 1-form, we have $d_B\omega = 0 = \delta_B\omega$. Hence the proof is complete. □

Proposition 5.3 *For any real-valued basic functions f and g on M, and for any spinor field $\Psi\in S(\mathcal{F})$, we have*

$$\begin{aligned}
|\overset{f,g}{\nabla}_{tr}\Psi|^2 =& |\nabla_{tr}\Psi|^2 + qf^2|\Psi|^2 + qg^2|\kappa|^2|\Psi|^2 + g|\kappa|^2|\Psi|^2 \\
& - 2f\mathrm{Re} < D_{tr}\Psi,\Psi > + 2g\mathrm{Re} < D_{tr}\Psi,\kappa\cdot\Psi > \\
& - 4g\mathrm{Re} < \nabla_\kappa\Psi,\Psi > .
\end{aligned} \tag{5.3}$$

Proof. Fix $x\in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$ for all a. Then we have at the point x that for any Ψ,

$$\begin{aligned}
|\overset{f,g}{\nabla}_{tr}\Psi|^2 =& \sum_a < \overset{f,g}{\nabla}_{E_a}\Psi, \overset{f,g}{\nabla}_{E_a}\Psi > \\
=& |\nabla_{tr}\Psi|^2 + qf^2|\Psi|^2 + qg^2|\kappa|^2|\Psi|^2 + g|\kappa|^2|\Psi|^2 \\
& - f\{< D_{tr}\Psi,\Psi > + < \Psi, D_{tr}\Psi >\} \\
& + g\{< D_{tr}\Psi,\kappa\cdot\Psi > + < \kappa\cdot\Psi, D_{tr}\Psi >\} \\
& - f\mathrm{Re} < \Psi,\kappa\cdot\Psi > - 4g\mathrm{Re} < \nabla_\kappa\Psi,\Psi >,
\end{aligned}$$

which means (5.3) together with the fact that $< X \cdot \Psi, \Psi >$ is pure imaginary. □

We now that for an appropriate choice of the real-valued basic functions f and g, one gets a sharp estimate of the first eigenvalue of the basic Dirac operator on compact foliated Riemannian manifolds.

Theorem 5.4 *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with an isoparametric transverse spin foliation of codimension $q > 1$ and bundle-like metric g_M with respect to $\mathcal{F}$. Assume that the mean curvature κ of $\mathcal{F}$ satisfies $\delta_B \kappa = 0$ and $K_\sigma^\nabla \geq 0$. Any eigenvalue λ of the transverse Dirac operator D_{tr} satisfies*

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M \Big(K_\sigma^\nabla + \frac{1}{q} |\kappa|^2 \Big). \tag{5.4}$$

Proof. Since $< X \cdot \Psi, \Psi >$ is purely imagnary, we have from (3.6), (3.14) and (5.2),

$$\begin{aligned} & -2 \int_M \mathrm{Re} < \nabla_\kappa \Psi, \Psi > \\ & = \int_M \mathrm{Re} < D_{tr}(\kappa \cdot \Psi), \Psi > + \int_M \mathrm{Re} < \kappa \cdot D_{tr} \Psi, \Psi > \\ & = \int_M \mathrm{Re} < \kappa \cdot \Psi, D_{tr} \Psi > + \int_M \mathrm{Re} < \kappa \cdot D_{tr} \Psi, \Psi > \\ & = 0. \end{aligned}$$

Hence from Proposition 5.3, we have

$$\int_M | \overset{f,g}{\nabla}_{tr} \Psi|^2 = \int_M \Big(\lambda^2 - \frac{1}{4} K_\sigma^\nabla + q f^2 - 2 f \lambda + q g^2 |\kappa|^2 + g |\kappa|^2 \Big) |\Psi|^2.$$

If we put $f = \dfrac{\lambda}{q}$ and $g = -\dfrac{1}{2q}$, we have

$$\int_M | \overset{f,g}{\nabla}_{tr} \Psi|^2 = \int_M \frac{q-1}{q} \Big(\lambda^2 - \frac{q}{4(q-1)} \{ K_\sigma^\nabla + \frac{1}{q} |\kappa|^2 \} \Big) |\Psi|^2, \tag{5.5}$$

which proves (5.4). □

Corollary 5.5 *In addition to assumptions in Theorem* 5.4, *if the transverse scalar curvature is zero, then we get*

$$\lambda^2 \geq \frac{q+1}{4(q-1)} \inf_M |\kappa|^2.$$

5.2 The limiting case

We define $\mathrm{Ric}_{\nabla}^{f,g} : \Gamma Q \otimes S \to S$ by

$$\mathrm{Ric}_{\nabla}^{f,g}(X \otimes \Psi) = \sum_a E_a \cdot R^{f,g}(X, E_a)\Psi, \tag{5.6}$$

where $R^{f,g}$ is the curvature tensor with respect to $\overset{f,g}{\nabla}$. Then we have the following lemma.

Lemma 5.6 *For any vector field* $X \in \Gamma Q$ *and spinor field* $\Psi \in \Gamma S(\mathcal{F})$,

$$\begin{aligned}
&\mathrm{Ric}_{\nabla}^{f,g}(X \otimes \Psi) \qquad (5.7)\\
&= -\frac{1}{2}\, \rho^{\nabla}(X)\Psi - qX(f)\Psi + 2(q-1)f^2 X \cdot \Psi - d_B f \cdot X \cdot \Psi\\
&+ (q-2)X(g)\kappa \cdot \Psi + (q-2)g\nabla_X \kappa \cdot \Psi + 2qfg g_Q(X,\kappa)\Psi + 2fg\kappa \cdot X \cdot \Psi\\
&+ 2(q-2)g^2|\kappa|^2 X \cdot \Psi - 2(q-2)g^2 g_Q(X,\kappa)\kappa \cdot \Psi - d_B g \cdot \kappa \cdot X \cdot \Psi\\
&+ g|\kappa|^2 X \cdot \Psi.
\end{aligned}$$

Proof. A direct calculation gives

$$\begin{aligned}
\overset{f,g}{\nabla}_X \overset{f,g}{\nabla}_{E_a} \Psi =& \overset{f,g}{\nabla}_X \{\nabla_{E_a}\Psi + fE_a \cdot \Psi + g\kappa \cdot E_a \cdot \Psi\}\\
&= \nabla_X \nabla_{E_a}\Psi + X(f)E_a \cdot \Psi + f\nabla_X E_a \cdot \Psi + fE_a \cdot \nabla_X \Psi\\
&+ X(g)\kappa \cdot E_a \cdot \Psi + g\nabla_X \kappa \cdot E_a \cdot \Psi + g\kappa \cdot \nabla_X E_a \cdot \Psi\\
&+ g\kappa \cdot E_a \cdot \nabla_X \Psi + fX \cdot \nabla_{E_a}\Psi + f^2 X \cdot E_a \cdot \Psi\\
&+ fgX \cdot \kappa \cdot E_a \cdot \Psi + g\kappa \cdot X \cdot \nabla_{E_a}\Psi + fg\kappa \cdot X \cdot E_a \cdot \Psi\\
&+ g^2 \kappa \cdot X \cdot \kappa \cdot E_a \cdot \Psi.
\end{aligned}$$

With the similar calculation, we have

$$\begin{aligned}
R^{f,g}(X,E_a)\Psi &= R^S(X,E_a)\Psi + X(f)E_a\cdot\Psi - X(g)E_a\cdot\kappa\cdot\Psi \\
&- 2X(g)g_Q(\kappa,E_a)\Psi - gE_a\cdot\nabla_X\kappa\cdot\Psi - 2gg_Q(\nabla_X\kappa,E_a)\Psi \\
&- 2f^2E_a\cdot X\cdot\Psi - 2f^2g_Q(X,E_a)\Psi \\
&- 2fgg_Q(X,\kappa)E_a\cdot\Psi + 2fgg_Q(E_a,\kappa)X\cdot\Psi \\
&+ g^2\kappa\cdot\{X\cdot\kappa\cdot E_a - E_a\cdot\kappa\cdot X\}\cdot\Psi - E_a(f)X\cdot\Psi \\
&- E_a(g)\kappa\cdot X\cdot\Psi - g\nabla_{E_a}\kappa\cdot X\cdot\Psi.
\end{aligned}$$

Note that

$$\begin{aligned}
X\cdot\kappa\cdot E_a - E_a\cdot\kappa\cdot X = 2\kappa\cdot E_a\cdot X + 2g_Q(X,E_a)\kappa - 2g_Q(X,\kappa)E_a \\
+ 2g_Q(E_a,\kappa)X.
\end{aligned}$$

Hence we have

$$\begin{aligned}
R^{f,g}(X,E_a)\Psi &= R^S(X,E_a)\Psi + X(f)E_a\cdot\Psi - X(g)E_a\cdot\kappa\cdot\Psi \\
&- 2X(g)g_Q(\kappa,E_a)\Psi - gE_a\cdot\nabla_X\kappa\cdot\Psi - 2gg_Q(\nabla_X\kappa,E_a)\Psi \\
&- 2f^2E_a\cdot X\cdot\Psi - 2f^2g_Q(X,E_a)\Psi - 2fgg_Q(X,\kappa)E_a\cdot\Psi \\
&+ 2fgg_Q(E_a,\kappa)X\cdot\Psi - 2g^2|\kappa|^2E_aX\cdot\Psi \\
&- 2g^2g_Q(X,E_a)|\kappa|^2\Psi + 2g^2g_Q(X,\kappa)E_a\cdot\kappa\cdot\Psi \\
&+ 4g^2g_Q(X,\kappa)g_Q(E_a,\kappa)\Psi + 2g^2g_Q(E_a,\kappa)\kappa\cdot X\cdot\Psi \\
&- E_a(f)X\cdot\Psi - E_a(g)\kappa\cdot X\cdot\Psi - g\nabla_{E_a}\kappa\cdot X\cdot\Psi.
\end{aligned}$$

From (3.12) and (5.6), the proof is completed. □

Let $D_b\Psi_1 = \lambda_1\Psi_1$ with $\lambda_1^2 = \dfrac{q}{4(q-1)}\inf_M\left(K_\sigma^\nabla + \dfrac{1}{q}|\kappa|^2\right)$. From (5.5), we see $\overset{f_1,g_1}{\nabla}_{tr}\Psi_1 = 0$, where $f_1 = \dfrac{\lambda_1}{q}$ and $g_1 = -\dfrac{1}{2q}$. Hence from (5.1), we have

$$\nabla_X\Psi_1 = -\frac{\lambda_1}{q}\,X\cdot\Psi_1 + \frac{1}{2q}\,\kappa\cdot X\cdot\Psi_1. \tag{5.8}$$

Note that

$$\sum_a E_a \cdot \nabla_{E_a} \Psi_1 = \lambda_1 \Psi_1 + \frac{1}{2}\ \kappa \cdot \Psi_1.$$

On the other hand, from (5.8)

$$\begin{aligned}\sum_a E_a \cdot \nabla_{E_a} \Psi_1 =& -\frac{\lambda_1}{q} \sum_a E_a \cdot E_a \cdot \Psi_1 + \frac{1}{2q} \sum_a E_a \cdot \kappa \cdot E_a \cdot \Psi_1 \\ =& \lambda_1 \Psi_1 + \frac{q-2}{2q}\ \kappa \cdot \Psi_1.\end{aligned}$$

Hence we have $\kappa \cdot \Psi_1 = 0$ which implies that $\kappa = 0$, i.e., $\mathcal{F}$ is minimal. If $\overset{f,g}{\nabla}_X \Psi = 0$ for any $X \in \Gamma Q$, then $\mathrm{Ric}^{f,g}_{\nabla} = 0$. From (5.7), we get

$$-\frac{1}{2}\ \rho^{\nabla}(X)\Psi - qX(f)\Psi + 2(q-1)f^2 X \cdot \Psi - d_B f \cdot X \cdot \Psi = 0. \tag{5.9}$$

If we put $X = d_B f$, then we have

$$< (-\frac{1}{2}\ \rho^{\nabla}(X) + 2(q-1)f^2 X) \cdot \Psi, \Psi >= (q-1)|d_B f|^2 |\Psi|^2. \tag{5.10}$$

Since for all $X \in \Gamma Q$ and $\Psi \in \Gamma S$, $< X \cdot \Psi, \Psi >$ is pure imaginary, the left-hand side of (5.10) is pure imaginary. But the right-hand side of (5.10) is real. Therefore, both sides are zero. Hence, if $q \geq 2$, then we have $d_B f = 0$. That is, $X(f) = 0$ for any $X \in \Gamma Q$. Since f is basic function, f is constant. So from (5.9), we have

$$\rho^{\nabla}(X)\Psi = \frac{4(q-1)}{q^2}\ \lambda_1^2 X \cdot \Psi. \tag{5.11}$$

This means that $\mathcal{F}$ is transversally Einsteinian with a constant transversal scalar curvature $\sigma^{\nabla} = \dfrac{4(q-1)}{q}\lambda_1^2$. Hence we have the following theorem.

Theorem 5.7 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q > 1$ and a bundle-like* *metric g_M. Assume that $K_\sigma^\nabla \geq 0$. If there exists an eigenspinor field Ψ_1 of the basic Dirac operator D_b for the eigenvalue λ_1 satisfying*

$$\lambda_1^2 = \frac{q}{4(q-1)} \inf_M \Big(K_\sigma^\nabla + \frac{1}{q}\ |\kappa|^2 \Big), \tag{5.12}$$

then $\mathcal{F}$ is minimal and transversally Einsteinian with a positive constant transversal scalar curvature σ^∇.

6 Eigenvalue estimate with the conformal change

6.1 Eigenvalue estimate

Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M such that κ is basic-harmonic. In this section, we estimate the eigenvalues of the basic Dirac operator by a transversally conformal change of the metric. Now, we consider, for any real basic function u on M, the transversally conformal metric $\bar g_Q = e^{2u} g_Q$. Let $\bar S(\mathcal{F})$ be its corresponding spinor bundle. For any tangent vector field X and any spinor field Ψ, we define the modified connection $\overset{f,g}{\bar\nabla}$ on $\bar S(\mathcal{F})$ by

$$\overset{f,g}{\bar\nabla}_X\ \bar\Psi = \bar\nabla_X \bar\Psi + f\pi(X) \bar\cdot \bar\Psi + g\kappa_{\bar g} \bar\cdot \pi(X) \bar\cdot \bar\Psi, \tag{6.1}$$

where f and g are real-valued basic functions on M.

Lemma 6.1 *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric g_M. Then for any basic-*

harmonic 1-*form* $\omega \in \Omega^1_B(\mathcal{F})$,

$$\bar{D}_{tr}(f\omega \,\bar{\cdot}\, \bar{\Psi}) = - f\omega \,\bar{\cdot}\, \bar{D}_{tr}\bar{\Psi} - 2f\bar{\nabla}_\omega\bar{\Psi} - (q+2)f\omega(u)\bar{\Psi} \qquad (6.2)$$
$$- 2f\overline{\omega \cdot d_B u \cdot \Psi} + \overline{d_B f \cdot \omega \cdot \Psi}$$

where f *is any basic function.*

Proof. Note that we have, from (3.23),

$$\begin{aligned} D_{tr}(f\omega \cdot \Psi) =& d_B f \cdot \omega \cdot \Psi + f D_{tr}(\omega \cdot \Psi) \\ =& - f\omega \cdot D_{tr}\Psi - 2f\nabla_\omega \Psi + d_B f \cdot \omega \cdot \Psi. \end{aligned}$$

From (3.20), (3.22) and (3.24), we have

$$\begin{aligned} &\bar{D}_{tr}(f\omega \,\bar{\cdot}\, \bar{\Psi}) \\ &= \bar{D}_{tr}(e^u \overline{f\omega \cdot \Psi}) = e^{-u}\overline{d_B e^u \cdot f\omega \cdot \Psi} + e^u \bar{D}_{tr}(\overline{f\omega \cdot \Psi}) \\ &= f\overline{d_B u \cdot \omega \cdot \Psi} + \overline{D_{tr}(f\omega \cdot \Psi)} + \frac{q-1}{2}\,\overline{d_B u \cdot f\omega \cdot \Psi} \\ &= - f\bar{\omega} \,\bar{\cdot}\, \overline{D_{tr}\Psi} - 2f\overline{\nabla_\omega \Psi} + \overline{d_B f \cdot \omega \cdot \Psi} + \frac{q+1}{2}\, f\overline{d_B u \cdot \omega \cdot \Psi} \\ &= - f\bar{\omega} \,\bar{\cdot}\, \Big(e^u \bar{D}_{tr}\bar{\Psi} - \frac{q-1}{2}\,\overline{d_B u \cdot \Psi}\Big) \\ &\quad - 2f\Big(\bar{\nabla}_\omega \bar{\Psi} + \frac{1}{2}\,\overline{\omega \cdot d_B u \cdot \Psi} + \frac{1}{2}\,\omega(u)\bar{\Psi}\Big) \\ &\quad + \overline{d_B f \cdot \omega \cdot \Psi} - \frac{q+1}{2}\, f\overline{\omega \cdot d_B u \cdot \Psi} - (q+1)f\omega(u)\bar{\Psi} \\ &= - f\omega \,\bar{\cdot}\, \bar{D}_{tr}\bar{\Psi} + \frac{q-1}{2}\, f\overline{\omega \cdot d_B u \cdot \Psi} - 2f\bar{\nabla}_\omega \bar{\Psi} \\ &\quad - f\overline{\omega \cdot d_B u \cdot \Psi} - f\omega(u)\bar{\Psi} + \overline{d_B f \cdot \omega \cdot \Psi} \\ &\quad - \frac{q+1}{2}\, f\overline{\omega \cdot d_B u \cdot \Psi} - (q+1)f\omega(u)\bar{\Psi} \\ &= - f\omega \,\bar{\cdot}\, \bar{D}_{tr}\bar{\Psi} - 2f\bar{\nabla}_\omega \bar{\Psi} \\ &\quad - 2f\overline{\omega \cdot d_B u \cdot \Psi} - (q+2)\omega(u)\bar{\Psi} + \overline{d_B f \cdot \omega \cdot \Psi}, \end{aligned}$$

which implies (6.2). □

Let $\mathcal{K}_u = \{u \in \Omega^0_B(\mathcal{F}) \mid \kappa(u) = 0\}$. Then we have the following corollary.

Corollary 6.2 *Assume that $u \in \mathcal{K}_u$. Then we have*

$$\bar{D}_{tr}(e^{-2u}\kappa\bar{\cdot}\,\bar{\Psi}) = -e^{-2u}\Big(\kappa\bar{\cdot}\,\bar{D}_{tr}\bar{\Psi} + 2\bar{\nabla}_{\kappa}\bar{\Psi}\Big). \tag{6.3}$$

Proposition 6.3 *For any real-valued basic functions f and g on M, and for any spinor field Ψ, we have*

$$\begin{aligned} |\overset{f,g}{\bar{\nabla}}_{tr}\,\bar{\Psi}|^2 =& |\bar{\nabla}_{tr}\bar{\Psi}|^2_{\bar{g}_Q} + qf^2|\bar{\Psi}|^2_{\bar{g}_Q} + qg^2|\kappa_{\bar{g}}|^2_{\bar{g}_Q}|\bar{\Psi}|^2_{\bar{g}_Q} + g|\kappa_{\bar{g}}|^2_{\bar{g}_Q}|\bar{\Psi}|^2_{\bar{g}_Q} \\ &- 2f\mathrm{Re} < \bar{D}_{tr}\bar{\Psi}, \bar{\Psi} >_{\bar{g}_Q} - f\mathrm{Re} < \kappa_{\bar{g}}\,\bar{\cdot}\,\bar{\Psi}, \bar{\Psi} >_{\bar{g}_Q} \\ &+ 2g\mathrm{Re} < \bar{D}_{tr}\bar{\Psi}, \kappa_{\bar{g}}\,\bar{\cdot}\,\bar{\Psi} >_{\bar{g}_Q} - 4g\mathrm{Re} < \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \bar{\Psi} >_{\bar{g}_Q} . \end{aligned} \tag{6.4}$$

Proof. This is a simple calculation. □

Let $D_b\Phi = \lambda\Phi$ ($\Phi \neq 0$) and $\bar{\Psi} = e^{-\frac{q+1}{2}u}\Phi$. Since $< X \cdot \Psi, \Psi >$ is pure imaginary, we have from (4.9)

$$\mathrm{Re} < \kappa_{\bar{g}}\,\bar{\cdot}\,\bar{\Psi}, \bar{\Psi} >_{\bar{g}_Q} = 0 \quad \text{and} \quad \mathrm{Re} < \bar{D}_{tr}\bar{\Psi}, \kappa_{\bar{g}}\,\bar{\cdot}\,\bar{\Psi} >_{\bar{g}_Q} = 0.$$

Hence the equation (6.4) gives

$$\begin{aligned} \int_M |\overset{f,g}{\bar{\nabla}}_{tr}\,\bar{\Psi}|^2 =& \int_M e^{-2u}\Big(\lambda^2 - 2fe^u\lambda - \frac{1}{4}\,e^{2u}K^{\bar{\nabla}}_{\sigma}\Big)|\bar{\Psi}|^2_{\bar{g}_Q} \\ &+ \int_M \Big(qf^2 + qg^2|\kappa_{\bar{g}}|^2_{\bar{g}_Q} + g|\kappa_{\bar{g}}|^2_{\bar{g}_Q}\Big)|\bar{\Psi}|^2_{\bar{g}_Q} \\ &- 4g\int_M \mathrm{Re} < \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \bar{\Psi} >_{\bar{g}_Q} . \end{aligned} \tag{6.5}$$

From (6.3) and (3.30), if $u \in \mathcal{K}_u$, then

$$
\begin{aligned}
&-2\int_M \mathrm{Re} < \bar{\nabla}_{\kappa_{\bar g}}\bar{\Psi}, \bar{\Psi} >_{\bar g_Q} \\
&= \int_M \mathrm{Re} < \bar{D}_{tr}(e^{-2u}\kappa \,\bar{\cdot}\, \bar{\Psi}), \bar{\Psi} >_{\bar g_Q} + \int_M e^{-2u}\mathrm{Re} < \kappa \,\bar{\cdot}\, \bar{D}_{tr}\bar{\Psi}, \bar{\Psi} >_{\bar g_Q} \\
&= \int_M \mathrm{Re} < e^{-2u}\kappa \,\bar{\cdot}\, \bar{\Psi}, \bar{D}_{tr}\bar{\Psi} >_{\bar g_Q} + \int_M e^{-2u}\mathrm{Re} < \kappa \,\bar{\cdot}\, \bar{D}_{tr}\bar{\Psi}, \bar{\Psi} >_{\bar g_Q} \\
&= \lambda \int_M e^{-2u}\mathrm{Re} < \kappa \cdot \Psi, \Psi >_{g_Q} + \lambda \int_M e^{-2u}\mathrm{Re} < \kappa \cdot \Psi, \Psi >_{g_Q} \\
&= 0.
\end{aligned}
$$

Therefore (6.5) yields

$$
\begin{aligned}
\int_M |\overset{f,g}{\bar{\nabla}}_{tr}\ \bar{\Psi}|^2 = &\int_M e^{-2u}\Big(\lambda^2 - 2fe^u\lambda - \frac{1}{4}\ e^{2u}K_\sigma^{\bar\nabla}\Big)|\bar{\Psi}|^2_{\bar g_Q} \\
&+ \int_M \Big(qf^2 + qg^2|\kappa_{\bar g}|^2_{\bar g_Q} + g|\kappa_{\bar g}|^2_{\bar g_Q}\Big)|\bar{\Psi}|^2_{\bar g_Q}.
\end{aligned}
$$

If we put $f = \dfrac{\lambda}{q}\ e^{-u}$ and $g = -\dfrac{1}{2q}$, we have

$$
\begin{aligned}
&\int_M |\overset{f,g}{\bar{\nabla}}_{tr}\ \bar{\Psi}|^2 \\
&= \frac{q-1}{q}\int_M\ e^{-2u}\Big(\lambda^2 - \frac{q}{4(q-1)}\ \{e^{2u}K_\sigma^{\bar\nabla} + \frac{1}{q}\ |\bar\kappa|^2\}\Big)|\bar{\Psi}|^2_{\bar g_Q}.
\end{aligned}
\tag{6.6}
$$

Hence we have the following theorem.

Theorem 6.4 *Let $(M, g_M, \mathcal{F})$ be a compact manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 2$ and bundle-like metric g_M such that $\kappa \in \Omega^1_B(\mathcal{F})$ and $\delta\kappa = 0$. Assume that $K_\sigma^{\bar\nabla} \geq 0$ for some transversally conformal metric $\bar g_Q = e^{2u}g_Q$. Then we have*

$$
\lambda^2 \geq \frac{q}{4(q-1)}\ \sup_{u\in\mathcal{K}_u}\inf_M\Big(e^{2u}K_\sigma^{\bar\nabla} + \frac{1}{q}\ |\bar\kappa|^2\Big), \tag{6.7}
$$

where $K_\sigma^{\bar\nabla} = \sigma^{\bar\nabla} + |\bar\kappa|^2$.

From (4.16), we have the following corollary.

Corollary 6.5 *Under the same condition as in Corollary* 4.10*, we have*

$$\lambda^2 \geq \begin{cases} \dfrac{q}{4(q-1)} \sup\limits_{u\in\mathcal{K}_u} \inf\limits_M \{K_\sigma^\nabla + 2(q-1)\Delta_B u \\ \qquad\qquad + (q-1)(2-q)|d_B u|^2 + \dfrac{1}{q}\ |\kappa|^2\} & \text{if } q \geq 2 \\ \dfrac{q}{4(q-1)} \sup\limits_{h\in\mathcal{K}_u} \inf\limits_M \{h^{-1}Y_b h + |\kappa|^2 + \dfrac{1}{q}\ |\kappa|^2\} & \text{if } q \geq 3, \end{cases}$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$.

Corollary 6.6 *Let* $(M, g_M, \mathcal{F})$ *be a compact Riemannian manifold with a transverse spin foliation* $\mathcal{F}$ *of codimension* $q \geq 3$ *and bundle-like metric* g_M *with* $\kappa \in \Omega_B^1(\mathcal{F})$ *and* $\delta\kappa = 0$*. If the transversal scalar curvature satisfies* $\sigma^\nabla \geq 0$*, then any eigenvalue* λ *of the Dirac operator corresponding to the eigenspinor* Ψ *satisfies*

$$\lambda^2 \geq \frac{q}{4(q-1)}\Big(\mu_1 + \frac{q+1}{q}\inf|\kappa|^2\Big), \tag{6.8}$$

where μ_1 *is the first eigenvalue of the basic Yamabe operator* Y_b *of* $\mathcal{F}$*.*

6.2 The limiting case

We define $\mathrm{Ric}_{\bar\nabla}^{f,g} : \Gamma Q \otimes \bar S(\mathcal{F}) \to \bar S(\mathcal{F})$ by

$$\mathrm{Ric}_{\bar\nabla}^{f,g}(X \otimes \bar\Psi) = \sum_a \bar E_a \bar{\cdot}\, \bar R^{f,g}(X, \bar E_a)\bar\Psi, \tag{6.9}$$

where $\bar{R}^{f,g}$ is the curvature tensor with respect to $\overset{f,g}{\bar\nabla}$. For $X \in \Gamma Q$ and $\Psi \in \Gamma S(\mathcal{F})$ we have the following by the direct calculation;

$$\begin{aligned}
\overset{f,g}{\bar\nabla}_X \overset{f,g}{\bar\nabla}_{\bar{E}_a} \bar\Psi =& \overset{f,g}{\bar\nabla}_X \left(\bar\nabla_{\bar{E}_a}\bar\Psi + f\bar{E}_a \,\bar\cdot\, \bar\Psi + g\kappa_{\bar g} \,\bar\cdot\, \bar{E}_a \,\bar\cdot\, \bar\Psi \right) \\
&= \bar\nabla_X \bar\nabla_{\bar{E}_a}\bar\Psi + fX \,\bar\cdot\, \bar\nabla_{\bar{E}_a}\bar\Psi + g\kappa_{\bar g} \,\bar\cdot\, X \,\bar\cdot\, \bar\nabla_{\bar{E}_a}\bar\Psi \\
&+ X(f)\bar{E}_a \,\bar\cdot\, \bar\Psi + f\bar\nabla_X \bar{E}_a \,\bar\cdot\, \bar\Psi + f\bar{E}_a \,\bar\cdot\, \bar\nabla_X\bar\Psi \\
&+ f^2 X \,\bar\cdot\, \bar{E}_a \,\bar\cdot\, \bar\Psi + fg\kappa_{\bar g} \,\bar\cdot\, X \,\bar\cdot\, \bar{E}_a \,\bar\cdot\, \bar\Psi \\
&+ X(g)\kappa_{\bar g} \,\bar\cdot\, \bar{E}_a \,\bar\cdot\, \bar\Psi + g\bar\nabla_X\kappa_{\bar g} \,\bar\cdot\, \bar{E}_a \,\bar\cdot\, \bar\Psi + g\kappa_{\bar g} \,\bar\cdot\, \bar\nabla_X \bar{E}_a \,\bar\cdot\, \bar\Psi \\
&+ g\kappa_{\bar g} \,\bar\cdot\, \bar{E}_a \,\bar\cdot\, \bar\nabla_X\bar\Psi + fgX \,\bar\cdot\, \kappa_{\bar g} \,\bar\cdot\, \bar{E}_a \,\bar\cdot\, \bar\Psi \\
&+ g^2\kappa_{\bar g} \,\bar\cdot\, X \,\bar\cdot\, \kappa_{\bar g} \,\bar\cdot\, \bar{E}_a \,\bar\cdot\, \bar\Psi.
\end{aligned}$$

With the similar calculation, we have

$$\begin{aligned}
&\bar{R}^{f,g}(X,\bar{E}_a)\bar\Psi \\
&= \bar{R}^S(X,\bar{E}_a)\bar\Psi + X(f)\bar{E}_a \,\bar\cdot\, \bar\Psi - 2f^2\bar{E}_a \,\bar\cdot\, X \,\bar\cdot\, \bar\Psi - 2f^2\bar{g}_Q(X,\bar{E}_a)\bar\Psi \\
&- 2fg\bar{g}_Q(\kappa_{\bar g}, X)\bar{E}_a \,\bar\cdot\, \bar\Psi - X(g)\bar{E}_a \,\bar\cdot\, \kappa_{\bar g} \,\bar\cdot\, \bar\Psi \\
&- 2X(g)\bar{g}_Q(\kappa_{\bar g},\bar{E}_a)\bar\Psi - g\bar{E}_a \,\bar\cdot\, \bar\nabla_X\kappa_{\bar g} \,\bar\cdot\, \bar\Psi - 2g\bar{g}_Q(\bar\nabla_X\kappa_{\bar g},\bar{E}_a)\bar\Psi - \bar{E}_a(f)X \,\bar\cdot\, \bar\Psi \\
&+ g^2\kappa_{\bar g} \,\bar\cdot\, \left(X \,\bar\cdot\, \kappa_{\bar g} \,\bar\cdot\, \bar{E}_a - \bar{E}_a \,\bar\cdot\, \kappa_{\bar g} \,\bar\cdot\, X \right) \,\bar\cdot\, \bar\Psi + 2fg\bar{g}_Q(\kappa_{\bar g},\bar{E}_a)X \,\bar\cdot\, \bar\Psi \\
&- \bar{E}_a(g)\kappa_{\bar g} \,\bar\cdot\, X \,\bar\cdot\, \bar\Psi - g\bar\nabla_{\bar{E}_a}\kappa_{\bar g} \,\bar\cdot\, X \,\bar\cdot\, \bar\Psi.
\end{aligned}$$

Note that

$$\begin{aligned}
X \,\bar\cdot\, \kappa_{\bar g} \,\bar\cdot\, \bar{E}_a - \bar{E}_a \,\bar\cdot\, \kappa_{\bar g} \,\bar\cdot\, X &= 2\kappa_{\bar g} \,\bar\cdot\, \bar{E}_a \,\bar\cdot\, X + 2\bar{g}_Q(X,\bar{E}_a)\kappa_{\bar g} \\
&- 2\bar{g}_Q(X,\kappa_{\bar g})\bar{E}_a + 2\bar{g}_Q(\kappa_{\bar g},\bar{E}_a)X.
\end{aligned}$$

Hence we have

$$
\begin{aligned}
&\bar{R}^{f,g}(X, \bar{E}_a)\bar{\Psi} \\
&= \bar{R}^S(X, \bar{E}_a)\bar{\Psi} + X(f)\bar{E}_a \bar{\cdot} \bar{\Psi} - 2f^2\bar{E}_a \bar{\cdot} X \bar{\cdot} \bar{\Psi} - 2f^2\bar{g}_Q(X, \bar{E}_a)\bar{\Psi} \\
&- 2fg\bar{g}_Q(\kappa_{\bar{g}}, X)\bar{E}_a \bar{\cdot} \bar{\Psi} - X(g)\bar{E}_a \bar{\cdot} \kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} - \bar{E}_a(f)X \bar{\cdot} \bar{\Psi} \\
&- 2X(g)\bar{g}_Q(\kappa_{\bar{g}}, \bar{E}_a)\bar{\Psi} - g\bar{E}_a \bar{\cdot} \bar{\nabla}_X\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} - 2g\bar{g}_Q(\bar{\nabla}_X\kappa_{\bar{g}}, \bar{E}_a)\bar{\Psi} \\
&- 2g^2|\kappa_{\bar{g}}|^2\bar{E}_a \bar{\cdot} X \bar{\cdot} \bar{\Psi} - 2g^2|\kappa_{\bar{g}}|^2\bar{g}_Q(X, \bar{E}_a)\bar{\Psi} + 2g^2\bar{g}_Q(X, \kappa_{\bar{g}})\bar{E}_a \bar{\cdot} \kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} \\
&+ 4g^2\bar{g}_Q(X, \kappa_{\bar{g}})\bar{g}_Q(\kappa_{\bar{g}}, \bar{E}_a)\bar{\Psi} + 2g^2\bar{g}_Q(\kappa_{\bar{g}}, \bar{E}_a)\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{\Psi} \\
&+ 2fg\bar{g}_Q(\kappa_{\bar{g}}, \bar{E}_a)X \bar{\cdot} \bar{\Psi} - \bar{E}_a(g)\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{\Psi} - g\bar{\nabla}_{\bar{E}_a}\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{\Psi}.
\end{aligned}
$$

By a simple calculation, we have, from (3.12) and (6.9),

$$
\begin{aligned}
&\mathrm{Ric}^{f,g}_{\bar{\nabla}}(X \otimes \bar{\Psi}) \qquad (6.10)\\
&= -\frac{1}{2}\,\rho^{\bar{\nabla}}(X) \bar{\cdot} \bar{\Psi} - qX(f)\bar{\Psi} + 2(q-1)f^2X \bar{\cdot} \bar{\Psi} + 2qfg\bar{g}_Q(\kappa_{\bar{g}}, X)\bar{\Psi} \\
&+ (q-2)X(g)\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} + (q-2)g\bar{\nabla}_X\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} \\
&- \overline{d_Bf} \bar{\cdot} X \bar{\cdot} \bar{\Psi} + 2(q-2)g^2|\kappa_{\bar{g}}|^2X \bar{\cdot} \bar{\Psi} - 2(q-2)g^2\bar{g}_Q(X, \kappa_{\bar{g}})\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi} \\
&- 2fg\kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{\Psi} - \overline{d_Bg} \bar{\cdot} \kappa_{\bar{g}} \bar{\cdot} X \bar{\cdot} \bar{\Psi} + g|\kappa_{\bar{g}}|^2X \bar{\cdot} \bar{\Psi}.
\end{aligned}
$$

On the other hand, we have the following.

Proposition 6.7 *If a non-zero spinor field Ψ satisfies $\overset{f,g}{\bar{\nabla}}_{tr}\ \bar{\Psi} = 0$, then*

$$
\begin{aligned}
\nabla_X\Psi &= -fe^u\pi(X)\cdot\Psi - g\kappa\cdot\pi(X)\cdot\Psi \qquad (6.11)\\
&\quad + \frac{1}{2}\,g_Q(d_Bu, \pi(X))\Psi + \frac{1}{2}\,\pi(X)\cdot d_Bu\cdot\Psi.
\end{aligned}
$$

Proof. From (6.1), we have

$$\bar{\nabla}_X\bar{\Psi} + f\pi(X) \bar{\cdot} \bar{\Psi} + g\kappa_{\bar{g}} \bar{\cdot} \pi(X) \bar{\cdot} \bar{\Psi} = 0.$$

Hence from (3.20), we have

$$\overline{\nabla_X\Psi} - \frac{1}{2}\,\overline{\pi(X)\cdot d_Bu\cdot\Psi} - \frac{1}{2}X(u)\bar{\Psi} + fe^u\overline{\pi(X)\cdot\Psi} + \overline{g\kappa\cdot\pi(X)\cdot\Psi} = 0.$$

Since $\tilde{I}_u$ is an isometry, the proof is completed. □

Theorem 6.8 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric g_M such that $\kappa \in \Omega_B^1(\mathcal{F})$ and $\delta\kappa = 0$. Assume that $\sigma^\nabla \geq 0$. If there exists an eigenspinor field Ψ_1 of the basic Dirac operator D_b for the eigenvalue $\lambda_1^2 = \dfrac{q}{4(q-1)}\Big(\mu_1 + \dfrac{q+1}{q}\inf|\kappa|^2\Big)$, then $\mathcal{F}$ is minimal, transversally Einsteinian with a positive constant transversal scalar curvature σ^∇.*

Proof. Let $D_b\Phi = \lambda_1\Phi$ with $\lambda_1^2 = \dfrac{q}{4(q-1)}\Big(\mu_1 + \dfrac{q+1}{q}\inf|\kappa|^2\Big)$ and $\Psi = e^{-\frac{q-1}{2}u}\Phi$. From (6.6), we see that $\overset{f_1,g_1}{\nabla}_{tr}\bar\Psi = 0$, where $f_1 = \dfrac{\lambda_1}{q}\, e^{-u}$ and $g_1 = -\dfrac{1}{2q}$. Hence we have that from (6.1)

$$\bar\nabla_{\bar E_a}\bar\Psi + f\bar E_a \,\bar\cdot\, \bar\Psi + g\kappa_{\bar g} \,\bar\cdot\, \bar E_a \,\bar\cdot\, \bar\Psi = 0.$$

Note that we have

$$\begin{aligned}\sum_a \bar E_a \,\bar\cdot\, \bar\nabla_{\bar E_a}\bar\Psi =& - f\sum_a \bar E_a \,\bar\cdot\, \bar E_a \,\bar\cdot\, \bar\Psi - g\sum_a \bar E_a \,\bar\cdot\, \kappa_{\bar g} \,\bar\cdot\, \bar E_a \,\bar\cdot\, \bar\Psi \\ =& qf\bar\Psi + g\sum_a\Big(\bar E_a \,\bar\cdot\, \bar E_a \,\bar\cdot\, \kappa_{\bar g} + 2\bar g_Q(\kappa_{\bar g}, \bar E_a)\bar E_a\Big) \,\bar\cdot\, \bar\Psi \\ =& qf\bar\Psi - (q-2)g\kappa_{\bar g} \,\bar\cdot\, \bar\Psi,\end{aligned}$$

and hence

$$\bar D_{tr}\bar\Psi + \frac{1}{2}\,\kappa_{\bar g} \,\bar\cdot\, \bar\Psi = qf\bar\Psi - (q-2)g\kappa_{\bar g} \,\bar\cdot\, \bar\Psi.$$

Since $\bar D_b\bar\Psi = \lambda_1 e^{-u}\bar\Psi$, we have

$$\lambda_1 e^{-u}\bar\Psi + \frac{1}{2}\,\kappa_{\bar g} \,\bar\cdot\, \bar\Psi \;= \lambda_1 e^{-u}\bar\Psi + \frac{q-2}{2q}\,\kappa_{\bar g} \,\bar\cdot\, \bar\Psi.$$

Hence we have $\kappa\cdot\Psi = 0$ which implies that $\kappa = 0$, i.e., $\mathcal{F}$ is minimal. If $\overset{f,g}{\nabla}_X\,\bar\Psi = 0$ for any $X \in \Gamma Q$, then $\mathrm{Ric}^{f,g}_{\bar\nabla} = 0$. Let $X = d_B f$. Then from (6.10), we get

$$< \Big(-\frac{1}{2}\,\rho^{\bar\nabla}(X) + 2(q-1)f^2X\Big) \,\bar\cdot\, \bar\Psi, \bar\Psi >_{\bar g_Q} = (q-1)|d_B f|^2|\bar\Psi|^2. \quad (6.12)$$

Hence the left hand side in the equation (6.12) is pure imaginary but the right hand side in the equation (6.12) is real, and so both sides are all zero. That is, $d_B f = 0$. So u is constnat. Also, we have from (6.10)

$$\rho^{\bar{\nabla}}(X) = 4(q-1)f^2 X \quad \text{for} \quad X \in \Gamma Q. \tag{6.13}$$

Since u is constant, we have from (2.20)

$$\rho^{\nabla}(X) = \frac{4(q-1)}{q^2}\lambda_1^2 X. \tag{6.14}$$

Hence $\mathcal{F}$ is transversally Einsteinian with a constant transversal scalar curvature $\sigma^{\nabla} = \dfrac{4(q-1)}{q}\lambda_1^2$. □

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<국 문 초 록>

Basic Dirac 연산자에 대한 고유치의 하한

엽층구조를 가지는 리만다양체상의 횡단적 공간이 spin 구조를 가질 때 횡단적 Dirac 연산자의 성질들을 공부하고 이를 이용하여 엽들의 좌표에 불변인 스피너들에 작용하는 basic Dirac 연산자의 고유치의 하한을 여러 방법으로 연구한다. 우선 횡단적으로 공형변환을 한 계량의 스피너 공간에서 횡단적 twistor 연산자를 정의하고 이것으로 basic Dirac 연산자의 고유치의 하한을 구한다. 또한 새로운 접속을 정의하고 그것의 크기를 이용하여 basic Dirac 연산자의 고유지치의 하한을 구하는 한편 새로운 접속을 공형변환하여 basic Dirac 연산자의 고유치의 하한을 연구한다. 모든 경우에 있어서 고유치가 하한일 때의 엽층은 극소적이고 횡단적 스칼라 곡률이 양의 상수인 횡단적 Einstein 공간이다.

감 사 의 글

이 과정을 통하여 배움에 대한 설레임과 고통, 신선함은 물론 풍요로움을 느끼곤 하였습니다. 그 때 그 때마다 따뜻한 격려와 세심한 지도를 아끼지 않으시고, 학문하는 자세를 가르쳐 주신 정승달 교수님께 깊은 감사를 드립니다. 그리고 그 동안 친절한 지도조언뿐만 아니라 희망과 용기를 북돋워주신 모든 교수님들께 감사를 드립니다.

이 과정 중에 함께 공부하면서 서로 격려하고 의지했던 문동주 선생님, 강상진 선생님, 김철준 선생님, 강경훈 선생님, 강문환 선생님, 그리고 만날 때마다 열심히 할 수 있도록 힘이 되어 주신 문영봉 선생님, 고연순 선생님, 정민주, 이금란 조교 선생님께도 고마운 마음을 전합니다.

그리고, 학교의 일과 진행의 어려움 속에서도 이 과정을 무사히 마칠 수 있도록 배려를 해 주신 한림중학교 교장, 교감, 교무부장 선생님, 특히, 3학년 진학지도를 함께한 담임 선생님들을 비롯한 모든 선생님들께 감사를 드리며, 주위에서 많은 격려와 용기를 주신 모든 분들께도 감사를 드립니다.

끝으로, 오늘날까지 정성과 사랑으로 늘 곁을 지켜주시는 어머님과 장모님, 여러 가지 어려운 여건 속에서도 한 번 내색하지 않고 인내와 사랑으로 묵묵히 도와준 소중한 아내, 채정애씨, 그리고 자기 발전을 위해 나날이 정진하고 있는 든든하고 건강한 두 아들 민성, 민규와 함께 이 기쁨을 나누고자 합니다.

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