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LINEAR OPERATORS THAT PRESERVE RANK OF THE BOOLEAN MATRIX PRODUCT

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<Abstract>

LINEAR OPERATORS THAT PRESERVE RANK OF THE BOOLEAN MATRIX PRODUCT

In this thesis, we consisted sets of pairs of matrices on Boolean Algebra.
these are pairs of matrices being shown naturally in the cases of the equality's extremum associated with ranks of two Boolean matrices' product.
The sets composed of these pairs are constructed of extremal cases on the equalities related to multiplication of ranks(coefficients) of two Boolean matrices.
immediately, It is made up of the following five kinds of sets

$$P_1(B)=\{(X,Y)\in M_n(B)^2 \mid r_B(XY)=\min\{r_B(X),r_B(Y)\}\}$$
$$P_2(B)=\{(X,Y)\in M_n(B)^2 \mid r_B(XY)=0\}$$
$$P_3(B)=\{(X,Y)\in M_n(B)^2 \mid r_B(XY)=1\}$$
$$P_4(B)=\{(X,Y)\in M_n(B)^2 \mid r_B(XY)=r_B(X)+r_B(Y)-n\}$$
$$P_5(B)=\{(X,Y,Z)\in M_n(B)^3 \mid r_B(XYZ)+r_B(X)=r_B(XY)+r_B(YZ)\}$$

we mapped the pairs of matrices as were stated above by Linear operators , studied the Linear operators that preserve the properties of the set, and identified the forms.
In a moment, we realized that the form of the Linear operators that preserve the sets of the matrix pairs show $T(X)=PXQ$ or $T(X)=PX^tQ$ and proved it.
Moreover, we researched the conditions equivalent of this Linear operators, and gave proof of equality of these ones.

CONTENTS

1 Introduction

During the past century a lot of literature has been devoted to investigations of semiring. Briefly, a semiring is essentially a ring where only the zero element is required to have an additive inverse. Therefore, all rings are also semirings.

A *semiring* $\mathcal{S}$ consists of a set $\mathcal{S}$ and two binary operations, addition and multiplication, such that:

- $\mathcal{S}$ is an s monoid under addition (identity denoted by 0);
- $\mathcal{S}$ is a semigroup under multiplication (identity, if any, denoted by 1);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

A semiring is called *antinegative* if the zero element is the only element with an additive inverse. For example, the set of nonnegative integers is an antinegative semiring with usual addition and multiplication.

Definition 1.1. A semiring $\mathcal{S}$ is called *Boolean* if $\mathcal{S}$ is equivalent to a set of subsets of a given set N, the sum of two subsets is their uion, and the product is their intersection. The zero element is the empty set and the identity element is the whole set N.

It is straightforward to see that Boolean semiring is commutative and antinegative. If $\mathcal{B}$ consists of only the lower and upper bounds and the N then it is called a binary Boolean algebra (or $\{0, 1\}$ -semiring) and is denoted by $\mathcal{S}$

A semiring S is called *chanin* if the set S is totally ordered under set inclusion with universal lower and upper bounds and the operations are defined by $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$.

It is straightforward to see that any chain semiring S is a Boolean semiring on the set of appropriate subsets of S. Consider the set N of all elements in S, and choose all those subsets that consist of all elements strictly lower than a given element.

Let $\mathcal{M}_{m,n}(\mathcal{B})$ denote the set of $m \times n$ matrices with enties form the binary Boolean algebra B. Matrix theory over semirings is an object of intensive study during the last decades, see for example [6, 7] and references therein. In particular, many authors have investigated various rank functions for matrices over Boolean algebra and their properties, see [1, 10, 11, 14]. Among the rank functions that have the most interesting applications is the well-Known notion of the factor rank.

Let $\mathcal{M}_{m,n}(\mathcal{B})$ be the set of $m \times n$ Boolean matrices. Throughout we assume that $m \leq n$. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I,J, and O, respectively. The matrix $E_{i,j}$, called a cell, denotes the matrix with exactly 1, that being a 1 in the (i,j) entry. Let R_i denote the matrix whose i^{th} row is all ones and is zero elsewhere, and C_j denote the matrix whose j^{th} column is all ones and is zero elsewhere. We let $|A|$ denote the number of nonzero entries in the matrix A.

Definition 1.2. The matrix $A \in \mathcal{M}_{m,n}(\mathcal{B})$ is said to be of *Boolean rank* k ($r_B(A) = k$) if there exist matries $B \in \mathcal{M}_{m,k}(\mathcal{B})$ and $C \in \mathcal{M}_{k,n}(\mathcal{B})$ such that $A = BC$ and k is the smallest positive integer such that a factorization exists. By definition the only matirx with Boolean rank equal to 0 is the zero matrix, O.

If B is considered as a subsemiring of a real field R then there is a real rank function $\rho(A)$ for any Boolean matrix $A \in \mathcal{M}_{m,n}(\mathcal{B})$.

Example 1.3. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \in \mathcal{M}_{4,4}(\mathcal{B}).$$

then $r_B(A) = 4$ from Example 2.3.1 [5]. But $\rho(A) = 3$

■

The example 1.3 shows that the Boolean rank and real rank of A are not equal. However, the inequality $r_B(A) \geq \rho(A)$ always holds.

The behavior of the function ρ with respect to matrix multiplication and addition is given by the following inequalities: The *rank-sum inequalities*:

$$\mid \rho(A) - \rho(B) \mid \leq \rho(A+B) \leq \rho(A) + \rho(AB) \leq \min\{\rho(A), \rho(B)\}$$

and the *Frobenius inequality*:

$$\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B),$$

where A, B, C are real matrices (see [8]).

Arithmetic properties of Boolean rank is restricted by the following list of inequalities established from [3] because Boolean algebra is antinegative semiring.

1. $r_B(A+B) \leq r_B(A) + r_B(B)$;

2. $r_B(AB) \leq \min\{r_B(A), r_B(B)\}$.

3. $r_B(A+B) \geq \begin{cases} r_B(A) & \text{if} \quad B=0 \\ r_B(B) & \text{if} \quad A=0 \\ \quad 1 & \text{if} \quad A \neq 0 \text{ and } B \neq 0 \end{cases}$;

4. $r_B(AB) \geq \begin{cases} 0 & if \quad r_B(A)+r_B(B) \leq n \\ 1 & if \quad r_B(A)+r_B(B) > n \end{cases}$.

If $\mathcal{B}$ is considered as a subsemiring of $\Re^+$, the positive real numbers, we have:

5. $r_B(A+B) \geq |\rho(A)-\rho(B)|$;

6. $r_B(AB) \geq \begin{cases} 0 & if \rho(A)+\rho(B) \leq n, \\ \rho(A)+\rho(B)-n & if \rho(A)+\rho(B) > n \end{cases}$;

7. $\rho(AB)+\rho(BC) \leq r_B(ABC)+r_B(B)$.

As was proved in [3] the inequalities $1 \sim 7$ are sharp and the best possible.

The natural question is to characterize the equality cases in the above inequalities. Even over fields this is an open problem, see [2] for more details. The structure of matrix varieties which arise as extremal cases in these inequalities is far from being understood over fields, as well as over Boolean algebra. A usual way to generate elements of such a variety is to find a tuple of matrices which belongs to it and to act on this tuple by various linear operators that preserve this variety. The linear operators that preserve cases of equalities in various matrix inequalities over fields were obtained in [8, 9]. For the details on linear operators preserving matrix invariants one can see [13] and references therein. The aim of the present thesis is to characterize linear operators that preserve the sets of matrix pairs which satisfies the Boolean rank equalities. Among those sets, we consider the sums of two Boolean matrices and their Boolean ranks. These rank equalities come from the extreme cases of the inequalities of Boolean ranks.

In section 2, we present the concrete sets of matrix pairs which come from the the extreme cases of the inequalities of Boolean ranks.

In section 3 to 7, we characterize the linear operators that preserve the sets of matrix pairs which come from the the extreme cases of the inequalities of Boolean ranks.

2 Preliminaries

Let B be the binary Boolean algebra. Consider following notation in order to denote sets of Boolean matrices that arise as extremal cases in the inequalities listed above:

$$\mathcal{P}_1(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_n(\mathcal{B})^2 \mid r_B(XY) = min\{r_B(X), r_B(Y)\}\};$$

$$\mathcal{P}_2(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_n(\mathcal{B})^2 \mid r_B(XY) = 0\};$$

$$\mathcal{P}_3(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_n(\mathcal{B})^2 \mid r_B(XY) = 1\};$$

$$\mathcal{P}_4(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_n(\mathcal{B})^2 \mid r_B(XY) = r_B(X) + r_B(Y) - n\};$$

$$\mathcal{P}_5(\mathcal{B}) = \{(X,Y,Z) \in \mathcal{M}_n(\mathcal{B})^3 \mid r_B(XYZ) + r_B(Y) = r_B(XY) + r_B(YZ)\};$$

Definition 2.1. We say an operator, T, *preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ implies that $T(X) \in \mathcal{P}$, or if P is a set of ordered pairs [triples], that $(X,Y) \in \mathcal{P}$ $[(X,Y,Z)] \in \mathcal{P}]$ implies $(T(X),T(Y)) \in \mathcal{P}$ $[(T(X),T(Y),T(Z)) \in \mathcal{P}]$.

Definition 2.2. An operator T *strongly preserves* a set $\mathcal{P}$ if $X \in \mathcal{P}$ if and only if $T(X) \in \mathcal{P}$, or, if $\mathcal{P}$ is a set of ordered pairs [triples], that $(X,Y) \in \mathcal{P}[(X,Y,Z) \in \mathcal{P}]$ if and only if $(T(X),T(Y)) \in \mathcal{P}$ $[(T(X),T(Y),T(Z)) \in \mathcal{P}]$.

Definition 2.3. An operator $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ is called a (P,Q)*-operator* if there exist permutation matrices P and Q of appropriate orders such that $T(X) = PXQ$ for all $X \in \mathcal{M}_{m,n}(\mathcal{B})$, or, if $m = n$, $T(X) = PX^tQ$ for all $X \in \mathcal{M}_{m,n}(\mathcal{B})$, where X^t denotes the transpose of X

Definition 2.4. A mapping $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ is called a *Boolean linear operator* if $T(O_{m,n}) = O_{m,n}$ and $T(X+Y) = T(X) + T(Y)$ for all $X, Y \in \mathcal{M}_{m,n}(\mathcal{B})$.

Definition 2.5. A matrix $A \in \mathcal{M}_{m,n}(\mathcal{B})$ is called *monomial* if it has exactly one nonzero element in each row and column.

Definition 2.6. A *line* of a matrix A is a row or a column of the matrix A.

Definition 2.7. We say that the matrix A *dominates* the matrix B if $b_{i,j} \neq 0$ implies that $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$.

Definition 2.8. If A and B are Boolean matrices and $A \geq B$ we let $A \backslash B$ denote the matrix C where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} = 1 \\ 1 & \text{if } b_{i,j} = 0 \end{cases}.$$

Definition 2.9. The matrix $X \circ Y$ denotes the *Hadamard* or *Schur product*, i.e., the (i,j) entry of $X \circ Y$ is $x_{i,j}y_{i,j}$.

The following theorem implies the characterizations of the surjective linear operator on $\mathcal{M}_{m,n}(\mathcal{B})$.

Theorem 2.10. [15] *Let $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ be a Boolean linear operator. Then the following are equivalent:*

1. T is bijective.

2. T is surjective.

3. There exists a permutation σ on $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$.

Proof. That 1) implies 2) and 3) implies 1) is straight forward. We now show that 2) implies 3).

We assume that T is surjective. Then, for any pair (i,j), there exists some X such that $T(X) = E_{i,j}$. Clearly $X \neq O$ by the linearity of T. Thus there is a pair of indices

(r,s) such that $X = E_{r,s} + X'$ where (r,s) entry of X' is zero and $T(E_{r,s}) \neq O$. Indeed, if $T(E_{r,s}) = O$ for all pairs (r,s), then $T(X) = O$ by linearity of T. Thus we have a contradiction. But $T(X) = E_{i,j} \neq O$. Hence

$$T(E_{r,s}) \leq T(E_{r,s}) + T(X \setminus (E_{r,s})) = T(X) = E_{i,j}.$$

That is, $T(E_{r,s}) \leq E_{i,j}$ and $T(E_{r,s}) = E_{i,j}$. Since the set $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$ is a finite set, T is injective since it is surjective.

Therefore there is some permutation σ on $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$ such that $T(E_{i,j}) = E_{\sigma(i,j)}$. ∎

Henceforth we will always assume that $m, n \geq 2$.

Lemma 2.11. [15] *Let $T : \mathcal{M}_{m,n}(\mathcal{B}) \to \mathcal{M}_{m,n}(\mathcal{B})$ be a Boolean operator which maps lines to lines and is defined by $T(E_{i,j}) = E_{\sigma(i,j)}$, where σ is a permutation on the set $\{(i,j) \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}$. Then T is a (P,Q)-operator.*

Proof. Since no combination of a rows and b columns can dominate J where $a+b=m$ unless $b=0$ (or if $m=n$, if $a=0$) we have that either the image of each row is a row and the image of each column is a column, or $m=n$ and the image of each row is a column and the image of each column is a row. Thus, there are permutation matrices P and Q such that $T(R_i) \leq PR_iQ$ and $T(C_j) \leq PC_jQ$ or, if $m=n$, $T(R_i) \leq P(R_i)^tQ$ and $T(C_j) \leq P(C_j)^tQ$. Since each cell lies in the intersection of a row and a column and T maps nonzero cells to nonzero (weighted) cells, it follows that $T(E_{i,j}) = PE_{i,j}Q$, or, if $m=n$, $T(E_{i,j}) = PE_{j,i}Q = P(E_{i,j})^tQ$. ∎

3 Linear preservers of $\mathcal{P}_1(\mathcal{B})$.

Recall that

$$\mathcal{P}_1(\mathcal{B}) = \{(X,Y) \in \mathcal{M}_n(\mathcal{B})^2 \mid r_B(XY) = min\{r_B(X), r_B(Y)\}\};$$

We begin with some general observations on Boolean linear operators of special types that preserve $\mathcal{P}_1(\mathcal{B})$.

Theorem 3.1. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ be a surjective linear operator which preserves $\mathcal{P}_1(\mathcal{B})$. Then there exists permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. By Theorem 2.10 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ which a permutation σ on $\{(i,j)|1 \leq i,j \leq n\}$. Consider, $(E_{i,j}, E_{j,k}) \in \mathcal{P}_1(\mathcal{B})$ since $r_B(E_{i,j}E_{j,k}) = r_B(E_{i,k}) = 1 = min\{r_B(E_{i,j}), r_B(E_{j,k})\}$. Since T preserves $\mathcal{S}_1(\mathcal{B})$, $(T(E_{i,j}), T(E_{j,k})) \in \mathcal{P}_1(\mathcal{B})$. Thus $r_B(T(E_{i,j})T(E_{j,k})) = min\{r_B(T(E_{i,j}), r_B(T(E_{j,k}))\} = 1$, but $r_B(T(E_{i,j})T(E_{j,k}))$ $=E_{\sigma(i,j)}E_{\sigma(j,k)}$. It follows that $E_{\sigma(j,k)}$ is in the same row as $E_{\sigma(j,1)}$ for all k. That is, T maps rows to rows. Similarly T maps columns to columns. By Lemma 2.11, we have that $T(X) = PXQ$ for some permutation matrices P, Q.

Let us show that $Q = P^t$. Let $T(E_{i,j}) = E_{\alpha(i)\beta(j)}$ where α and β are permutation corresponding to P and Q^t, respectively.

It follows from $(E_{1,i}, E_{i,1}) \in \mathcal{S}_1(\mathcal{B})$ that $(T(E_{1,i}), T(E_{i,1}))= (E_{\alpha(1)\beta(i)}, E_{\alpha(i)\beta(1)}) \in \mathcal{S}_1(\mathcal{B})$. Hence $\alpha(i) = \beta(i)$ for all i and so $\alpha = \beta$. Thus $Q = P^t$ ■

Corollary 3.2. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ be a linear operator. Then T strongly preserves $\mathcal{P}_1(\mathcal{B})$ if and only if there exists permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. It is easily shown that all operators of the form $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$ strongly preserves $\mathcal{P}_1(\mathcal{B})$.

Conversely suppose that T strongly preserves $\mathcal{P}_1(\mathcal{B})$. Our aim is to show that T is surjective on $\mathcal{M}_n(\mathcal{B})$. Equivalently, for each cell $E_{i,j}$, there exists $Y \in \mathcal{M}_n(\mathcal{B})$ such that $T(Y) = E_{i,j}$. If not, there exists $M \in \mathcal{M}_n(\mathcal{B})$ with $m_{r,s} = 0$ such that$T(M) = T(J)$. Let $A = J \backslash E_{r,s}$, then $(A, A) \notin \mathcal{P}_1(\mathcal{B})$. Since T strongly preserves $\mathcal{P}_1(\mathcal{B})$, $(T(A), T(A)) \notin \mathcal{S}_1(\mathcal{B})$. Since $M \leq A \leq J$, $T(J) = T(M) \leq J(A) \leq T(J)$, and so $T(A) = T(J)$. Since $(T(J), T(J)) \in \mathcal{P}_1(\mathcal{B})$, $(T(A), T(A)) \in \mathcal{P}_1(\mathcal{B})$, a contradiction.

Thus, T is a surjective. By Theorem 3.1, there exists permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$. ■

Corollary 3.3. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ be an operator defined by $T(X) = PXQ$ for some permutation matrices P, Q. Then, T strongly preserves $\mathcal{P}_1(\mathcal{B})$ if and only if $Q = P^t$.*

Proof. Suppose that T strongly preserves $\mathcal{P}_1(\mathcal{B})$. Let X be an arbitrary matrix in $\mathcal{M}_n(\mathcal{B})$. Then there exists $P^tXQ^t \in \mathcal{M}_n(\mathcal{B})$ such that $T(P^tXQ^t) = X$. Thus T is surjective. By Theorem 3.1, $Q = P^t$.

Conversely assume that $Q = P^t$. It follows from Corollary 3.2 that T strongly preserve $\mathcal{P}_1(\mathcal{B})$. ■

4 Linear preservers of $\mathcal{P}_2(\mathcal{B})$.

Recall that

$$\mathcal{P}_2(\mathcal{B}) = \{(X, Y) \in \mathcal{M}_n(\mathcal{B})^2 \mid r_B(XY) = 0\};$$

Theorem 4.1. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ be a nonsingular(that is, $T(X) = 0 \Rightarrow X = 0$) additive operator. If $T(J) = J$, then T preserves $\mathcal{P}_2(\mathcal{B})$ if and only if there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. It is easy to see that operators of the form $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$ preserves $\mathcal{P}_2(\mathcal{B})$.

Assume now that T preserves $\mathcal{P}_2(\mathcal{B})$. Since $T(J) = J$, there are n different cells whose images have nonzero entries in every column.

Suppose that these cells can be chosen such that their nonzero entries are in fewer than n columns. Say $X = E_1 + E_2 + \cdots + E_n$ is the sum of n such cells and that X has no nonzero entry in k^{th}column. Then $(X, R_k) \in \mathcal{P}_2(\mathcal{B})$, and hence $(T(X), (T(R_k)) \in \mathcal{P}_2(\mathcal{B})$. But $(T(X), T(R_k)) \neq 0$, a contradiction.

Thus, T must map columns to columns, and further, T induces a permutation on the set of columns. Similarly T induces a permutation on the set of rows. That is, $T(X) = PXQ$ for all $X \in \mathcal{M}_n(\mathcal{B})$ for some permutation matrices P and Q. Let us show that $Q = P^t$. Indeed we have that $T(E_{i,j}) = E_{\alpha(i)\beta(j)}$ where α, β are permutations corresponding P and Q^t, respectively. If $Q \neq P^t$ then $\alpha \neq \beta$, and so $\alpha(i) \neq \beta(i)$ for some i, hence, for some $j \neq i, \alpha(j) = \beta(i)$. Here, $(E_{i,i}, E_{j,i}) \in \mathrm{P}_2(\mathcal{B})$, but $(T(E_{i,i}), T(E_{j,i})) \notin \mathcal{P}_2(\mathcal{B})$, a contradiction. Therefore $Q = P^t$ ∎

Corollary 4.2. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ be a surjective linear operator. Then T preserves $\mathcal{P}_2(\mathcal{B})$ if and only if there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. Since T is surjective, T is nonsingular. By Theorem 4.1 the result follows. ∎

Theorem 4.3. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ be a linear operator. Then T strongly preserves $\mathcal{P}_2(\mathcal{B})$ if and only if there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. It is easy to see that operators of the form $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$ strongly preserve $\mathcal{P}_2(\mathcal{B})$.

Assume now that T strongly preserves $\mathcal{P}_2(\mathcal{B})$. Let us check that $T(J) = J$, Assume in the contrary that $T(J)$has a zero column (all considerations in the case of zero row are quite similar). Up to a multiplication with permutational matrices we may assume that there are nonzero elements in columns $1, 2, \cdots, t$ of $T(J)$ and all elements in the columns $(t+1), \cdots, n$ are zero. Then, there exist column matrices $C_{j1}, C_{j2}, \cdots, C_{js}$ whose images have nonzero entries in columns 1 through t. Let $l \neq j_k$ for all k, $1 \leq k \leq s$. Thus $(C_{j1}, C_{j2}, \cdots, C_{js})R_l = 0$. Since T preserves $\mathcal{P}_2(\mathcal{B})$ it follows that $T(C_{j1}, C_{j2}, \cdots, C_{js})T(R_l) = 0$. Then $T(R_l)$ has no nonzero entries in rows 1 through t, since in each of the first t columns of $T(C_{j1}, C_{j2}, \cdots, C_{js})$ there is a nonzero element. Therefore, $T(E_{l,l})$ has nonzero entries only in rows $t+1, \cdots, n$ and only in columns $1, \cdots, t$. Thus $T(E_{l,l})^2 = 0$, ie, $(T(E_{l,l}), T(E_{l,l})) \in \mathcal{P}_2(\mathcal{B})$. This is a contradiction sine T strongly preserves $\mathcal{P}_2(\mathcal{B})$ and $(E_{l,l}, E_{l,l}) \notin \mathcal{P}_2(\mathcal{B})$. Thus, $T(J)$ has neither a zero row, nor a zero column. Let us check that T is nonsingular. Assume that there exists $0 \neq X$ such that $T(X) = 0$. Thus $(T(X), T(I)) \in \mathcal{P}_2(\mathcal{B})$ as far as $(X, I) \notin \mathcal{P}_2(\mathcal{B})$. This contradicts with T strongly preserves $\mathcal{P}_2(\mathcal{B})$. Hence Theorem 4.1 is applicable and by this $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$ for some permutation matrix P. The Theorem follows. ■

5 Linear preservers of $\mathcal{P}_3(\mathcal{B})$.

Recall that

$$\mathcal{P}_3(\mathcal{B}) = \{(X, Y) \in \mathcal{M}_n(\mathcal{B})^2 \mid r_B(XY) = 1\};$$

Corollary 5.1. *Let $\mathcal{B}$ be a Boolean semiring, $T(E_{i,j}) = E_{\sigma(i,j)}$ for some permutation, σ, of $\{(i,j) \mid 1 \le i,j \le n\}$. Then T preserves $\mathcal{P}_3(\mathcal{B})$ if and only if there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. Clearly operators of the form $T(X) = PXP^t$ preserve $\mathcal{P}_3(\mathcal{B})$.

Suppose that T preserves $\mathcal{P}_3(\mathcal{B})$. Consider, $(E_{i,i}, E_{i,k}) \in \mathcal{P}_3(\mathcal{B})$ for all k. If $T(E_{i,i}) = E_{r,s}$ for some r, s, then $T(E_{i,k}) = E_{s,\tau(k)}$ for some permutation τ. That is, $T(R_i) \le R_s$. Thus, T induces a permutation on the rows. Similarly, T induces a permutation on the columns. Thus, for some permutations π and τ, $T(E_{i,j}) = E_{\pi(i)\tau(j)}$. Now, $r_B(T(E_{i,i})T(E_{i,j}))$ must be 1, and so, $\pi(i) = \tau(i)$. That is, $\pi = \tau$, and we have that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$ where P is the permutation corresponding to π. ∎

Theorem 5.2. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ be a surjective linear operator. Then T preserves $\mathcal{P}_3(\mathcal{B})$ if and only if there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. By Theorem 2.10 we have that for all $i, j, 1 \le i, j \le n$, $T(E_{i,j}) = E_{\sigma(i,j)}$. Thus by Corollary 5.1 the result follows. ∎

Theorem 5.3. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ be a linear operator. Then T strongly preserves $\mathcal{P}_3(\mathcal{B})$ if and only if there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. Since operators of the form $T(X) = PXP^t$ preserve $\mathcal{P}_3(\mathcal{B})$, we assume that T strongly preserves $\mathcal{P}_3(\mathcal{B})$ and show that T is of the form $T(X) = PXP^t$. Let $M \in \mathcal{M}_n(\mathcal{B})$ such that $|T(M)| = |T(J)|$, and if $|T(N)| = |T(J)|$ then $|M| \leq |N|$(that is, M is a minimal matrix), so that $T(M) = T(J)$. Assume that there exists an index j such that j^{th}column of M is zero. Then $ME_{j,k} = 0$, thus $(M, E_{j,k}) \notin \mathcal{P}_3(\mathcal{B})$. Since T preserves $\mathcal{P}_3(\mathcal{B})$, $(T(M), T(E_{j,k})) = (T(J), T(E_{j,k})) \notin \mathcal{P}_3(\mathcal{B})$, a contradiction. Thus M has no zero column. Similarly, M has no zero row. Now $(J, I) \in \mathcal{P}_3(\mathcal{B})$, so that $(T(J), T(I)) = (T(M), T(I)) \in \mathcal{P}_3(\mathcal{B})$. Since T strongly preserves $\mathcal{P}_3(\mathcal{B})$, $(M, I) \in \mathcal{P}_3(\mathcal{B})$. Thus $r_B(M) = 1$ and hence $M = J$. Since M define, T induces a bijection on the set of cells, that is, $T(E_{i,j}) = E_{\sigma(ij)}$ for some permutation, σ. By Corollary 5.1, the theorem follows. ■

6 Linear preservers of $\mathcal{P}_4(\mathcal{B})$.

Recall that

$$\mathcal{P}_4(\mathcal{B}) = \{(X, Y) \in \mathcal{M}_n(\mathcal{B})^2 \mid r_B(XY) = r_B(X) + r_B(Y) - n\};$$

Theorem 6.1. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ is a surjective linear operator which preserves $\mathcal{P}_4(\mathcal{B})$. Then there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. By Theorem 2.10, we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ which a permutation σ on $\{(i,j)|1 \le i,j \le n\}$. If $r_B(A) = n$ then $(E_{i,j}, A) \in \mathcal{P}_4(\mathcal{B})$. Since T preserves $\mathcal{P}_4(\mathcal{B})$, $(T(E_{i,j}), T(A)) \in \mathcal{P}_4(\mathcal{B})$. That is, $r_B(T(A)) = n$. Thus T preserves $r_B - n$ matrix.
If image of a row is not dominated by any line then there are cells, $E_{i,j}, E_{s,q}$ such that $T(E_{i,j} + E_{s,q}) \le E_{r,z} + E_{r,w}$ and $i \ne s, j \ne q$. By extending $E_{i,j} + E_{s,q}$ to a permutation matrix by adding $n-2$ cells, we find a matrix which is the image of a permutation matrix. but is dominated by $n-1$ lines, a contradiction, since T preserves $r_B - n$ matrix. Thus the pre-image of every row is a row or column and similarly, the pre-image of every column is a row or column. Hence T maps lines to lines. By Theorem 2.11, we have that T is (P,Q)-operator. Since $(E_{1,1} + E_{2,1} + \cdots + E_{n,n-1}) \in \mathcal{P}_4(\mathcal{B})$, while $(T(E_{1,1} + E_{1,2} + \cdots + E_{n-1,n}) \notin \mathcal{P}_4(\mathcal{B})$, we have that the transpose operator does not preserve $\mathcal{P}_4(\mathcal{B})$, thus, there exist permutation matrices P and Q such that $T(X) = PXQ$.
Without loss of generality we may assume that $P = I$. If $Q \ne I$, say Q corresponds to the permutation π, and $\pi(1) \ne 1$. Without loss of generality, $T(E_{1,1}) = E_{1,2}$. Then, $(E_{1,1} + E_{2,2} + \cdots + E_{n,n}) \in \mathcal{P}_4(\mathcal{B})$, while $(T(E_{1,1} + E_{2,2} + \cdots + E_{n,n}) \notin \mathcal{P}_4(\mathcal{B})$,since

$(E_{1,1})(E_{2,\pi(2)} + E_{3,\pi(3)} + \cdots + E_{n,\pi(n)}) = E_{1,2}E_{2,\pi(2)} \neq 0$. This contradiction gives that $Q = P^t$, that is, $T(X) = PXP^t$. ∎

Corollary 6.2. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ is defined by $T(X) = PXQ$ for all $X \in \mathcal{M}_n(\mathcal{B})$ where $P, Q$are permutation matrix. Then T preserves $\mathcal{P}_4(\mathcal{B})$ if and only if $Q = P^t$.*

Proof. Suppose that $Q = P^t$ then $r_B(T(X)T(Y)) = r_B(PXP^tPYP^T) = r_B(XY) = r_B(X) + r_B(Y) - n= r_B(PXP^t) + r_B(PYP^t) - n=r_B(T(X)) + r_B(T(Y)) - n$. Conversely, Let T preserves $\mathcal{P}_4(\mathcal{B})$, for $X \in \mathcal{M}_n(\mathcal{B})$ there exists $P^tXQ^t \in \mathcal{M}_n(\mathcal{B})$ such that $T(P^tXQ^t) = X$. Thus T is a surjective. By Theorem 6.1 we have done. ∎

Theorem 6.3. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ is a surjective linear operator. Then T preserves $\mathcal{P}_4(\mathcal{B})$ if and only if there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. It is easily shown that all operators of the form $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$ preserves $\mathcal{P}_4(\mathcal{B})$. Conversely suppose that T preserves $\mathcal{P}_4(\mathcal{B})$. Since T is a surjective, by Theorem 2.13, we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ which a permutation σ on $\{(i,j)|1 \leq i,j \leq n\}$. By Theorem 6.1, there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$. ∎

7 Linear preservers of $\mathcal{P}_5(\mathcal{B})$.

Recall that

$$\mathcal{P}_5(\mathcal{B}) = \{(X, Y, Z) \in \mathcal{M}_n(\mathcal{B})^3 \mid r_B(XYZ) + r_B(Y) = r_B(XY) + r_B(YZ)\};$$

Corollary 7.1. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ be a surjective linear preserver of $\mathcal{P}_5(\mathcal{B})$. Then there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. By Theorem 2.10 we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for a certain permutation σ on $\{(i,j) \mid 1 \le i,j \le n\}$. It can be easily checked that $(T(E_{i,j}), T(E_{j,k}), T(E_{k,l})) \in \mathcal{P}_5(\mathcal{B})$ for all l and for arbitrary fixed i,j,k. Thus $r_B(T(E_{i,j})T(E_{j,k})T(E_{k,l})) + r_B(T(E_{j,k}))=$ $r_B(T(E_{i,j})T(E_{j,k})) + r_B(T(E_{j,k})T(E_{k,l}))$

By Theorem 2.10, it follows that $T(E_{i,j}) = E_{p,q}, T(E_{j,k}) = E_{r,s}, T(E_{k,l}) = E_{u,v}$, indices p,q,r,s,u,v. Since $r_B(T(E_{j,k})) = 1 \neq 0$ it follows from the equality that either $r = q$ or $s = u$ or *both*.

If for all $l = 1, \dots, n$ both equalities hold then for fixed i,j,k all matrices $T(E_{k,l}), l = 1, \dots, n$, have their nonzero elements lying in one row. Thus T maps rows to rows. Similarly it is easy to see that T maps columns to columns.

Assume now that there exists an index l such that the only one of the above equalities holds for the triple $(E_{i,j}, E_{j,k}, E_{k,l})$. Without loss of generality assume that $s = u$ and $r \neq q$. Thus for arbitrary m, $1 \le m \le n$ one has that $(E_{i,j}, E_{j,k}, E_{k,l}) \in \mathcal{P}_5(\mathcal{B})$. By Theorem 2.10, $T(E_{k,m}) = E_{w,z}$ for certain w, z depending on k, m. In the above notations obtain that rows are transformed to rows. By the same arguments with the first matrix it is easy to see that columns are transformed to rows. By the same arguments with the first matrix it is easy to see that rows are transformed to columns and columns to rows. By Lemma 2.11, it follows that there exists a permutation P and Q such that $T(X) = PXQ$ or $T(X) = PX^tQ$ for all $X \in \mathcal{M}_n(\mathcal{B})$.

In order to show that transposition operator does not preserve $\mathcal{P}_5(\mathcal{B})$ it suffices to note that $(E_{i,j}, I, I \backslash E_{j,j}) \in \mathcal{P}_5(\mathcal{B})$ while $(E_{j,i}, I, I \backslash E_{j,j}) \in \mathcal{P}_5(\mathcal{B})$.

In order to show that $Q = P^t$ it suffices to note that $(E_{i,j}, E_{j,j}, E_{j,i}) \in \mathcal{P}_5(\mathcal{B})$.

Therefore, $(E_{\sigma(i),\tau(j)}, E_{\sigma(j),\tau(j)}, E_{\sigma(j),\tau(i)}) \in \mathrm{P}_5(\mathcal{B})$. Thus $\sigma = \tau$, that is, $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$. ■

Corollary 7.2. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ be defined by $T(X) = PXQ$ for all $X \in \mathcal{M}_n(\mathcal{B})$ where $P, Q \in \mathcal{M}_n(\mathcal{B})$ are permutation matrices. Then T preserves $\mathcal{P}_5(\mathcal{B})$ if and only if $Q = P^t$.*

Proof. Suppose that $Q = P^t$ then $r_B(T(X)T(Y)T(Z)) + r_B(T(Y))$ $= r_B(PXYZP^t) + r_B(PYP^t) = r_B(XYZ) + r_B(Y) = r_B(XY) + r_B(YZ)$ $= r_B(T(X)T(Y)) + r_B(T(Y)T(Z))$. Conversely, for $X \in \mathcal{M}_n(\mathcal{B})$ there exists $P^tXP \in \mathcal{M}_n(\mathcal{B})$ such that $T(P^tXP) = X$. Thus T is a surjective, By Theorem 2.10, Lemma 7.1, there exists a permutation P such that $T(X) = PXP^t$, that is, $Q = P^t$. ■

Theorem 7.3. *Let $\mathcal{B}$ be a Boolean semiring, $T : \mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ be a surjective linear operator. Then T preserves $\mathcal{P}_5(\mathcal{B})$ if and only if there exists a permutation P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$.*

Proof. It is to see that operators of the form $T(X) = PXP^t$ preserves $\mathcal{P}_5(\mathcal{B})$.

Conversely suppose that T preserves $\mathcal{P}_5(\mathcal{B})$. Since T is surjective, by Theorem 2.10, we have that $T(E_{i,j}) = E_{\sigma(i,j)}$ for a certain permutation σ on $\{(i,j) \mid 1 \leq i, j \leq n\}$.Then, by Corollary 7.1, there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{M}_n(\mathcal{B})$. ■

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<국문초록>

부울행렬 곱의 계수를 보존하는 선형 연산자

본 논문에서는 부울 대수상의 행렬의 짝들로 구성되는 집합들을 구성하였다.

이 집합들은 두 부울행렬들의 곱의 계수와 관련된 부등식이 극치인 경우들에서 자연스럽게 나타나는 행렬 짝들의 집합들이다. 이 행렬 짝들의 집합들은 두 부울 행렬들의 계수의 곱과 관련된 부등식들에서 극치인 경우들로 구성하였다.

곧, 다음과 같은 5가지 집합들로 구성하였다.

$$P_1(B) = \{(X, Y) \in M_n(B)^2 \mid r_B(XY) = \min\{r_B(X), r_B(Y)\}\}$$

$$P_2(B) = \{(X, Y) \in M_n(B)^2 \mid r_B(XY) = 0\}$$

$$P_3(B) = \{(X, Y) \in M_n(B)^2 \mid r_B(XY) = 1\}$$

$$P_4(B) = \{(X, Y) \in M_n(B)^2 \mid r_B(XY) = r_B(X) + r_B(Y) - n\}$$

$$P_5(B) = \{(X, Y, Z) \in M_n(B)^3 \mid r_B(XYZ) + r_B(X) = r_B(XY) + r_B(YZ)\}$$

이상의 행렬짝들의 집합을 선형 연산자로 보내어 그 집합의 성질들을 보존하는 선형 연산자를 연구하여 그 형태를 규명하였다. 곧, 이러한 행렬 짝들의 집합을 보존하는 선형 연산자의 형태는 $T(X) = PXQ$ 또는 $T(X) = PX^tQ$ 로 나타냄을 보이고, 이들을 증명하였다. 그리고 이 선형 연산자와 동치가 되는 조건들을 찾고, 이 조건들이 동등함을 증명하였다.