碩 士 學 位 論 文

Integral formulas and vanishing theorems on a Riemannian Foliation

濟州大學校 大學院

數 學 科

李 今 蘭

2005年 12月

Integral formulas and vanishing theorems on a Riemannian Foliation

指導教授　鄭　承　達

李　今　蘭

이 論文을 理學 碩士學位 論文으로 提出함

2005年 12月

李今蘭의 理學 碩士學位 論文을 認准함

審査委員長 ______________ 印

委　　員 ______________ 印

委　　員 ______________ 印

濟州大學校 大學院

2005年 12月

Integral formulas and vanishing theorems on a Riemannian Foliation

Keum Ran Lee
(Supervised by professor Seoung Dal Jung)

A thesis submitted in partial fulfillment of the requirement for the degree of Master of Science

2005. 12.

Department of Mathematics
GRADUATE SCHOOL
CHEJU NATIONAL UNIVERSITY

CONTENTS

<**Abstract**>

Integral formulas and vanishing theorems on a Riemannian Foliation

In this thesis, we study infinitesimal automorphisms of a compact Riemannian manifold with non-minimal foliations. In particular, we establish the integral formulas for infinitesimal automorphisms and prove vanishing theorems of transversal Killing field, transversal affine Killing field, transversal projective Killing field and transversal conformal Killing field under some transversal Ricci curvature conditions and mean curvature conditions.

1 Introduction

Let $(M, g_M, \mathcal{F})$ be a closed, oriented, connected Riemannian manifold of dimension $p+q$ with a transversally oriented foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Let L be the tangent bundle of $\mathcal{F}$ and $Q = TM/L$ the normal bundle of $\mathcal{F}$. A vector field Y on M is called an infinitesimal automorphism of $\mathcal{F}$ if the flow generated by Y preserves the foliation, that is, maps leaves into leaves. In other words, for any $Z \in \Gamma L, [Y, Z] \in \Gamma L$. There has been extensive studies of geometric infinitesimal automorphisms of a minimal Riemannian foliation by many differential geometers([6,7,8,10,11,15]). For the point foliation, such infinitesimal automorphisms of a Riemannian manifold. In this paper, we extend well-known results concerning infinitesimal automorphisms on a Riemannian manifold to a foliated version. Among geometric infinitesimal automorphisms, transversal Killing, affine, projective, conformal fields have been the objets of main interest. Many results about those infinitesimal automorphisms on a minimal foliation have obtained([6,7,10,11,15]). In this paper, we extend many results on a minimal foliation to the general case, that is, the case where the foliation is non-minimal. This paper is organized as the followings. In Chapter 2, we review the known facts on the foliated Riemannian manifold. In Chapter 3, we study the basic Laplacian. In Chapter 4, we have integral formulas about infinitesimal automorphism. In Chapter 5, we have vanishing theorems of infinitesimal automorphisms on a Riemannian foliation.

2 Riemannian foliation

Let M be a smooth manifold of dimension $p+q$.

Definition 2.1 A codimension q *foliation* $\mathcal{F}$ on M is given by an open cover $\mathcal{U} = (U_i)_{i\in I}$ and for each i, a diffeomorphism $\varphi_i : \mathbb{R}^{p+q} \to U_i$ such that, on $U_i \cap U_j \neq \emptyset$, the coordinate change $\varphi_j^{-1} \circ \varphi_i : \varphi_i^{-1}(U_i \cap U_j) \to \varphi_j^{-1}(U_i \cap U_j)$ has the form

$$\varphi_j^{-1} \circ \varphi_i(x,y) = (\varphi_{ij}(x,y), \gamma_{ij}(y)). \tag{2.1}$$

From Definition 2.1, the manifold M is decomposed into connected submanifolds of dimension p. Each of these submanifolds is called a *leaf* of $\mathcal{F}$. Coordinate patches (U_i, φ_i) are said to be *distinguished* for the foliation $\mathcal{F}$. The tangent bundle L of $\mathcal{F}$ is the subbundle of TM, consisting of all vectors tangent to the leaves of $\mathcal{F}$. The normal bundle Q of $\mathcal{F}$ on M is the quotient bundle $Q = TM/L$. Equivalently, Q appears in the exact sequence of vector bundles

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0. \tag{2.2}$$

If $(x_1, \ldots, x_p; y_1, \ldots, y_q)$ are local coordinates in a distinguished chart U, then the bundle $Q|U$ is framed by the vector fields $\pi\frac{\partial}{\partial y_1}, \ldots, \pi\frac{\partial}{\partial y_q}$. For a vector field $Y \in \Gamma TM$, we denote also $\overline{Y} = \pi Y \in \Gamma Q$.

Definition 2.2 A vector field Y on U is *projectable,* if $Y = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_\alpha b_\alpha \frac{\partial}{\partial y_\alpha}$ with $\frac{\partial b_\alpha}{\partial x_i} = 0$ for all $\alpha = 1, \ldots, q$ and $i = 1, \ldots, p$.

Definition 2.2 means that the functions $b_\alpha = b_\alpha(y)$ are independent of x. Then $\overline{Y} = \sum_\alpha b_\alpha \frac{\bar{\partial}}{\partial y_\alpha}$ with b_α independent of x. This property is preserved under the change of distinguished charts. Note that every projectable vector field preserves the leaves in sense of $[Y, Z] \in \Gamma L$ for any $Z \in \Gamma L$.

Let $V(\mathcal{F})$ be the space of all projectable vector fields on M, i.e.,

$$V(\mathcal{F}) = \{Y \in TM | [Y, Z] \in \Gamma L, \quad \forall Z \in \Gamma L\}. \tag{2.3}$$

An element of $V(\mathcal{F})$ is called an *infinitesimal automorphism* of $\mathcal{F}$. Now we put

$$\bar{V}(\mathcal{F}) = \{\bar{Y} = \pi(Y) \in \Gamma Q | Y \in V(\mathcal{F})\}. \tag{2.4}$$

The *transversal geometry* of a foliation is the geometry infinitesimally modeled by Q, while the tangential geometry is infinitesimally modeled by L. A key fact of the transversal geometry is the existence of the *Bott connection* in Q defined by

$$\overset{\circ}{\nabla}_X s = \pi([X, Y_s]), \quad \forall X \in \Gamma L, \tag{2.5}$$

where $Y_s \in TM$ is any vector field projecting to s under $\pi : TM \to Q$. It is a partial connection along L. The right hand side in (2.5) is independent of the choice of Y_s. Namely, the difference of two such choices is a vector field $X' \in \Gamma L$ and $[X, X'] \in \Gamma L$, which implies $\pi([X, X']) = 0$.

Definition 2.3 A Riemannian metric g_Q on the normal bundle Q of a foliation $\mathcal{F}$ is *holonomy invariant* if

$$\theta(X) g_Q = 0, \quad \forall X \in \Gamma L, \tag{2.6}$$

where $\theta(X)$ is the transversal Lie derivative, which is defined by $\theta(X)s = \pi[X, Y_s]$.

Here $\theta(X)g_Q$ is defined by

$$(\theta(X)g_Q)(s,t) = Xg_Q(s,t) - g_Q(\theta(X)s, t) - g_Q(s, \theta(X)t) \quad \forall s, t \in \Gamma Q.$$

Definition 2.4 A *Riemannian foliation* is a foliation $\mathcal{F}$ with a holonomy invariant transversal metric g_Q. A metric g_M is a *bundle-like* if the induced metric g_Q on Q is holonomy invariant.

The study of a Riemannian foliation was initiated by Reinhart in 1959([14]). A simple example of a Riemannian foliation is given by a nonsingular Killing vector field X on (M, g_M), because $\theta(X)g_M = 0$.

Definition 2.5 An *adapted connection* in Q is a connection restricting along L to the partial Bott connection $\overset{\circ}{\nabla}$.

To show that such connections exist, consider a Riemannian metric g_M on M. Then TM splits orthogonally as $TM = L \oplus L^\perp$. This means that there is a bundle map $\sigma : Q \to L^\perp$ splitting the exact sequence (2.2), i.e., satisfying $\pi \circ \sigma = identity$. This metric g_M on TM is then a direct sum

$$g_M = g_L \oplus g_{L^\perp}.$$

With $g_Q = \sigma^* g_{L^\perp}$, the splitting map $\sigma : (Q, g_Q) \to (L^\perp, g_{L^\perp})$ is a metric isomorphism. Let ∇^M be the Levi-Civita connection associated to the Riemannian metric g_M. Then the adapted connection ∇ in Q is given by([5,15])

$$\nabla_X s = \begin{cases} \overset{\circ}{\nabla}_X s = \pi([X, Y_s]) & \forall X \in \Gamma L, \\ \pi(\nabla^M_X Y_s) & \forall X \in \Gamma L^\perp, \end{cases} \tag{2.7}$$

where $s \in \Gamma Q$ and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $Q \cong L^\perp$. For any connection ∇ in Q, there is a torsion T_∇ defined by

$$T_\nabla(Y, Z) = \nabla_Y \pi(Z) - \nabla_Z \pi(Y) - \pi([Y, Z]) \tag{2.8}$$

for any $Y, Z \in \Gamma TM$. Then we have the following proposition ([15]).

Proposition 2.6 *For any metric g_M on M and the adapted connection ∇ in Q defined by (2.7) the torsion is free, i.e., $T_\nabla = 0$.*

Proof. For any vector fields $X \in \Gamma L$, $Y \in \Gamma TM$, we have

$$T_\nabla(X, Y) = \nabla_X \pi(Y) - \pi([X, Y]) = 0.$$

For any vector fields $Z, Z' \in \Gamma L^\perp$, we have

$$T_\nabla(Z, Z') = \pi(\nabla^M_Z Z') - \pi(\nabla^M_{Z'} Z) - \pi([Z, Z']) = \pi(T_{\nabla^M}(Z, Z')) = 0,$$

where T_{∇^M} is the (vanishing) torsion of ∇^M. Finally the bilinearity and skew symmetry of T_∇ imply the desired result. □

The curvature R^∇ of ∇ is defined by

$$R^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \forall X,\ Y \in TM. \tag{2.9}$$

From the adapted connection ∇ in Q defined by (2.7), its curvature R^∇ coincides with $\overset{\circ}{R}$ for $X, Y \in \Gamma L$, hence $R^\nabla(X, Y) = 0$ for $X, Y \in \Gamma L$. And we have the following proposition ([4,5,15]).

Proposition 2.7 *Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and bundle-like metric g_M with respect to $\mathcal{F}$. Let ∇ be the connection defined by (2.7) in Q with curvature R^∇. Then for $X \in \Gamma L$ the following holds:*

$$i(X) R^\nabla = \theta(X) R^\nabla = 0. \tag{2.10}$$

By Proposition 2.7, we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature σ^∇ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla(s, E_a) E_a, \quad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a), \tag{2.11}$$

where $\{E_a\}_{a=1,\cdots,q}$ is a local orthonormal basic frame of Q.

Definition 2.8 The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space N is Einsteinian, that is,

$$\rho^{\nabla} = \frac{1}{q}\sigma^{\nabla} \cdot id \tag{2.12}$$

with constant transversal scalar curvature σ^{∇}.

Definition 2.9 *The mean curvature vector* $\kappa^{\sharp}$ of $\mathcal{F}$ is defined by

$$\kappa^{\sharp} = \pi\big(\sum_{i=1}^{p} \nabla^{M}_{E_i} E_i\big), \tag{2.13}$$

where $\{E_i\}$ is a local orthonormal basis of L. The foliation $\mathcal{F}$ is said to be *minimal* if $\kappa^{\sharp} = 0$.

For the later use, we recall the divergence theorem on a foliated Riemannian manifold ([19]).

Theorem 2.10 *Let $(M, g_M, \mathcal{F})$ be a closed, oriented, connected Riemannian manifold with a transversally orientable foliation $\mathcal{F}$ and a bundle-like metric g_M with respect to $\mathcal{F}$. Then*

$$\int_M div_{\nabla}(X) = \int_M g_Q(X, \kappa^{\sharp}) \tag{2.14}$$

for all $X \in \Gamma Q$, where $div_{\nabla}(X)$ denotes the transversal divergence of X with respect to the connection ∇ defined by (2.7).

Proof. Let $\{E_i\}$ and $\{E_a\}$ be orthonormal basis of L and Q, respectively. Then for any $X \in \Gamma Q$,

$$\begin{aligned} div(X) &= \sum_i g_M(\nabla^M_{E_i} X, E_i) + \sum_a g_M(\nabla^M_{E_a} X, E_a) \\ &= \sum_i -g_M(X, \pi(\nabla^M_{E_i} E_i)) + \sum_a g_M(\pi(\nabla^M_{E_a} X), E_a) \\ &= -g_Q(X, \kappa^\sharp) + \sum_a g_Q(\nabla_{E_a} X, E_a) \\ &= -g_Q(X, \kappa^\sharp) + div_\nabla(X). \end{aligned}$$

By Green's Theorem on an ordinary manifold M, we have

$$0 = \int_M div(X) = \int_M div_\nabla(X) - \int_M g_Q(X, \kappa^\sharp). \quad \Box$$

Corollary 2.11 *If $\mathcal{F}$ is minimal, then we have that for any $X \in \Gamma Q$,*

$$\int_M div_\nabla(X) = 0. \tag{2.15}$$

3 The basic Laplacian

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M.

Definition 3.1 Let $\mathcal{F}$ be an arbitrary foliation on a manifold M. A differential form $\omega \in \Omega^r(M)$ is *basic* if

$$i(X)\omega = 0, \ \theta(X)\omega = 0, \quad \forall X \in \Gamma L. \tag{3.1}$$

In a distinguished chart $(x_1, \ldots, x_p; y_1, \ldots, y_q)$ of $\mathcal{F}$, a basic 1-form w is expressed by

$$\omega = \sum_{a_1 < \cdots < a_r} \omega_{a_1 \cdots a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r},$$

where the functions $\omega_{a_1 \cdots a_r}$ are independent of x, i.e. $\frac{\partial}{\partial x_i} \omega_{a_1 \cdots a_r} = 0$. Let $\Omega_B^r(\mathcal{F})$ be the set of all basic r-forms on M. The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega_B^1(\mathcal{F})$, where κ is a g_Q-dual 1-form $\kappa^\sharp$. Then we have the well known theorem([9,15]).

Theorem 3.2 *Let $\mathcal{F}$ be an isoparametric Riemannian foliation on M. Then the mean curvature form κ is closed, i.e., $d\kappa = 0$.*

We now define the star operator $\bar{*} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{q-r}(\mathcal{F})$ naturally associated to g_Q. The relationships between $\bar{*}$ and $*$ are characterized by

$$\bar{*}\phi = (-1)^{p(q-r)} * (\phi \wedge \chi_{\mathcal{F}}), \tag{3.2}$$

$$*\phi = \bar{*}\phi \wedge \chi_{\mathcal{F}} \tag{3.3}$$

for $\phi \in \Omega_B^r(\mathcal{F})$, where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ and $*$ is the Hodge star operator([15]). Then the inner product $< \ , \ >_B$ on $\Omega_B^r(\mathcal{F})$ is defined by

$< \phi, \psi >_B = \phi \wedge \bar{*}\psi \wedge \chi_{\mathcal{F}}$ for any $\phi, \psi \in \Omega_B^r$ and the global inner product is given by

$$\ll \phi, \psi \gg_B = \int_M < \phi, \psi >_B . \tag{3.4}$$

With respect to this scalar product, the adjoint $\delta_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r-1}(\mathcal{F})$ of d_B is given by

$$\delta_B \phi = (-1)^{q(r+1)+1} \bar{*}(d_B - \kappa \wedge)\bar{*}\phi. \tag{3.5}$$

Then the *basic Laplacian* is given by

$$\Delta_B = d_B \delta_B + \delta_B d_B. \tag{3.6}$$

Lemma 3.3 ([1,2]) *On the Riemannian foliation $\mathcal{F}$, we have*

$$d_B \phi = \sum_a E^a \wedge \nabla_{E_a} \phi, \quad \delta_B \phi = \sum_a -i(E_a)\nabla_{E_a}\phi + i(\kappa^\sharp)\phi, \tag{3.7}$$

when $\{E_a\}$ is a local orthonormal basic frame on Q and $\{E^a\}$ its g_Q-dual 1-form.

Definition 3.4 For any vector field $Y \in V(\mathcal{F})$, we define an operator $A_Y : \Gamma Q \to \Gamma Q$ as

$$A_Y s = \theta(Y)s - \nabla_Y s. \tag{3.8}$$

Remark. Let $Y_s \in \Gamma TM$ with $\pi(Y_s) = s$. Then it is trivial that

$$A_Y s = -\nabla_{Y_s} \pi(Y). \tag{3.9}$$

So A_Y depends only on $s = \pi(Y)$ and is a linear operator. Moreover, A_Y extends in an obvious way to tensors of any type on Q (see [6] for details). Namely, we can define the following.

Definition 3.5 For any basic 1-form $\phi \in \Omega^1_B(\mathcal{F})$, the operator A_Y is given by

$$(A_Y\phi)(X) = -\phi(A_Y X) \quad \forall X \in \Gamma Q. \tag{3.10}$$

Now, we introduce the operator $\nabla^*_{tr}\nabla_{tr} : \Omega^*_B(\mathcal{F}) \to \Omega^*_B(\mathcal{F})$ as

$$\nabla^*_{tr}\nabla_{tr}\phi = -\sum_a \nabla^2_{E_a,E_a}\phi + \nabla_{\kappa^\sharp}\phi, \tag{3.11}$$

where $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla^M_X Y}$ for any $X, Y \in TM$. Then we have the following.

Proposition 3.6 ([2]) *On the Riemannian foliation $\mathcal{F}$ on a compact manifold M, the operator $\nabla^*_{tr}\nabla_{tr}$ satisfies*

$$\ll \nabla^*_{tr}\nabla_{tr}\phi_1, \phi_2 \gg_B = \ll \nabla\phi_1, \nabla\phi_2 \gg_B \tag{3.12}$$

for all $\phi_1,\ \phi_2 \in \Omega^*_B(\mathcal{F})$, where $< \nabla\phi_1, \nabla\phi_2 >_B = \sum_a < \nabla_{E_a}\phi_1, \nabla_{E_a}\phi_2 >_B$.

By the straight calculation, we have the following theorem.

Theorem 3.7 *On the Riemannian foliation $\mathcal{F}$, we have*

$$\Delta_B\phi = \nabla^*_{tr}\nabla_{tr}\phi + A_{\kappa^\sharp}\phi + F(\phi) \tag{3.13}$$

for $\phi \in \Omega^r_B(\mathcal{F})$, where $F(\phi) = \sum_{a,b} E^a \wedge i(E_b)R^\nabla(E_b, E_a)\phi$. In particular, if ϕ is a basic 1-form, then $F(\phi)^\sharp = \rho^\nabla(\phi^\sharp)$.

Proof. Fix $x \in M$ and let $\{E_a\}$ be an orthonormal basis for Q with $(\nabla E_a)_x = 0$. Then from (3.7) we have

$$\begin{aligned}
d_B\delta_B\phi &= \sum_{a,b}(E^a \wedge \nabla_{E_a})(-i(E_b)\nabla_{E_b}\phi + i(\kappa^\sharp)\phi) \\
&= -\sum_{a,b} E^a \wedge \nabla_{E_a}\{i(E_b)\nabla_{E_b}\phi\} + \sum_a E^a \wedge \nabla_{E_a} i(\kappa^\sharp)\phi \\
&= -\sum_{a,b} E^a \wedge i(E_b)\nabla_{E_a}\nabla_{E_b}\phi + d_B i(\kappa^\sharp)\phi
\end{aligned}$$

and

$$\begin{aligned}
\delta_B d_B\phi &= \sum_{a,b} -i(E_b)\nabla_{E_b}\{E^a\wedge\nabla_{E_a}\phi\} + i(\kappa^\sharp)d_B\phi \\
&= \sum_{a,b} -(i(E_b)E^a)\nabla_{E_b}\nabla_{E_a}\phi + i(\kappa^\sharp)d_B\phi \\
&\quad + \sum_{a,b} E^a\wedge i(E_b)\nabla_{E_b}\nabla_{E_a}\phi \\
&= \sum_a -\nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b} E^a\wedge i(E_b)\nabla_{E_b}\nabla_{E_a}\phi + i(\kappa^\sharp)d_B\phi.
\end{aligned}$$

Summing up the above two equations, we have

$$\begin{aligned}
\Delta_B\phi &= d_B\delta_B\phi + \delta_B d_B\phi \\
&= d_B i(\kappa^\sharp)\phi + i(\kappa^\sharp)d_B\phi - \sum_a \nabla_{E_a}\nabla_{E_a}\phi \\
&\quad + \sum_{a,b} E^a\wedge i(E_b)R^\nabla(E_b,E_a)\phi \\
&= \theta(\kappa^\sharp)\phi - \sum_a \nabla_{E_a}\nabla_{E_a}\phi + \sum_{a,b} E^a\wedge i(E_b)R^\nabla(E_b,E_a)\phi \\
&= -\sum_a \nabla_{E_a}\nabla_{E_a}\phi + F(\phi) + A_{\kappa^\sharp}\phi + \nabla_{\kappa^\sharp}\phi \\
&= -\sum_a \nabla^2_{E_a,E_a}\phi + \nabla_{\kappa^\sharp}\phi + F(\phi) + A_{\kappa^\sharp}\phi \\
&= \nabla^*_{tr}\nabla_{tr}\phi + F(\phi) + A_{\kappa^\sharp}\phi.
\end{aligned}$$

The proof is completed. On the other hand, let ϕ be a basic 1-form and $\phi^\sharp$ its g_Q-dual vector field. Then

$$\begin{aligned}
g_Q(F(\phi),E^c) &= \sum_{a,b} g_Q(E^a\wedge i(E_b)R^\nabla(E_b,E_a)\phi, E^c) \\
&= \sum_b i(E_b)R^\nabla(E_b,E_c)\phi = \sum_b g_Q(R^\nabla(E_b,E_c)\phi^\sharp, E_b) \\
&= \sum_b g_Q(R^\nabla(\phi^\sharp,E_b)E_b,E_c) = g_Q(\rho^\nabla(\phi^\sharp),E_c).
\end{aligned}$$

This yields that for any basic 1-form ϕ, $F(\phi)^\sharp = \rho^\nabla(\phi^\sharp)$. $\square$

From (3.10) and Theorem 3.7, we have the following corollary.

Corollary 3.8 *On the Riemannian foliation, we have that for any* $X \in \Gamma Q$

$$\Delta_B X = \nabla_{tr}^* \nabla_{tr} X + \rho^\nabla(X) - A_{\kappa^\sharp}^t X. \tag{3.14}$$

Lemma 3.9 *Let* $\mathcal{F}$ *be a Riemannian foliation. For any vector fields* $Y, Z \in V(\mathcal{F})$ *and* $s \in \Gamma Q$, *we have*

$$(\theta(Y)\nabla)(Z, s) = R^\nabla(Y, Z)s - (\nabla_Z A_Y)s. \tag{3.15}$$

where $(\theta(Y)\nabla)(Z, s) = \theta(Y)\nabla_Z s - \nabla_{\theta(Y)Z} s - \nabla_Z \theta(Y)s$ *and*
$(\nabla_Z A_Y)s = -\nabla_Z \nabla_Y \pi(Y) + \nabla_{\nabla_Z s}\pi(Y)$.

Proof. By a direct calculation, we have that for any $Y, Z \in V(\mathcal{F})$

$$(\theta(Y)\nabla)(Z, s) - [\nabla_Y, \nabla_Z]s = (\theta(Y) - \nabla_Y)\nabla_Z s - \nabla_Z(\theta(Y) - \nabla_Y)s - \nabla_{[Y,Z]}s. \quad \square$$

4 Integral formulas

Let (M, g_M, F) be a closed, oriented, connected Riemannian manifold with a foliation F of codimension q and a bundle-like metric g_M. Let $\{E_a\}$ be a local orthonormal basic frame of Q such that $(\nabla E_a)_x = 0$ at $x \in M$.

Proposition 4.1 *For any basic function f on M, it holds that*

$$\int_M \Delta_B f = 0. \tag{4.1}$$

Proof. From (3.6) and Lemma 3.3, we have

$$\Delta_B f = \delta_B d_B f = -\sum_a i(E_a)\nabla_{E_a} d_B f + i(\kappa^\sharp) d_B f = -div_\nabla(d_B f) + i(\kappa^\sharp) d_B f.$$

By integrating the above equation and using the divergence theorem (2.14), we have

$$\begin{aligned}\int_M \Delta_B f &= -\int_M div(d_B f) + \int_M g_Q(\kappa^\sharp, d_B f)\\ &= -\int_M g_Q(\kappa^\sharp, d_B f) + \int_M g_Q(\kappa^\sharp, d_B f)\\ &= 0. \quad \square\end{aligned}$$

Note that, the direct calculation gives

$$\frac{1}{2}\Delta_B f^2 = (\Delta_B f)f - |\nabla_{tr} f|^2, \tag{4.2}$$

which yields

$$\int_M \{(\Delta_B f)f - |\nabla_{tr} f|^2\} = 0. \tag{4.3}$$

Hence we have the following proposition.

Proposition 4.2 *On the Riemannian foliation $\mathcal{F}$ on M, if a basic function f satisfies $\Delta_B f \geq 0$ (or$\Delta_B f \leq 0$), then f is constant on M.*

Proof. By Proposition 4.1, if $\Delta_B f \geq 0$, then $\Delta_B f = 0$. So f is constant from (4.3). □

Proposition 4.3 *For any basic function f and a constant λ on M, if $\Delta_B f = \lambda f$, then λ is positive.*

Proof. From (4.3), if $\Delta_B f = \lambda f$, then

$$\int_M [(\lambda f)f - |\nabla_{tr} f|^2] = 0$$

which implies $\lambda > 0$. □

Lemma 4.4 *For any vector $X \in \bar{V}(\mathcal{F})$, it holds that*

$$\begin{aligned} TrA_X A_X &= -\frac{1}{2}|d_B\xi|^2 + |\nabla X|^2 \\ &= \frac{1}{2}|\theta(X)g_Q|^2 - |\nabla X|^2, \end{aligned}$$

where ξ is g_Q-dual 1-form of X.

Proof. For any basic 1-form ϕ, it is well-known that

$$(d_B\phi)(Y, Z) = Y\phi(Z) - Z\phi(Y) - \phi([Y, Z]), \quad \forall X, Y \in \Gamma Q.$$

Since $[E_a, E_b] = 0$ at $x \in M$, we have that at $x \in M$

$$\begin{aligned} |d_B\xi|^2 =& \sum_{a,b}\{(d_B\xi)(E_a, E_b)\}^2 \\ =& \sum_{a,b}\{E_a\xi(E_b) - E_b\xi(E_a)\}^2 = \sum_{a,b}\{g_Q(\nabla_{E_a}X, E_b) - g_Q(\nabla_{E_b}X, E_a)\}^2 \\ =& 2|\nabla X|^2 - 2\sum_{a,b} g_Q(\nabla_{E_a}X, E_b)g_Q(\nabla_{E_b}X, E_a). \end{aligned} \tag{4.4}$$

On the other hand, from (3.9) it is trivial that

$$TrA_XA_X = \sum_{a,b} g_Q(\nabla_{E_a}X, E_b)g_Q(\nabla_{E_b}X, E_a). \tag{4.5}$$

Hence the first equation in Lemma 4.4 is proved from (4.4) and (4.5). Next, it is well-known that

$$\begin{aligned} TrA_XA_X =& - TrA_X^tA_X + \frac{1}{2}Tr(A_X + A_X^t)^2 \\ =& - |\nabla_{tr}X|^2 + \frac{1}{2}Tr(A_X + A_X^t)^2. \end{aligned} \tag{4.6}$$

Moreover, from (3.9) we have

$$\begin{aligned} |\theta(X)g_Q|^2 =& \sum_{a,b}\{g_Q(\nabla_{E_a}X, E_b) + g_Q(\nabla_{E_b}X, E_a)\}^2 \\ =& \sum_{a,b} g_Q((A_X + A_X^t)E_a, E_b)^2 = Tr(A_X + A_X^t)^2. \end{aligned} \tag{4.7}$$

From (4.6) and (4.7), the second equation is proved. □

Proposition 4.5 *On the Riemannian foliation $\mathcal{F}$ on M, any vector field $X \in \bar{V}(\mathcal{F})$ satisfies*

$$\begin{aligned} &-div_\nabla(A_XX) - div_\nabla(div_\nabla(X)X) \\ &\quad = g_Q(\rho^\nabla(X), X) + \frac{1}{2}|\theta(X)g_Q|^2 - |\nabla_{tr}X|^2 - (\delta_TX)^2 \\ &\quad = g_Q(\rho^\nabla(X), X) - \frac{1}{2}|d_B\xi|^2 + |\nabla_{tr}X|^2 - (\delta_TX)^2. \end{aligned}$$

Proof. By a direct calculation with (3.9), it holds that for any $X \in \bar{V}(\mathcal{F})$

$$\begin{aligned} div_\nabla(A_XX) =& - g_Q(\nabla_{E_a}\nabla_XX, E_a), \\ div_\nabla(div_\nabla(X)X) =& Xdiv_\nabla(X) + (div_\nabla(X))^2. \end{aligned}$$

Since $Xdiv_\nabla(X) = Xg_Q(\nabla_{E_a}X, E_a) = g_Q(\nabla_X\nabla_{E_a}, E_a)$, we have

$$\begin{aligned}
&div_\nabla(div_\nabla(X)X) + div_\nabla(A_XX)\\
&\quad = g_Q(\nabla_X\nabla_{E_a}X - \nabla_{E_a}\nabla_XX, E_a) + (div_\nabla(X))^2\\
&\quad = g_Q(R^\nabla(X, E_a)X + \nabla_{[X,E_a]}X, E_a) + (div_\nabla(X))^2\\
&\quad = -g_Q(\rho^\nabla(X), X) + g_Q(\nabla_{[X,E_a]}X, E_a) + (div_\nabla(X))^2\\
&\quad = -g_Q(\rho^\nabla(X), X) - g_Q(A_X[X, E_a], E_a) + (div_\nabla(X))^2\\
&\quad = -g_Q(\rho^\nabla(X), X) - g_Q(A_XA_XE_a, E_a) + (div_\nabla(X))^2.
\end{aligned}$$

From Lemma 4.4, the proof is completed. □

Corollary 4.6 *On the Riemannian foliation $\mathcal{F}$ on M, any vector field $X \in \bar{V}(\mathcal{F})$ satisfies*

$$\begin{aligned}
&\int_M \{g_Q(\rho^\nabla(X), X) + \frac{1}{2}|\theta(X)g_Q|^2 - |\nabla_{tr}X|^2 - (\delta_TX)^2\}\\
&\qquad + \int_M \{div_\nabla(A_XX) + div_\nabla(div_\nabla(X)X)\} = 0
\end{aligned} \tag{4.8}$$

or

$$\begin{aligned}
&\int_M \{g_Q(\rho^\nabla(X), X) - \frac{1}{2}|d_B\xi|^2 + |\nabla_{tr}X|^2 - (\delta_TX)^2\}\\
&\qquad + \int_M \{div_\nabla(A_XX) + div_\nabla(div_\nabla(X)X)\} = 0.
\end{aligned} \tag{4.9}$$

Lemma 4.7 *On the Riemannian foliation $\mathcal{F}$ on M, any vector field $X \in \bar{V}(\mathcal{F})$ satisfies*

$$\int_M g_Q(\Delta_BX, X) = \int_M |\nabla_{tr}X|^2 + \int_M g_Q((\rho^\nabla - A^t_{\kappa^\sharp})(X), X).$$

Proof. It is trivial from (3.14). □

From (4.8) and (4.9), we have the following corollary.

Corollary 4.8 *On the Riemannian foliation $\mathcal{F}$ on M, if $\Box_B X = \Delta_B X - 2\rho^{\nabla}(X)$ for any $X \in \bar{V}(\mathcal{F})$, then*

$$\int_M \{g_Q(\Box_B X, X) - \frac{1}{2}|\theta(X)g_Q|^2 + (\delta_T X)^2\} \tag{4.10}$$
$$+ \int_M \{g_Q(A_{\kappa^\sharp}X, X) - div_\nabla(A_X X) - div_\nabla(div_\nabla(X)X)\} = 0,$$

or

$$\int_M \{g_Q(\Delta_B X, X) - \frac{1}{2}|d_B\xi|^2 - (\delta_T X)^2\} \tag{4.11}$$
$$+ \int_M \{g_Q(A_{\kappa^\sharp}X, X) + div_\nabla(A_X X) + div_\nabla(div_\nabla(X)X)\} = 0.$$

Lemma 4.9 *On the Riemannian foliation $\mathcal{F}$ on M, any vector field $X \in \bar{V}(\mathcal{F})$ satisfies*

$$|\theta(X)g_Q + \frac{2}{q}(\delta_T X)|^2 = |\theta(X)g_Q|^2 - \frac{4}{q}(\delta_T X)^2 \quad \forall X \in \bar{V}(\mathcal{F}).$$

Proof. A direct calculation gives

$$\begin{aligned}
|\theta(X)g_Q + \frac{2}{q}(\delta_T X)|^2 =& |\theta(X)g_Q|^2 + \frac{4}{q}(\delta_T X)^2 + \frac{4}{q}(\delta_T X)\sum_a(\theta(X)g_Q)(E_a, E_a) \\
=& |\theta(X)g_Q|^2 + \frac{4}{q}(\delta_T X)^2 - \frac{8}{q}(\delta_T X)^2 \\
=& |\theta(X)g_Q|^2 - \frac{4}{q}(\delta_T X)^2. \quad \Box
\end{aligned}$$

From Corollary 4.8 and Lemma 4.9, we have the following.

Corollary 4.10 *On the Riemannian foliation $\mathcal{F}$ on M, any vector field $X \in \bar{V}(\mathcal{F})$ satisfies*

$$\int_M \{g_Q(\Box_B X, X) - \frac{1}{2}|\theta(X)g_Q + \frac{2}{q}(\delta_T X)|^2 + \frac{q-2}{q}(\delta_T X)^2\}$$
$$+ \int_M \{g_Q(A_{\kappa^\sharp}X, X) - div_\nabla(A_X X) - div_\nabla(div_\nabla(X)X)\} = 0. \tag{4.12}$$

Lemma 4.11 *On the Riemannian foliation $\mathcal{F}$ on M, any vector field $X \in \bar{V}(\mathcal{F})$ satisfies*

$$\int_M \{g_Q(A_{\kappa^\sharp}X, X) + div_\nabla(A_X X)\} = -\int_M X g_Q(\kappa^\sharp, X), \tag{4.13}$$

$$\int_M div_\nabla(div_\nabla(X)X) = -\int_M (\delta_T X) g_Q(X, \kappa^\sharp). \tag{4.14}$$

Proof. The second equation is followed from the divergence theorem. From (3.4) and divergence theorem, the first equation is proved. □

Now we denote $VK^\perp(\mathcal{F})$ by

$$VK^\perp(\mathcal{F}) = \{X \in \bar{V}(\mathcal{F}) | g_Q(X, \kappa^\sharp) = 0\}. \tag{4.15}$$

Then we have the following theorem.

Theorem 4.12 *Let $(M, g_M, \mathcal{F})$ be a closed Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M. For any vector field $X \in VK^\perp(\mathcal{F})$ we have*

$$\int_M \{g_Q(\Box_B X, X) - \frac{1}{2}|\theta(X)g_Q + \frac{2}{q}(\delta_T X)|^2 + 2g_Q(A_{\kappa^\sharp}X, X) + \frac{q-2}{q}(\delta_T X)^2\} = 0. \tag{4.16}$$

Proof. From Corollary 4.10 and Lemma 4.11, it is trivial. □

5 Vanishing theorems for infinitesimal automorphisms

Let $(M, g_M, \mathcal{F})$ be a closed, oriented, connected Riemannian manifold of dimension $p+q$ with a transversally oriented foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$.

5.1 Transversal Killing fields

Definition 5.1 A vector field $X \in \bar{V}(\mathcal{F})$ is a *transversal Killing field* if it satisfies

$$\theta(X)g_Q = 0, \tag{5.1}$$

equivalently,

$$g_Q(\nabla_Y X, Z) + g_Q(\nabla_Z X, Y) = 0, \quad \forall Y, Z \in \Gamma Q. \tag{5.2}$$

From (5.2), we have the following proposition.

Proposition 5.2 *Let $X \in VK^{\perp}(\mathcal{F})$ be a transversal Killing field on M. Then we have*

$$\int_M div_\nabla(A_X X) = -\int_M g_Q(A_X\kappa^\sharp, X) = -\int_M g_Q(A_{\kappa^\sharp}X, X). \tag{5.3}$$

Proof. Let X be a transversal Killing field with $g_Q(X, \kappa^\sharp) = 0$. From (5.2), we have

$$g_Q(A_X Y, Z) + g_Q(Y, A_X Z) = 0, \quad \forall Y, Z \in \Gamma Q. \tag{5.4}$$

Hence the divergence theorem with (5.4) implies that

$$\int_M div_\nabla(A_X X) = \int_M g_Q(A_X X, \kappa^\sharp) = -\int_M g_Q(X, A_X\kappa^\sharp). \tag{5.5}$$

On the other hand, the second equality follows from Lemma 4.11. □

Theorem 5.3 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M. Any infinitesimal automorphism $X \in \bar{V}(\mathcal{F})$ is a transversal Killing field if and only if*

$$(1)\ \Box_B X + A^t_{\kappa^\sharp} X + A_X \kappa^\sharp = 0,$$

$$(2)\ \delta_T X = -div_\nabla X = 0,$$

$$(3)\ \int_M g_Q((A_X + A^t_X)X, \kappa^\sharp) = 0.$$

Proof. Let $X \in \bar{V}(\mathcal{F})$ be a transversal Killing field. Then it holds that

$$g_Q(\nabla_{E_a} X, E_b) + g_Q(\nabla_{E_b} X, E_a) = 0. \tag{5.6}$$

Hence (2) is trivial from (5.6). For the proof of (1), we have from (5.6)

$$\begin{aligned} g_Q(\rho^\nabla(X), E_b) =& g_Q(\sum_a R(X, E_a)E_a, E_b) = g_Q(R(E_a, E_b)X, E_a) \\ =& g_Q(\nabla_{E_a}\nabla_{E_b} X, E_a) - g_Q(\nabla_{E_b}\nabla_{E_a} X, E_a) \\ =& E_a g_Q(\nabla_{E_b} X, E_a) - E_b g_Q(\nabla_{E_a} X, E_a) \\ =& -g_Q(\nabla_{E_a}\nabla_{E_a} X, E_b). \end{aligned}$$

Hence

$$\rho^\nabla(X) = -\nabla_{E_a}\nabla_{E_a} X.$$

From (3.10) and (3.13), (1) is proved. Next (4.10) together with the properties (1)and (2) gives (3). Conversely, if $X \in \bar{V}(\mathcal{F})$ satisfies (1), (2) and (3), then $\theta(X) g_Q = 0$ in (4.10). So X is a transversal Killing field. □

Corollary 5.4 *Let $\mathcal{F}$ be a minimal foliation on M. Then any infinitesimal automorphism X is a transversal Killing field if and only if*

$$\Box_B X = 0 \quad \text{and} \quad \delta_T X = 0.$$

Proposition 5.5 *Let $X \in VK^{\perp}(\mathcal{F})$ be a transversal Killing field on M. If $X = d_B f$ for some basic function f, then $X = 0$.*

Proof. Since X is a transversal Killing field with $g_Q(X, \kappa^{\sharp}) = 0$, we have $\delta_B X = \delta_T X$. Hence

$$\Delta_B f = \delta_B d_B f = \delta_B X = \delta_T X = 0.$$

From Proposition 4.2, f is constant, which implies $X = 0$. □

Theorem 5.6 *Let $X \in VK^{\perp}(\mathcal{F})$ be a transversal Killing field on M, Then*

$$\begin{cases} \text{if } \rho^{\nabla}(X) - A_X\kappa^{\sharp} \le 0, \text{ then } \nabla_{tr}X = 0, \ i.e\ \nabla X = 0 \\ \text{if} \rho^{\nabla}(X) - A_X\kappa^{\sharp} \le 0 \text{ and } < 0 \text{ at some point, then } X = 0. \end{cases}$$

Proof. Let X be the transversal Killing field with $g_Q(X, \kappa^{\sharp}) = 0$. Since $\delta_T X = 0$, Corollary 4.6 yields

$$\int_M [g_Q(\rho^{\nabla}(X) - A_X\kappa^{\sharp}, X) - |\nabla_{tr}X|^2] = 0.$$

Therefore the proof is completed. □

Corollary 5.7 ([6]) *Let $\mathcal{F}$ be a minimal foliation and X a transversal Killing field on M. If the transversal Ricci curvature is non-positive, then X is parallel. If the transversal Ricci curvature is quasi-negative, then X is trivial.*

5.2 Transversal affine Killing fields

Definition 5.8 A vector field $X \in \bar{V}(\mathcal{F})$ is a *transversal affine Killing field* if it satisfies

$$\theta(X)\nabla = 0, \quad That\ is, \quad R^{\nabla}(X, E_a)E_b + \nabla_{E_a}\nabla_{E_b}X = 0, \tag{5.7}$$

where

$$(\theta(X)\nabla)(Y,Z) = \theta(X)\nabla_Y Z - \nabla_{\theta(X)Y} Z - \nabla_Y \theta(X) Z, \quad \forall Y, Z \in \Gamma Q.$$

Theorem 5.9 *On the Riemannian foliation $\mathcal{F}$, if $X \in \bar{V}(\mathcal{F})$ is a transversal affine Killing field, then*

$$\Box_B X + A^t_{\kappa^\sharp} X + A_X \kappa^\sharp = 0 \quad \text{and} \quad d_B \delta_T X = 0. \tag{5.8}$$

Proof. From (5.7), $\rho^\nabla(X) + \nabla_{E_a}\nabla_{E_a} = 0$, which means the first equation. Next, (5.7) implies that

$$0 = g_Q(\nabla_{E_a}\nabla_{E_b} X, E_b) = E_a g_Q(\nabla_{E_b} X, E_b) = E_a div_\nabla(X).$$

Therefore the second equation is proved. □

Theorem 5.10 *If any transversal affine Killing field $X \in VK^\perp(\mathcal{F})$ satisfies*

$$\int_M g_Q((A_X + A^t_X)X, \kappa^\sharp) = 0, \tag{5.9}$$

then X is a transversal Killing field.

Proof. Since $g_Q(X, \kappa^\sharp) = 0$, $\delta_B X = \delta_T X$. Hence

$$0 = \int_M g_Q(d_B \delta_T X, X) = \int_M |\delta_T X|^2,$$

which yields $\delta_T X = 0$. By Theorems 5.3 and 5.9, the proof is completed. □

Theorem 5.11 *On the Riemannian foliation $\mathcal{F}$ on M, any transversal affine Killing field $X \in VK^\perp(\mathcal{F})$ satisfies*

$$\int_M \{2g_Q(\rho^\nabla(X), X) - g_Q(A_X\kappa^\sharp + A_{\kappa^\sharp}X, X) - |\delta_T X|^2 - \frac{1}{2}|d_B\xi|^2\} = 0. \tag{5.10}$$

Proof. Since $g_Q(X, \kappa^\sharp) = 0$, we have

$$g_Q(A_X X, \kappa^\sharp) = -g_Q(A_{\kappa^\sharp} X, X).$$

Hence (5.10) is proved from (4.11) and (5.8). □

If $\kappa^\sharp$ of $\mathcal{F}$ is a transversal Killing field, then

$$g_Q(A_X \kappa^\sharp + A_{\kappa^\sharp} X, X) = g_Q(A_X \kappa^\sharp, X), \tag{5.11}$$

because $g_Q(A_{\kappa^\sharp} X, X) = 0$. Hence we have the following theorem.

Theorem 5.12 *Let $(M, g_M, \mathcal{F})$ be a closed, oriented, connected Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M such that $\kappa^\sharp$ is a transversal Killing field. If any transversal affine Killing field $X \in VK^\perp(\mathcal{F})$ satisfies $\rho^\nabla(X) - A_X \kappa^\sharp \leq 0$ on M, then $d_B \xi = \delta_T X = 0$, which implies $\nabla X = 0$. If any transversal affine Killing field $X \in VK^\perp(\mathcal{F})$ satisfies $\rho^\nabla(X) - A_X \kappa^\sharp \leq 0$ and < 0 at some point, then $X = 0$.*

Proof. It is trivial from (3.14) and Theorem 5.11. □

Corollary 5.13 ([6]) *Under the same assumption as in Theorem 5.12 except for $\mathcal{F}$ is minimal,if $\rho^\nabla \leq 0$, then every transversal affine Killing field is a transversal Killing field.*

Remark. If $\rho^\nabla \leq 0$, then every transversal affine Killing field is parallel. If $\rho^\nabla \leq 0$ and < 0 at some point, then every transversal affine Killing field is trivial.

5.3 Transversal projective Killing fields

Definition 5.14 A vector field $X \in \bar{V}(\mathcal{F})$ is a *transversal projective Killing field* if it satisfies

$$(\theta(X)\nabla)(Y,Z) = \alpha(Y)Z + \alpha(Z)Y, \quad Y,Z \in \Gamma Q, \tag{5.12}$$

where α is a basic 1-form on M.

Proposition 5.15 *Let $X \in \bar{V}(\mathcal{F})$ be a transversal projective Killing field on M. Then it holds*

$$\Box_B X + A^t_{\kappa^\sharp} X + A_X \kappa^\sharp = -2\alpha^\sharp. \tag{5.13}$$

Proof. From (5.7), it is well-known that

$$\sum_a (\theta(X)\nabla)(E_a, E_a) = \rho^\nabla(X) - \nabla^*_{tr}\nabla_{tr} X - A_X\kappa^\sharp. \tag{5.14}$$

From (5.12), we have $\sum_a (\theta(X)\nabla)(E_a, E_a) = 2\alpha(E_a)E_a = 2\sum_a \alpha^\sharp$. So we have

$$\rho^\nabla(X) - \nabla^*_{tr}\nabla_{tr} X - A_X\kappa^\sharp = 2\alpha^\sharp, \tag{5.15}$$

which prove (5.13) by virtue of (3.14). □

Lemma 5.16 *Let $X \in \bar{V}(\mathcal{F})$ be a transversal projective Killing field on M. Then it holds*

$$d_B(\delta_T X) = -(q+1)\alpha. \tag{5.16}$$

Proof. From Lemma 3.3, we have

$$\begin{aligned} d_B(\delta_T X) &= \sum_{a,b} E_a \wedge \nabla_{E_a}[-i(E_b)\nabla_{E_b} X] = -\sum_{a,b} E_a \wedge \nabla_{E_a} g_Q(\nabla_{E_b} X, E_b) \\ &= -\sum_{a,b} g_Q(\nabla_{E_a}\nabla_{E_b} X, E_b)E_a. \end{aligned} \tag{5.17}$$

From (5.7) and (5.12), we have

$$R^{\nabla}(X, E_a)E_b + \nabla_{E_a}\nabla_{E_b}X = \alpha(E_a)E_b + \alpha(E_b)E_a. \tag{5.18}$$

From (5.17) and (5.18), it follows (5.16). $\square$

From (5.13) and (5.16), we have the following proposition.

Proposition 5.17 *Let $X \in \bar{V}(\mathcal{F})$ be a transversal projective Killing field on M. Then it holds*

$$\square_B X + A^t_{\kappa^\sharp}X + A_X\kappa^\sharp = \frac{2}{q+1}d_B\delta_T X. \tag{5.19}$$

From (4.11) and (5.19), we have

$$\begin{aligned} 0 = \int_M \{&2g_Q(\rho^{\nabla}(X), X) - \frac{1}{2}|d_B\xi|^2 - (\delta_T X)^2 + \frac{2}{q+1}g_Q(d_B\delta_T X, X) \\ &+ g_Q(A_X X, \kappa^\sharp) - g_Q(A_X\kappa^\sharp, X) + div_{\nabla}(div_{\nabla}(X)X)\}. \end{aligned} \tag{5.20}$$

Hence we have the following theorem.

Theorem 5.18 *On the Riemannian foliation $\mathcal{F}$ on M, any transversal projective Killing field $X \in VK^{\perp}(\mathcal{F})$ satisfies*

$$\int_M \{2g_Q(\rho^{\nabla}(X), X) - g_Q(A_X\kappa^\sharp + A_{\kappa^\sharp}X, X) - \frac{q-1}{q+1}|\delta_T X|^2 - \frac{1}{2}|d_B\xi|^2\} = 0. \tag{5.21}$$

Proof. Since $g_Q(X, \kappa^\sharp) = 0$, we have

$$\delta_T X = \delta_B X, \quad g_Q(A_X X, \kappa^\sharp) = -g_Q(A_{\kappa^\sharp}X, X).$$

Hence (5.21) is proved from (5.20). $\square$

If $\kappa^\sharp$ of $\mathcal{F}$ is a transversal Killing field, then we have the following theorem.

Theorem 5.19 *Let $(M, g_M, \mathcal{F})$ be a closed, oriented, connected Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M such that $\kappa^\sharp$ is a transversal Killing field. If any transversal projective Killing field $X \in VK^\perp(\mathcal{F})$ satisfies $\rho^\nabla(X) - A_X\kappa^\sharp \le 0$ on M, then $d_B\xi = \delta_T X = 0$, which implies $\nabla X = 0$. If any transversal projective Killing field $X \in VK^\perp(\mathcal{F})$ satisfies $\rho^\nabla(X) - A_X\kappa^\sharp \le 0$ and < 0 at some point, then $X = 0$.*

Proof. It is trivial from (3.14), (5.11) and Theorem 5.18. □

Corollary 5.20 ([6]) *Let $(M, g_M, \mathcal{F})$ be a closed, oriented, connected Riemannian manifold with a minimal foliation $\mathcal{F}$ and a bundle like metric g_M. If $\rho^\nabla \le 0$, then every transversal projective Killing field is a transversal Killing field.*

Remark. If $\rho^\nabla \le 0$, then every transversal projective Killing field is parallel. If $\rho^\nabla \le 0$ and < 0 at some point, then every transversal projective Killing field is trivial.

5.4 Transversal conformal Killing fields

Definition 5.21 A vector field $X \in \bar{V}(\mathcal{F})$ is a *transversal conformal Killing field* if it satisfies

$$\theta(X)g_Q = 2fg_Q, \tag{5.22}$$

where $f > 0$ is a basic function. In fact, $f = -\frac{1}{q}\delta_T X$.

Proposition 5.22 *On the Riemannian foliation $\mathcal{F}$, any transversal conformal Killing field $X \in \bar{V}(\mathcal{F})$ with $\theta(X)g_Q = 2fg_Q$ satisfies*

$$(\theta(X)\nabla)(Y, Z) = (d_Bf)(Y)Z + (d_Bf)(Z)Y - g_Q(Y, Z)d_Bf, \quad \forall\ Y, Z \in \Gamma Q. \tag{5.23}$$

Proof. Lemma 3.9 implies that for any $Z, W \in \Gamma Q$,

$$\begin{aligned}
&\nabla_Y(\theta(X)g_Q)(Z,W) + \nabla_Z(\theta(X)g_Q)(Y,W) - \nabla_W(\theta(X)g_Q)(Y,Z)\\
=&g_Q(\nabla_Y\nabla_Z X, W) + g_Q(\nabla_Y\nabla_W X, Z) + g_Q(\nabla_Z\nabla_Y X, W) + g_Q(\nabla_Z\nabla_W X, Y)\\
&- g_Q(\nabla_W\nabla_Y X, Z) - g_Q(\nabla_W\nabla_Z X, Y)\\
=&g_Q(R^\nabla(Y,W)X, Z) + g_Q(R^\nabla(Z,W)X, Y) + g_Q(R^\nabla(Y,Z)X, W) + 2g_Q(\nabla_Z\nabla_Y X, W)\\
=&-2g_Q(R^\nabla(Z,X)Y, W) + 2g_Q(\nabla_Z\nabla_Y X, W)\\
=&2g_Q((\theta(X)\nabla)(Y,Z), W).
\end{aligned}$$

On the other hand, since $X \in \bar{V}(\mathcal{F})$ is a transversal conformal Killing field, (5.22) implies

$$\begin{aligned}
&\nabla_Y(\theta(X)g_Q)(Z,W) + \nabla_Z(\theta(X)g_Q)(Y,W) - \nabla_W(\theta(X)g_Q)(Y,Z)\\
=&Y(f)g_Q(Z,W) + Z(f)g_Q(Y,W) - W(f)g_Q(Y,Z)\\
=&g_Q(Y(f)Z, W) + g_Q(Z(f)Y, W) - g_Q(d_B f, W)g_Q(Y,Z)
\end{aligned}$$

From the above two equations, the proof is completed. □

From (4.8), we have the following proposition.

Proposition 5.23 *On a Riemannian foliation $\mathcal{F}$, any transversal conformal Killing field $X \in \bar{V}(\mathcal{F})$ with $\theta(X)g_Q = 2fg_Q$ satisfies*

$$\int_M \{g_Q(\rho^\nabla(X), X) - \frac{q-2}{q}|\delta_T X|^2 - |\nabla_{tr}X|^2 + div_\nabla(A_X X) + div_\nabla(div_\nabla(X)X)\} = 0. \tag{5.24}$$

Theorem 5.24 *Let $(M, g_M, \mathcal{F})$ be a closed, oriented, connected Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M. For any transversal conformal Killing field $X \in VK^\perp(\mathcal{F})$, if holds*

$$\Box_B X + A^t_{\kappa^\sharp}X + A_X\kappa^\sharp = -\frac{q-2}{q}d_B\delta_T X. \tag{5.25}$$

conversely, if $\kappa^\sharp$ is a the Killing field, any vector field $X \in VK^\perp(\mathcal{F})$ satisfies (5.25) is a transversal conformal Killing field.

Proof. Let $X \in VK^\perp(\mathcal{F})$ be a transversal conformal Killing field. Then by (5.23) and Lemma 3.9, we have

$$R^\nabla(X,Y)Z - \nabla_{\nabla_Y Z}X + \nabla_Y\nabla_Z X = (d_Bf)(Y)Z + (d_Bf)(Z)Y - g_Q(Y,Z)d_Bf.$$

From this equation, we have

$$\rho^\nabla(X) - \nabla^*_{tr}\nabla_{tr}X - A_X^{\kappa^\sharp} = 2\sum_a (d_Bf)(E_a)E_a - qd_Bf = -(q-2)d_Bf.$$

$$\Box_B X + A^t_{\kappa^\sharp}X + A_X\kappa^\sharp = -\frac{q-2}{q}d_B\delta_T X.$$

Conversely, since $\kappa^\sharp$ is a transversal Killing field, $g_Q(A_{\kappa^\sharp}X,X) = 0$. By Theorem 4.12, if X satisfies (5.25), then X is a transversal conformal Killing field. □

From (4.11) and Theorem 5.24, we have

$$\begin{aligned} 0 = \int_M \{&2g_Q(\rho^\nabla(X),X) - \frac{1}{2}|d_B\xi|^2 - (\delta_T X)^2 - \frac{q-2}{q}g_Q(d_B\delta_T X, X) \\ &+ g_Q(A_X X, \kappa^\sharp) - g_Q(A_X\kappa^\sharp, X) + div_\nabla(div_\nabla(X)X)\}. \end{aligned} \tag{5.26}$$

Hence we have the following theorem.

Theorem 5.25 *On the Riemannian foliation $\mathcal{F}$ on M, any transversal conformal Killing field $X \in VK^\perp(\mathcal{F})$ satisfies*

$$\int_M \{2g_Q(\rho^\nabla(X),X) - g_Q(A_X\kappa^\sharp + A_{\kappa^\sharp}X, X) - \frac{2q-2}{q}|\delta_T X|^2 - \frac{1}{2}|d_B\xi|^2\} = 0. \tag{5.27}$$

Proof. Since $g_Q(X,\kappa^\sharp)=0$, we have

$$\delta_T X=\delta_B X,\quad g_Q(A_X X,\kappa^\sharp)=-g_Q(A_{\kappa^\sharp}X,X).$$

Hence (5.27) is proved from (5.26). $\square$

If $\kappa^\sharp$ of $\mathcal{F}$ is a transversal Killing field, then we have the following theorem by (5.11)

Theorem 5.26 *Let $(M,g_M,\mathcal{F})$ be a closed, oriented, connected Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g_M such that $\kappa^\sharp$ is a transversal Killing field. If any transversal conformal Killing field $X\in VK^\perp(\mathcal{F})$ satisfies $\rho^\nabla(X)-A_X\kappa^\sharp\le 0$ on M, then $d_B\xi=\delta_T X=0$, which implies $\nabla X=0$. If any transversal conformal Killing field $X\in VK^\perp(\mathcal{F})$ satisfies $\rho^\nabla(X)-A_X\kappa^\sharp\le 0$ and <0 at some point, then $X=0$.*

Proof. It is trivial from (3.14) and Theorem 5.25. $\square$

Corollary 5.27 ([6]) *Let $(M,g_M,\mathcal{F})$ be a closed, oriented, connected Riemannian manifold with a minimal foliation $\mathcal{F}$ and a bundle like metric g_M. If $\rho^\nabla\le 0$, then every transversal conformal Killing field is a transversal Killing field.*

Remark. If $\rho^\nabla\le 0$, then every transversal conformal Killing field is parallel. If $\rho^\nabla\le 0$ and <0 at some point, then every transversal conformal Killing field is trivial.

References

[1] J. A. Alvarez López, *The basic component of the mean curvature of Riemannian foliations*, Ann. Global Anal. Geom. 10(1992), 179-194.

[2] M. J. Jung, *Riemannain foliation admitting an infinitesimal conformal transformation*, In preprient.

[3] S. D. Jung, *The first eigenvalue of the transversal Dirac operator*, J. Geom. Phys. 39(2001), 253-264.

[4] S. D. Jung, *Transversal infinitesimal automorphisms for non-harmonic Kähler foliation*, Far East J. Math. Sci. Special Volume, Part II(2000), 169-177.

[5] F. W. Kamber and Ph. Tondeur, *Foliated bundles and Characteristic classes*, Lecture Notes in Math. 493, Springer-Verlag, Berlin, 1975.

[6] F. W. Kamber and Ph. Tondeur, *Harmonic foliations*, Proc. National Science Foundation Conference on Harmonic Maps, Tulane, Dec. 1980, Lecture Notes in Math. 949, Springer-Verlag, New-York, 1982, 87-121.

[7] F. W. Kamber and Ph. Tondeur, *Infinitesimal automorphisms and second variation of the energy for harmonic foliations*, Tohoku Math. J. 34(1982), 525-538.

[8] P. March, M. Min-Oo and E. A. Ruh, *Mean curvature of Riemannian foliations*, Canad. Math. Bull. 39(1996), 95-105.

[9] M. Obata, *Conformal transformations of Compact Riemannian manifolds*, Illinois J. Math. 6(1962), 292-295.

[10] J. S. Pak and S. Yorozu, *Transverse fields on foliated Riemannian manifolds*, J. Korean Math. Soc. 25(1988), 83-92.

[11] J. H. Park and S. Yorozu, *Transversal conformal fields of foliations*, Nihonkai Math. J. 4(1933), 73-85.

[12] B. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. 69(1959), 119-132.

[13] Ph. Tondeur, *Foliations on Riemannian manifolds*, Springer-Verlag, New-York, 1988.

[14] Ph. Tondeur, *Geometry of foliations*, Birkhäuser-Verlag, Basel; Boston; Berlin, 1997.

[15] Ph. Tondeur and G. Toth, *On transversal infinitesimal automorphisms for harmonic foliations*, Geometriae Dedicata, 24(1987), 229-236.

[16] K. Yano. *Integral Formulas in Riemannian Geometry*, Marcel Dekker Inc, 1970.

[17] S. Yorozu and T. Tanemura, *Green's theorem on a foliated Riemannian manifold and its applications*, Acta Math. Hungar. 56(1990), 239-245.

<국문 초록>

엽층이 극소부분이 아닌 컴팩트 리만 다양체들로 구성되어 있을 때 무한소의 자기동형에 대한 연구

본 논문에서는 엽층이 극소부분이 아닌 컴팩트 리만 다양체들로 구성되어 있을 때 무한소의 자기동형에 대해 연구하였다.

특히, 무한소의 자기동형에 필요한 적분 공식을 찾았고, 횡단적 Ricci 곡률과 횡단적 Killing 장, 횡단적 의사 Killing 장, 횡단적 사형 Killing 장과 횡단적 공형 Killing 장의 소멸정리를 증명하였다.

감사의 글

대학교를 졸업하고 무엇을 할까 고민을 하던 차에 공부를 선택하게 되었습니다. 미분기하학을 하면서 중간에 포기할까 라는 생각도 해 보았고, 울기도 정말 많이 울었습니다. 끝까지 포기하지 않게 옆에서 지켜봐주시고 이 자리까지 설 수 있게 해주신 정승달 지도 교수님께 깊은 감사의 마음을 전합니다.

수학이 단순한 계산이 아니라 이해와 노력이 필요하다는 것을 깨닫게 해주신 방은숙, 양영오, 송석준, 윤용식, 유상욱 교수님께도 감사의 마음을 전합니다. 그리고 옆에서 많은 격려와 응원을 해주신 고연순, 문영봉, 김순찬, 문동주 선생님과 대학원 2년 동안 저의 든든한 버팀목이 되어주었던 정말 고마운 내 친구 민주에게도 이 마음을 전합니다.

공부를 하겠다는 저의 마음을 이해해주시고 도와주신 부모님과 제 동생 옥란이, 힘들 때 위로해준 경윤, 제연, 주미언니, 9기 친구들, 고등학교 친구들, 후배들, 그 많은 이름을 담지 못하지만 이 글을 빌어 고맙다는 말을 전합니다.

2005년 12월