博士學位論文

# Eigenvalue estimates of the basic Dirac operator on a Riemannian foliation



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# Eigenvalue estimates of the basic Dirac operator on a Riemannian foliation

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# 엽층구조를 가지는 리만다양체에서의 basic Dirac 연산자의 고유치 계산


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< **Abstract** >


## Eigenvalue estimates of the basic Dirac operator on a Riemannian foliation

On a foliated Riemannian manifold with a transverse spin structure, we give a lower bound for the square of the eigenvalues of the basic Dirac operator in terms of the transversal scalar curvature and of the norm of an appropriate endomorphism of the normal bundle $Q$ of $\mathcal{F}$. We study the limiting case.

# 1 Introduction

The Dirac operator on a Riemannian spin manifold, which were introduced by M.F. Atiyah and I.M. Singer([2]), were studied by many authors([3,4,7,9,10,15]).

In 1963, A. Lichnerowicz([17]) proved that on a Riemannian spin manifold the square of the Dirac operator $D$ is given by

$$D^2 = \nabla^*\nabla + \frac{R}{4},$$

where $\nabla^*\nabla$ is the positive spinor Laplacian and $R$ the scalar curvature. In particular, the first sharp estimate for the eigenvalues $\lambda$ of the Dirac operator $D$ was proved by Th. Friedrich ([7]) in 1980. Using a suitable Riemannian spin connection, he proved the inequality



$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R \tag{1.1}$$

on manifolds $(M^n, g)$ with positive scalar curvature $R > 0$. He also proved, in the limiting case, that the manifold is an Einstein. The inequality (1.1) has been improved in several directions by many authors ([9,10,11,12]).

In 1988, J.Brüning and F.W.Kamber([5]) introduced the transversal spin structure on the normal bundle $Q$ of the foliation $\mathcal{F}$. Let $S(\mathcal{F})$ be a foliated spinor bundle on $(M, \mathcal{F})$. Then the transversal Dirac operator $D_{tr} : S(\mathcal{F}) \to S(\mathcal{F})$ is defined by

$$D_{tr}\Psi = \sum_a E_a \cdot \nabla_{E_a}\Psi - \frac{1}{2}\kappa \cdot \Psi, \tag{1.2}$$

where $\kappa$ is a mean curvature form of $\mathcal{F}$. If $\mathcal{F}$ is isoparametric and $\kappa$ is divergence-free, i.e. $\delta\kappa = 0$, then the Lichnerowicz type formula is given by

$$D_{tr}^2\Psi = \nabla_{tr}^*\nabla_{tr}\Psi + \frac{1}{4}K^\sigma, \tag{1.3}$$

where $K^\sigma = \sigma^\nabla + |\kappa|^2$, $\sigma^\nabla$ is the transversal scalar curvature of $\mathcal{F}$.

In 2001, Jung([11]) studied the eigenvalue of the basic Dirac operator and the limiting foliation. Namely, let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with an isoparametric transverse spin folition $\mathcal{F}$ of codimension $q > 1$ and bundle-like metric $g_M$ with respect to $\mathcal{F}$. Assume that the mean curvature $\kappa$ of $\mathcal{F}$ satisfies $\delta\kappa = 0$ and $K^\sigma \geq 0$. Then the eigenvalue $\lambda$ of the basic Dirac operator $D_b$ satisfies

$$\lambda^2 \geq \frac{1}{4}\frac{q}{q-1}\inf_M K^\sigma. \tag{1.4}$$

In the limiting case, it is proved that $\mathcal{F}$ is minimal, transversally Einsteinian with constant transversal scalar curvature.



In 2004, Jung et. al([12]) gave new lower bound for the eigenvalues of $D_b$ by the first eigenvalue of the basic Yamabe operator $Y_b$, which is defined by

$$Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^\nabla, \tag{1.5}$$

where $\Delta_B$ is a basic Laplacian acting on basic functions. Namely, let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric $g_M$ such that $\kappa \in \Omega_B^1(\mathcal{F})$ and $\delta\kappa = 0$. Then

$$\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf|\kappa|^2), \tag{1.6}$$

where $\mu_1$ is the first eigenvalue of $Y_b$.

In this thesis, we give a lower bound for the square of the eigenvalues of the basic Dirac operator in terms of the transversal scalar curvature and of the norm of an appropriate endomorphism of the normal bundle $Q$ of $\mathcal{F}$.

This article is organized as followings. In Chapter 2, we review the known fact on the foliated Riemannian manifold. In Chapter 3, we study some basic properties of the transversal Dirac operator $D_{tr}$. In Chapter 4, we estimate the conformal lower bound for the eigenvalues of the basic Dirac operator by the modified new connection. We apply some techniques and concerning conformal changes of the Riemannian metric to get a sharper estimate than the theorem in terms of the first eigenvalues of the Yamabe operator. Namely,



**Theorem 1.1** *Let $(M, \tilde{g}_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and a bundle-like metric $\tilde{g}_M$ . Then, any eigenvalue $\lambda$ of the basic Dirac operator corresponding to the eigenspinor $\Psi \in \Gamma S(\mathcal{F})$ satisfies*

$$\lambda^2 \geq \frac{1}{4}(\mu_1 + \inf_M |\kappa|^2) + \inf_M |l_\Psi|^2 \tag{1.7}$$

*where $\mu_1$ is the first eigenvalue of the basic Yamabe operator $Y_b$ of $\mathcal{F}$ and $l_\Psi$ is a symmetric endomorphism associated with $F_\Psi$.*

In Chapter 5, we prove, in the limiting case, that the foliation $\mathcal{F}$ is minimal.

**Theorem 1.2** *Let $(M, \tilde{g}_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and a bundle-like metric*

*$\tilde{g}_M$. Assume that an eigenvalue $\lambda$ of $D_b$ corresponding to the eigenspinor $\Psi$ satisfies*

$$\lambda^2 = \frac{1}{4}(\mu_1 + \inf_M |\kappa|^2) + \inf_M |l_\Psi|^2.$$

*Then $|l_\Psi|$ is constant and the foliation $\mathcal{F}$ is minimal. Moreover*

$$(div_\nabla l_\Psi)(X) = (1-q)g_Q(l_\Psi(X), grad_\nabla(u)) \tag{1.8}$$

*for any $X \in \Gamma Q$.*

Throughout this paper, we consider the bundle-like metric $\tilde{g}_M$ for $(M, \mathcal{F})$ such that the mean curvature form of $\mathcal{F}$ is basic-harmonic. The existence of the bundle-like metric $g_M$ for $(M, \mathcal{F})$ such that $\kappa$ is basic, i.e. $\kappa \in \Omega^1_B(\mathcal{F})$, is proved in [6]. In [18,19], it is proved that for any bundle-like metric $g_M$ with $\kappa \in \Omega^1_B(\mathcal{F})$ there exists another bundle-like metric $\tilde{g}_M$ for which the mean curvature form is basic-harmonic.

# 2 Riemannian foliation

## 2.1 Definition of foliations

Let $M$ be a smooth manifold of dimension $p+q$.

**Definition 2.1** A codimension $q$ *foliation* $\mathcal{F}$ on $M$ is given by an open cover $\mathcal{U} = (U_i)_{i\in I}$ and for each $i$, a diffeomorphism $\varphi_i : \mathbb{R}^{p+q} \to U_i$ such that, on $U_i \cap U_j \neq \emptyset$, the coordinate change $\varphi_j^{-1} \circ \varphi_i : \varphi_i^{-1}(U_i \cap U_j) \to \varphi_j^{-1}(U_i \cap U_j)$ has the form

$$\varphi_j^{-1} \circ \varphi_i(x, y) = (\varphi_{ij}(x, y), \gamma_{ij}(y)). \tag{2.1}$$

Equivalently, we have the following another definition. Let $f_i = pr \circ \varphi_i^{-1} : U_i \to \mathbb{R}^q$ be a submersion, where $pr : \mathbb{R}^{p+q} \to \mathbb{R}^q$ is a projection.



**Definition 2.2** A codimension $q$ *foliation* $\mathcal{F}$ on $M$ is given by an open cover $\mathcal{U} = (U_i)_{i\in I}$, submersions $f_i : U_i \to N$ over $q$ dimensional model manifold $N$ and for $U_i \cap U_j \neq \emptyset$, a diffeomorphism (transition function) $\gamma_{ij} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ satisfying

$$f_j(x) = \gamma_{ij} \circ f_i(x) \quad \textit{for } x \in U_i \cap U_j. \tag{2.2}$$

From Definition 2.1(or 2.2), the manifold $M$ is decomposed into connected submanifolds of dimension $p$. Each of these submanifolds is called a *leaf* of $\mathcal{F}$. Coordinate patches $(U_i, \varphi_i)$ are said to be *distinguished* for the foliation $\mathcal{F}$. The tangent bundle $L$ of a foliation is the subbundle of $TM$, consisting of all vectors tangent to the leaves of $\mathcal{F}$. The normal bundle $Q$ of a codimension $q$ foliation $\mathcal{F}$ on $M$ is the quotient bundle $Q = TM/L$. Equivalently $Q$ appears in the exact sequence of vector bundles

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0. \tag{2.3}$$

If $(x_1, \ldots, x_p; y_1, \ldots, y_q)$ are local coordinates in a distinguished chart $U$, the bundle $Q|U$ is framed by the vector fields $\pi\frac{\partial}{\partial y_1}, \ldots, \pi\frac{\partial}{\partial y_q}$. For a vector field $Y \in \Gamma TM$ , we denote also $\overline{Y} = \pi Y \in \Gamma Q$. A vector field $Y$ on $U$ is *projectable*, if $Y = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_\alpha b_\alpha \frac{\partial}{\partial y_\alpha}$ with $\frac{\partial b_\alpha}{\partial x_i} = 0$ for all $\alpha = 1, \ldots, q$ and $i = 1, \ldots, p$. This means that the functions $b_\alpha = b_\alpha(y)$ are independent of $x$. Then $\overline{Y} = \sum_\alpha b_\alpha \frac{\bar{\partial}}{\partial y_\alpha}$ with $b_\alpha$ independent of $x$. This property is preserved under change of distinguished charts, hence makes intrinsic sense.



The *transversal geometry* of a foliation is the geometry infinitesimally modeled by $Q$, while the tangential geometry is infinitesimally modeled by $L$. A key fact is the existence of the *Bott connection* in $Q$ defined by

$$\overset{\circ}{\nabla}_X s = \pi([X, Y_s]) \quad \text{for } X \in \Gamma L, \tag{2.4}$$

where $Y_s \in TM$ is any vector field projecting to $s$ under $\pi : TM \to Q$. It is a partial connection along $L$. The right hand side in (2.4) is independent of the choice of $Y_s$. Namely, the difference of two such choices is a vector field $X' \in \Gamma L$ and $[X, X'] \in \Gamma L$ so that $\pi[X, X'] = 0$.

A Riemannian metric $g_Q$ on the normal bundle $Q$ of a foliation $\mathcal{F}$ is *holonomy invariant*, if

$$\theta(X) g_Q = 0 \quad \text{for all } X \in \Gamma L, \tag{2.5}$$

where $\theta(X)$ is Lie derivative. Here we have by definition for $s, t \in \Gamma Q$,

$$(\theta(X)g_Q)(s,t) = Xg_Q(s,t) - g_Q(\theta(X)s,t) - g_Q(s,\theta(X)t).$$

**Definition 2.3** A *Riemannian foliation* is a foliation $\mathcal{F}$ with a holonomy invariant transversal metric $g_Q$. A metric $g_M$ is a *bundle-like,* if the induced metric $g_Q$ on $Q$ is holonomy invariant.

The study of Riemannian foliations was initiated by Reinhart in 1959([21]). A simple example of a Riemannian foliation is given by a nonsingular Killing vector field $X$ on $(M, g_M)$. This means that $\theta(X)g_M = 0$.

An *adapted connection* in $Q$ is a connection restricting along $L$ to the partial Bott connection $\overset{\circ}{\nabla}$. To show that such connections exist, consider a Riemannian metric $g_M$ on $M$. Then $TM$ splits orthogonally as $TM = L \oplus L^\perp$. This means that there is a bundle map $\sigma : Q \to L^\perp$ splitting the exact sequence (2.3), i.e., satisfying $\pi \circ \sigma = identity$. This metric $g_M$ on $TM$ is then a direct sum



$$g_M = g_L \oplus g_{L^\perp}.$$

With $g_Q = \sigma^* g_{L^\perp}$, the splitting map $\sigma : (Q, g_Q) \to (L^\perp, g_{L^\perp})$ is a metric isomorphism. Let now $\nabla^M$ be the Levi-Civita connection associated to the Riemannian metric $g_M$. Then the adapted connection $\nabla$ in $Q$ is defined by

$$\begin{cases} \nabla_X s = \overset{\circ}{\nabla}_X s = \pi([X, Y_s]) & \text{for } X \in \Gamma L, \\ \nabla_X s = \pi(\nabla^M_X Y_s) & \text{for } X \in \Gamma L^\perp, \end{cases} \tag{2.6}$$

where $s \in \Gamma Q$ and $Y_s \in \Gamma L^\perp$ corresponding to $s$ under the canonical isomorphism $Q \cong L^\perp$. For any connection $\nabla$ on $Q$, there is a torsion $T_\nabla$

defined by

$$T_\nabla(Y,Z) = \nabla_Y \pi(Z) - \nabla_Z \pi(Y) - \pi[Y,Z] \tag{2.7}$$

for any $Y, Z \in \Gamma TM$. Then we have the following proposition ([22]).

**Proposition 2.4** *For any metric $g_M$ on $M$, and the adapted connection $\nabla$ on $Q$ defined by (2.6), we have $T_\nabla = 0$.*

**Proof.** For $X \in \Gamma L$, $Y \in \Gamma TM$ we have $\pi(X) = 0$ and

$$T_\nabla(X,Y) = \nabla_X \pi(Y) - \pi[X,Y] = 0.$$

For $Z, Z' \in \Gamma L^\perp$ we have

$$T_\nabla(Z,Z') = \pi(\nabla^M_Z Z') - \pi(\nabla^M_{Z'} Z) - \pi[Z,Z'] = \pi(T_{\nabla^M}(Z,Z')) = 0,$$

where $T_{\nabla^M}$ is the (vanishing) torsion of $\nabla^M$. Finally the bilinearity and skew symmetry of $T_\nabla$ imply the desired result. □



The curvature $R^\nabla$ of $\nabla$ is defined by

$$R^\nabla(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for } X,\ Y \in TM.$$

From an adapted connection $\nabla$ in $Q$ defined by (2.6), its curvature $R^\nabla$ coincides with $\overset{\circ}{R}$ for $X, Y \in \Gamma L$, hence $R^\nabla(X,Y) = 0$ for $X, Y \in \Gamma L$. And we have the following proposition ([13,14,22]).

**Proposition 2.5** *Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q$ and bundle-like metric $g_M$ with respect to $\mathcal{F}$. Let $\nabla$ be a connection defined by (2.6) in $Q$ with curvature $R^\nabla$. Then for $X \in \Gamma L$ the following holds:*

$$i(X)R^\nabla = \theta(X)R^\nabla = 0. \tag{2.8}$$

**Proof.** (i) Let $Y \in \Gamma TM$ and $s \in \Gamma Q$. Then

$$\begin{aligned} R^\nabla(X,Y)s &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \\ &= \theta(X)\nabla_Y s - \nabla_Y \theta(X)s - \nabla_{\theta(X)Y} s \\ &= (\theta(X)\nabla)_Y s = 0. \end{aligned}$$

(ii) Let $Y, Z \in \Gamma TM$ and $s \in \Gamma Q$. Then

$$\begin{aligned} (\theta(X)R^\nabla)(Y,Z)s &= \theta(X)R^\nabla(Y,Z)s - R^\nabla(\theta(X)Y,Z)s \\ &\quad -R^\nabla(Y,\theta(X)Z)s - R^\nabla(Y,Z)\theta(X)s \\ &= \theta(X)\{\nabla_Y\nabla_Z s - \nabla_Z\nabla_Y s - \nabla_{[Y,Z]}s\} \\ &\quad -\{\nabla_{\theta(X)Y}\nabla_Z s - \nabla_Z\nabla_{\theta(X)Y} s - \nabla_{[\theta(X)Y,Z]}s\} \\ &\quad -\{\nabla_Y\nabla_{\theta(X)Z} s - \nabla_{\theta(X)Z}\nabla_Y s - \nabla_{[Y,\theta(X)Z]}s\} \\ &\quad -\{\nabla_Y\nabla_Z\theta(X)s - \nabla_Z\nabla_Y\theta(X)s - \nabla_{[Y,Z]}\theta(X)s\} \\ &= \nabla_Y(\theta(X)\nabla_Z s) - \nabla_Z(\theta(X)\nabla_Y s) - \nabla_{\theta(X)[Y,Z]}s \\ &\quad +\nabla_Z\nabla_{\theta(X)Y}s + \nabla_{[\theta(X)Y,Z]}s - \nabla_Y\nabla_{\theta(X)Z}s \\ &\quad +\nabla_{[Y,\theta(X)Z]}s - \nabla_Y\nabla_Z\theta(X)s + \nabla_Z\nabla_Y\theta(X)s \\ &= -\nabla_{\theta(X)[Y,Z]}s + \nabla_{[\theta(X)Y,Z]}s + \nabla_{[Y,\theta(X)Z]}s \\ &= (-\nabla_{[X,[Y,Z]]} + \nabla_{[[X,Y],Z]} + \nabla_{[Y,[X,Z]]})s = 0. \qquad \square \end{aligned}$$

By Proposition 2.5, we can define the (transversal) Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the (transversal) scalar curvature $\sigma^\nabla$ of $\mathcal{F}$ by

$$\rho^\nabla(s) = \sum_a R^\nabla(s, E_a)E_a, \quad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a), \tag{2.9}$$

where $\{E_a\}_{a=1,\cdots,q}$ is an orthonormal basis of $Q$.

**Definition 2.6** The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space $N$ is Einsteinian, that is,

$$\rho^{\nabla} = \frac{1}{q}\sigma^{\nabla} \cdot id \tag{2.10}$$

with constant transversal scalar curvature $\sigma^{\nabla}$.

## 2.2 Mean curvature form and basic Laplacian

The *second fundamental form* $\alpha$ of $\mathcal{F}$ is given by

$$\alpha(X,Y) = \pi(\nabla^M_X Y) \quad \text{for } X, Y \in \Gamma L. \tag{2.11}$$

**Proposition 2.7** *$\alpha$ is $Q$-valued, bilinear and symmetric.*

**Proof.** By definition, it is trivial that $\alpha$ is $Q$-valued and bilinear. Next, by torsion freeness of $\nabla^M$, we have that for any $X, Y \in \Gamma L$,



$$\alpha(X,Y) = \pi(\nabla^M_X Y) = \pi(\nabla^M_Y X) - \pi([X,Y]).$$

Since $[X,Y] \in \Gamma L$ for any $X, Y \in \Gamma L$, we have

$$\alpha(X,Y) = \pi(\nabla^M_Y X) = \alpha(Y,X). \qquad \square$$

**Definition 2.8** The *mean curvature vector field* of $\mathcal{F}$ is then defined by

$$\tau = \sum_i \alpha(E_i, E_i) = \sum_i \pi(\nabla^M_{E_i} E_i), \tag{2.12}$$

where $\{E_i\}_{i=1,\cdots,p}$ is an orthonormal basis of $L$. The dual form $\kappa$, the *mean curvature form* for $L$, is then given by

$$\kappa(X) = g_Q(\tau, X) \quad \textit{for } X \in \Gamma Q. \tag{2.13}$$

The foliation $\mathcal{F}$ is said to be *minimal* (or *harmonic*) if $\kappa = 0$.

**Definition 2.9** Let $\mathcal{F}$ be an arbitrary foliation on a manifold $M$. A differential form $\omega \in \Omega^r(M)$ is *basic* , if

$$i(X)\omega = 0,\ \theta(X)\omega = 0,\ \textit{for}\ X \in \Gamma L. \tag{2.14}$$

In a distinguished chart $(x_1, \ldots, x_p; y_1, \ldots, y_q)$ of $\mathcal{F}$, a basic form $w$ is expressed by

$$\omega = \sum_{a_1 < \cdots < a_r} \omega_{a_1 \cdots a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r},$$

where the functions $\omega_{a_1 \cdots a_r}$ are independent of $x$, i.e. $\frac{\partial}{\partial x_i}\omega_{a_1 \cdots a_r} = 0$. Let $\Omega_B^r(\mathcal{F})$ be the set of all basic $r$-forms on $M$. The exterior derivative $d$ preserves basic forms, since $\theta(X)d\omega = d\theta(X)\omega = 0,\ i(X)d\omega = \theta(X)\omega - di(X)\omega = 0$ for a basic form $\omega$. Hence $\Omega_B^r(\mathcal{F})$ constitutes a subcomplex

$$d : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r+1}(\mathcal{F})$$



of the De Rham complex $\Omega^*(M)$ and the restriction $d_B = d|_{\Omega_B^*(\mathcal{F})}$ is well defined. Its cohomology

$$H_B(\mathcal{F}) = H(\Omega_B^*(\mathcal{F}), d_B)$$

is the *basic cohomology* of $\mathcal{F}$. It plays the role of the De Rham cohomology of the leaf space $M/\mathcal{F}$ of the foliation. Let $\delta_B$ the formal adjoint operator of $d_B$. Then we have the following proposition ([1,11]).

**Proposition 2.10** *On a Riemannian foliation $\mathcal{F}$, we have*

$$d_B = \sum_a \theta_a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B), \tag{2.15}$$

*where $\kappa_B$ is the basic component of $\kappa$, $\{E_a\}$ is a local orthonormal basic frame in $Q$ and $\{\theta_a\}$ its $g_Q$-dual 1-form.*

The foliation $\mathcal{F}$ is said to be *isoparametric* if $\kappa \in \Omega_B^1(\mathcal{F})$. We already know that $\kappa$ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric ([18]).

**Definition 2.11** The *basic Laplacian* acting on $\Omega_B^*(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B. \tag{2.16}$$

The following theorem is proved in the same way as the corresponding usual result in De Rham-Hodge Theory.

**Theorem 2.12** ([22]) *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed oriented manifold $(M, g_M)$. Assume $g_M$ to be bundle-like metric with $\kappa \in \Omega_B^1(\mathcal{F})$. Then*

$$H_B^r(\mathcal{F}) \cong \mathcal{H}_B^r(\mathcal{F}),$$

*where $\mathcal{H}_B^r(\mathcal{F}) = \{\omega \in \Omega^r(M) | \Delta_B\omega = 0\}$.*



If $\mathcal{F}$ is the foliation by points of $M$, the basic Laplacian is the ordinary Laplacian. In the more general case, the basic Laplacian and its spectrum provide information about the transverse geometry of $(M, \mathcal{F})$([16]).

## 2.3 Transversal divergence theorem

For the later use, we recall the divergence theorem on a foliated Riemannian manifold ([23]).

**Theorem 2.13** *Let $(M, g_M, \mathcal{F})$ be a closed, oriented, connected Riemannian manifold with a transversally orientable foliation $\mathcal{F}$ and a bundle-like metric $g_M$ with respect to $\mathcal{F}$. Then*

$$\int_M div_\nabla(X) = \int_M g_Q(X, \tau) \tag{2.17}$$

*for all $X \in \Gamma Q$, where $div_\nabla(X)$ denotes the transverse divergence of $X$ with respect to the connection $\nabla$ defined by (2.6).*

**Proof.** Let $\{E_i\}$ and $\{E_a\}$ be orthonormal basis of $L$ and $Q$, respectively. Then for any $X \in \Gamma Q$,

$$\begin{aligned} div(X) &= g_M(\nabla^M_{E_i} X, E_i) + g_M(\nabla^M_{E_a} X, E_a) \\ &= -g_M(X, \pi(\nabla^M_{E_i} E_i)) + g_M(\pi(\nabla^M_{E_a} X), E_a) \\ &= -g_Q(X, \tau) + g_Q(\nabla_{E_a} X, E_a) \\ &= -g_Q(X, \tau) + div_\nabla(X). \end{aligned}$$

By Green's Theorem on an ordinary manifold $M$, we have

$$0 = \int_M div(X) d_M = \int_M div_\nabla(X) d_M - \int_M g_Q(X, \tau).$$



This completes the proof of this Theorem. □

**Corollary 2.14** *If $\mathcal{F}$ is minimal, then we have that for any $X \in \Gamma Q$,*

$$\int_M div_\nabla(X) = 0. \tag{2.18}$$

# 3 Transverse spin structure

## 3.1 Clifford algebras

**Definition 3.1** Let $V$ be a vector space over a field $K = \{\mathbb{R}, \mathbb{C}\}$ of dimension $n$ and $g$ a non-degenerate bilinear form on $V$. The *Clifford algebra* $Cl(V, g)$ associated to $g$ on $V$ is the algebra over $K$ generated by $V$ with the relation

$$v \cdot w + w \cdot v = -2g(v, w)1 \tag{3.1}$$

for $v, w \in V$. The product "$\cdot$" is called the *Clifford multiplication.*

Equivalently, the Clifford algebra of $V$ is given by the following universal problem(for detail, see [15]).



**Proposition 3.2** *(Universal property) Let $A$ be an associative algebra with unit and $f : V \to A$ a linear map such that for all $v \in V$*

$$f(v)^2 = -g(v, v)1.$$

*Then $f$ uniquely extends to a $K$-algebra homomorphism*

$$\widetilde{f}\colon Cl(V, g) \to A.$$

**Remark.** The Clifford algebra may be realized as the quotient

$$Cl(V, g) := T(V)/I(V, g)$$

where $T(V)$ is the tensor algebra of $V$, and $I(V, g)$ the ideal generated by all elements of the form $v \otimes v + g(v, v)1$, for $v \in V$.

**Remark.** (1) If $(E_1, \cdots E_n)$ is a g-orthonormal basis of $V$, then

$$\{E_{i_1} \cdot \ldots \cdot E_{i_k} | 1 \leq i_1 < \ldots < i_k \leq n, \quad 0 \leq k \leq n\}$$

is a basis of $Cl(V, g)$ , thus $\dim Cl(V, g) = 2^n$.

(2) There is a canonical isomorphism of vector spaces, between the exterior algebra and the Clifford algebra of $(V, g)$ which is given by :

$$\wedge^* V \overset{\simeq}{\to} Cl(V, g)$$

$$E_{i_1} \wedge \ldots \wedge E_{i_k} \longmapsto E_{i_1} \cdot \ldots \cdot E_{i_k}.$$

This isomorphism does not depend on the choice of the basis. Let us denote $Cl_n = Cl(\mathbb{R}^n, < \ , \ >)$. Then we have the following proposition ([15]).



**Proposition 3.3** *For all $v \in \mathbb{R}^n$ and all $\varphi \in Cl_n$ , we have*

$$v \cdot \varphi \simeq v \wedge \varphi - i(v)\varphi \quad \textit{and} \quad \varphi \cdot v \simeq (-1)^p (v \wedge \varphi + i(v)\varphi),$$

*where $\wedge$ denotes the exterior, $i(v)$ the interior product and $\varphi \in \wedge^p \mathbb{R}^n \subset \wedge^* \mathbb{R}_n \simeq Cl_n$.*

**Proof.** Let $v = E_j$ and $\varphi = E_{i_1} \cdot \ldots \cdot E_{i_p}$.

1. If there exists $i_k$ such that $j = i_k$ then $v \wedge \varphi = 0$ and

$$\begin{aligned} i(v)\varphi &= (-1)^{k-1} E_{i_1} \wedge \ldots \wedge E_{i_{k-1}} \wedge E_{i_{k+1}} \wedge \ldots E_{i_p} \\ &\simeq (-1)^{k-1} E_{i_1} \cdot \ldots \cdot E_{i_{k-1}} \cdot E_{i_{k+1}} \cdot \ldots \cdot E_{i_p} \\ &= -v \cdot \varphi \\ &= (-1)^p \varphi \cdot v. \end{aligned}$$

2. If $j \notin \{i_1, \ldots, i_p\}$ then $i(v)\varphi = 0$ and

$$\begin{aligned} v \wedge \varphi &= E_j \wedge E_{i_1} \wedge \ldots \wedge E_{i_p} \simeq E_j \cdot E_{i_1} \cdot \ldots \cdot E_{i_p} \\ &= v \cdot \varphi \\ &= (-1)^p \varphi \cdot v. \end{aligned}$$

As the equalities of the assertion are bilinear, the proposition is proved. □

**Definition 3.4** The *Pin group* $Pin(V)$ is defined by

$$Pin(V) = \{a \in Cl(V) | a = a_1 \cdots a_k, ||a_i|| = 1\}. \tag{3.2}$$

The *Spin group* is defined by



$$Spin(V) = \{a \in Pin(V) | aa^t = 1\}, \tag{3.3}$$

*where* $a^t = a_k \cdots a_1$ *for any* $a = a_1 \cdots a_k$. Equivalently, $Spin(V) = \{e_1 \cdots e_{2k} | \quad |e_i| = 1\}$.

Let $V$ be a real vector space. Then $Spin(V)$ is a compact and connected Lie group, and for $\dim V \geq 3$, it is also simply connected. Thus, for $\dim V \geq 3$, $Spin(V)$ is the universal cover of $SO(V)$ (for detail, see [15]).

## 3.2 Transversal Dirac operator

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transversally oriented Riemannian foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$ with respect to $\mathcal{F}$. Let $SO(q) \to P_{SO} \to M$ be the principal bundle of (oriented) transverse orthonormal framings. Then a *transverse spin structure* is a principal $Spin(q)$-bundle $P_{Spin}$ together with two sheeted covering $\xi : P_{Spin} \to P_{SO}$ such that $\xi(p \cdot g) = \xi(p)\xi_0(g)$ for all $p \in P_{Spin},\ g \in Spin(q)$, where $\xi_0 : Spin(q) \to SO(q)$ is a covering. In this case, the foliation $\mathcal{F}$ is called a *transverse spin foliation.* We then define the *foliated spinor bundle* $S(\mathcal{F})$ associated with $P_{Spin}$ by

$$S(\mathcal{F}) = P_{Spin} \times_{Spin(q)} S_q, \tag{3.4}$$

where $S_q$ is the irreducible spinor space associated to $Q$. The Hermitian metric $<,>$ on $S(\mathcal{F})$ induced from $g_Q$ satisfies the following relation:

$$< \varphi, \psi > = < v \cdot \varphi, v \cdot \psi > \tag{3.5}$$

for every $v \in Q,\ g_Q(v, v) = 1$ and $\varphi, \psi \in S_q$. And the Riemannian connection $\nabla$ on $P_{SO}$ defined by (2.6) can be lifted to one on $P_{Spin}$, in particular, to one on $S(\mathcal{F})$, which will be denoted by the same letter.



**Proposition 3.5** ([11, 15]) *The spinorial covariant derivative on $S(\mathcal{F})$ is given locally by:*

$$\nabla \Psi_\alpha = \frac{1}{4} \sum_{a,b} g_Q(\nabla E_a, E_b) E_a \cdot E_b \cdot \Psi_\alpha, \tag{3.6}$$

*where $\Psi_\alpha$ is an orthonormal basis of $S_q$. And the curvature transform $R^S$ on $S(\mathcal{F})$ is given as*

$$R^S(X, Y)\Phi = \frac{1}{4} \sum_{a,b} g_Q(R^\nabla(X, Y) E_a, E_b) E_a \cdot E_b \cdot \Phi \quad \text{for } X, Y \in TM. \tag{3.7}$$

*where $\{E_a\}$ is an orthonormal basis of the normal bundle $Q$.*

**Proposition 3.6** ([15]) (Compatibility of $\nabla$ with " $\cdot$ " and $< \cdot, \cdot >$)

$$(1) \quad X < \psi, \varphi > = < \nabla_X \psi, \varphi > + < \psi, \nabla_X \varphi >, \qquad X \in \Gamma TM \tag{3.8}$$

$$(2) \quad \nabla_X (Y \cdot \psi) = (\nabla_X Y) \cdot \psi + Y \cdot \nabla_X \psi, \qquad Y \in \Gamma Q. \tag{3.9}$$

**Theorem 3.7** ([11,12]) *On the foliated spinor bundle $S(\mathcal{F})$, we have*

$$\sum_a E_a \cdot R^S(X, E_a)\Phi = -\frac{1}{2}\rho^\nabla(\pi(X)) \cdot \Phi, \tag{3.10}$$

$$\sum_{a<b} E_a \cdot E_b \cdot R^S(E_a, E_b)\Phi = \frac{1}{4}\sigma^\nabla \Phi \tag{3.11}$$

*for $X \in TM$.*

Taking $\hat{\pi}$ to denote the projection

$$\hat{\pi} : C^\infty(T^*M \otimes S(\mathcal{F})) \to C^\infty(Q^* \otimes S(\mathcal{F})) \cong C^\infty(Q \otimes S(\mathcal{F}))$$

we define the *transversal Dirac Operator* $D'_{tr}$([5,8]) by



$$D'_{tr} = \cdot \circ \hat{\pi} \circ \nabla.$$

If $\{E_a\}_{a=1,\cdots,q}$ is taken to be a local orthonormal basic frame in $Q$, then

$$D'_{tr} = \sum_a E_a \cdot \nabla_{E_a}.$$

In [5,8] it was shown that the formal adjoint ${D'_{tr}}^*$ is given by ${D'_{tr}}^* = D'_{tr} - \kappa\cdot$ and that therefore

$$D_{tr} = D'_{tr} - \frac{1}{2}\kappa\cdot \tag{3.12}$$

is a symmetric, transversally elliptic differential operator, with symbol $\sigma_{D_{tr}}$ satisfying $\sigma_{D_{tr}}(x, \xi) = \xi$ for $\xi \in Q^*_x$ and $\sigma_{D_{tr}}(x, \xi) = 0$ for $\xi \in L^*_x$. Then the transversal Dirac operator $D_{tr}$ is locally defined by

$$D_{tr}\Psi = \sum_a E_a \cdot \nabla_{E_a}\Psi - \frac{1}{2}\kappa \cdot \Psi \quad \text{for } \Psi \in \Gamma S(\mathcal{F}), \tag{3.13}$$

where $\{E_a\}$ is a local orthonormal basic frame of $Q$. Then we have the Lichnerowicz-type formula on $\mathcal{F}$.

**Theorem 3.8** *On an isoparametric transverse spin foliation $\mathcal{F}$ with $\delta\kappa = 0$, it is well-known* ([8,11]) *that*

$$D_{tr}^2\Psi = \nabla_{tr}^*\nabla_{tr}\Psi + \frac{1}{4}K^\sigma\Psi, \tag{3.14}$$

*where $K^\sigma = \sigma^\nabla + |\kappa|^2$ and*

$$\nabla_{tr}^*\nabla_{tr}\Psi = -\sum_a \nabla^2_{E_a,E_a}\Psi + \nabla_\kappa\Psi. \tag{3.15}$$

The operator $\nabla_{tr}^*\nabla_{tr}$ is non-negative and formally self-adjoint ([11]). In fact, we have the following proposition.

**Proposition 3.9** ([11]) *Let $(M, g_M, \mathcal{F}, S(\mathcal{F}))$ be a compact Riemannian manifold with the transverse spin foliation $\mathcal{F}$ and a bundle-like metric $g_M$ with respect to $\mathcal{F}$. Then*



$$\ll \nabla_{tr}^*\nabla_{tr}\Phi, \Psi \gg = \ll \nabla_{tr}\Phi, \nabla_{tr}\Psi \gg$$

*for all $\Phi, \Psi \in \Gamma E$, where $\ll \Phi, \Psi \gg = \int_M < \Phi, \Psi >$ is the inner product on $S(\mathcal{F})$.*

We define the subspace $\Gamma_B(S(\mathcal{F}))$ of *basic* or *holonomy invariant* sections of $S(\mathcal{F})$ by

$$\Gamma_B(S(\mathcal{F})) = \{\Psi \in \Gamma S(\mathcal{F}) | \nabla_X\Psi = 0 \quad \text{for } X \in \Gamma L\}.$$

Trivially, we see that $D_{tr}$ leaves $\Gamma_B(S(\mathcal{F}))$ invariant if and only if the foliation $\mathcal{F}$ is isoparametric, i.e., $\kappa \in \Omega_B^1(\mathcal{F})$. Let $D_b = D_{tr}|_{\Gamma_B(S(\mathcal{F}))} : \Gamma_B(S(\mathcal{F})) \to \Gamma_B(S(\mathcal{F}))$. This operator $D_b$ is called the *basic Dirac operator* on (smooth) basic sections. It is well known that $D_b$ and $D_b^2$ have a discrete spectrum, respectively.

## 3.3 Transversal Dirac operators of transversally conformally related metrics

Now, we consider, for any real basic function $u$ on $M$, the transversally conformal metric $\bar{g}_Q = e^{2u} g_Q$. Let $\bar{P}_{so}(\mathcal{F})$ be the principal bundle of $\bar{g}_Q$ -orthogonal frames. Locally, the section $\bar{s}$ of $\bar{P}_{so}(\mathcal{F})$ corresponding a section $s = (E_1, \cdots, E_q)$ of $P_{so}(\mathcal{F})$ is $\bar{s} = (\bar{E}_1, \cdots, \bar{E}_q)$, where $\bar{E}_a = e^{-u} E_a$ $(a = 1, \cdots, q)$. This isometry will be denoted by $I_u$. Thanks to the isomorphism $I_u$ one can define a transverse spin structure $\bar{P}_{spin}(\mathcal{F})$ on $\mathcal{F}$ in such a way that the diagram

$$\begin{array}{ccc} P_{spin}(\mathcal{F}) & \xrightarrow{\tilde{I}_u} & \bar{P}_{spin}(\mathcal{F}) \\ \downarrow & & \downarrow \\ P_{so}(\mathcal{F}) & \xrightarrow{I_u} & \bar{P}_{so}(\mathcal{F}) \end{array}$$



commutes.

Let $\bar{S}(\mathcal{F})$ be the foliated spinor bundles associated with $\bar{P}_{spin}(\mathcal{F})$. For any section $\Psi$ of $S(\mathcal{F})$, we write $\bar{\Psi} \equiv I_u \Psi$. If $< , >_{g_Q}$ and $< , >_{\bar{g}_Q}$ denote respectively the natural Hermitian metrics on $S(\mathcal{F})$ and $\bar{S}(\mathcal{F})$, then for any $\Phi,\ \Psi \in \Gamma S(\mathcal{F})$

$$< \Phi, \Psi >_{g_Q} = < \bar{\Phi}, \bar{\Psi} >_{\bar{g}_Q}, \tag{3.16}$$

and the Clifford multiplication in $\bar{S}(\mathcal{F})$ is given by

$$\bar{X} \,\bar{\cdot}\, \bar{\Psi} = \overline{X \cdot \Psi} \quad \text{for } X \in \Gamma Q. \tag{3.17}$$

Let $\bar{\nabla}$ be the metric and torsion free connection corresponding to $\bar{g}_Q$. Then we have the following proposition([12]).

**Proposition 3.10** *On the Riemannian foliation, we have that for* $X, Y \in \Gamma TM$,

$$\bar{\nabla}_X \pi(Y) = \nabla_X \pi(Y) + X(u)\pi(Y) + Y(u)\pi(X) - g_Q(\pi(X), \pi(Y)) grad_\nabla(u), \tag{3.18}$$

*where* $grad_\nabla(u) = \sum_a E_a(u) E_a$ *is a transversal gradient of* $u$ *and* $X(u)$ *is the Lie derivative of the function* $u$ *in the direction of* $X$.

**Proof.** Since $\bar{\nabla}$ is the metric and torsion free connection with respect to $\bar{g}_Q$ on $Q$, we have

$$\begin{aligned} 2\bar{g}_Q(\bar{\nabla}_X s, t) &= X\bar{g}_Q(s,t) + Y\bar{g}_Q(\pi(X), t) - Z_t \bar{g}_Q(\pi(X), s) \\ &= \bar{g}_Q(\pi[X, Y_s], t) + \bar{g}_Q(\pi[Z_t, X], s) - \bar{g}_Q(\pi[Y_s, Z_t], \pi(X)), \end{aligned}$$

where $\pi(Y_s) = s$ and $\pi(Z_t) = t$. From this formula, the proof is completed. □



From (3.18), we have the following proposition([12]).

**Proposition 3.11** *The connection* $\nabla$ *and* $\bar{\nabla}$ *acting repectively on the sections of* $S(\mathcal{F})$ *and* $\bar{S}(\mathcal{F})$, *are related, for any vector field* $X$ *and any spinor field* $\Psi$ *by*

$$\bar{\nabla}_X \bar{\Psi} = \overline{\nabla_X \Psi} - \frac{1}{2}\overline{\pi(X) \cdot grad_\nabla(u) \cdot \Psi} - \frac{1}{2} g_Q(grad_\nabla(u), \pi(X))\bar{\Psi}. \tag{3.19}$$

**Proof.** Let $\{E_a\}$ be an orthonormal basis of $Q$ and denote by $\omega$ and $\bar{\omega}$, the connection forms corresponding to $g_Q$ and $\bar{g}_Q$. That is, for any vector field $X \in TM$,

$$\nabla_X E_b = \sum_c \omega_{bc}(\pi(X)) E_c, \quad \bar{\nabla}_X \bar{E}_b = \sum_c \bar{\omega}_{bc}(\pi(X)) \bar{E}_c. \tag{3.20}$$

From (3.18), we have

$$\bar{\omega}_{bc}(\pi(X)) = \omega_{bc}(\pi(X)) + g_Q(\pi(X), E_c)E_b(u) - g_Q(\pi(X), E_b)E_c(u). \tag{3.21}$$

Let $\{\Psi_A\}(A = 1, \cdots, 2^{[\frac{q}{2}]})$ be a local frame field of $S(\mathcal{F})$. Then the spinor covariant derivative of $\Psi_A$ is given ([12]) by

$$\nabla_X \Psi_A = \frac{1}{2}\sum_{b<c} \omega_{bc}(\pi(X)) E_b \cdot E_c \cdot \Psi_A. \tag{3.22}$$

With respect to $\bar{g}_Q$, we have

$$\begin{aligned}
\bar{\nabla}_X \bar{\Psi}_A &= \frac{1}{2}\sum_{b<c} \bar{\omega}_{bc}(\pi(X)) \bar{E}_b \bar{\cdot} \bar{E}_c \bar{\cdot} \bar{\Psi}_A \\
&= \frac{1}{2}\sum_{b<c} \{\omega_{bc}(\pi(X)) + g_Q(\pi(X), E_c)E_b(u) \\
&\quad - g_Q(\pi(X), E_b)E_c(u)\} \bar{E}_b \bar{\cdot} \bar{E}_c \bar{\cdot} \bar{\Psi}_A \\
&= \overline{\nabla_X \Psi_A} - \frac{1}{2}\sum_{b \neq c} g_Q(\pi(X), E_c)E_b(u) \bar{E}_c \bar{\cdot} \bar{E}_b \bar{\cdot} \bar{\Psi}_A \\
&= \overline{\nabla_X \Psi_A} - \frac{1}{2}\overline{\pi(X) \cdot grad_\nabla(u) \cdot \Psi_A} - \frac{1}{2} g_Q(grad_\nabla(u), \pi(X))\bar{\Psi}_A.
\end{aligned}$$

□

Let $\bar{D}_{tr}$ be the transversal Dirac operator associated with the metric $\bar{g}_Q = e^{2u} g_Q$ and acting on the sections of the foliated spinor bundle $\bar{S}(\mathcal{F})$. Let $\{E_a\}$ be a local frame of $P_{SO}(\mathcal{F})$ and $\{\bar{E}_a\}$ a local frame of $\bar{P}_{SO}(\mathcal{F})$. Locally, $\bar{D}_{tr}$ is expressed by

$$\bar{D}_{tr}\bar{\Psi} = \sum_a \bar{E}_a \bar{\cdot} \bar{\nabla}_{\bar{E}_a} \bar{\Psi} - \frac{1}{2}\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi}, \tag{3.23}$$

where $\kappa_{\bar{g}}$ is the mean curvature form associated with $\bar{g}_Q$, which satisfies $\kappa_{\bar{g}} = e^{-2u}\kappa$. Using (3.19), we have that for any $\Psi$,

$$\bar{D}_{tr}\bar{\Psi} = e^{-u}\{\overline{D_{tr}\Psi} + \frac{q-1}{2}\overline{grad_\nabla(u) \cdot \Psi}\}. \tag{3.24}$$

Now, for any function $f$, we have $D_{tr}(f\Psi) = grad_{\nabla}(f) \cdot \Psi + fD_{tr}\Psi$. Hence we have

$$\bar{D}_{tr}(f\bar{\Psi}) = e^{-u}\overline{grad_{\nabla}(f) \cdot \Psi} + f\bar{D}_{tr}\bar{\Psi}. \tag{3.25}$$

From (3.24) and (3.25), we have the following proposition.

**Proposition 3.12** *Let $\mathcal{F}$ be the transverse spin foliation of codimension $q$. Then the transverse Dirac operators $D_{tr}$ and $\bar{D}_{tr}$ satisfy*

$$\bar{D}_{tr}(e^{-\frac{q-1}{2}u}\bar{\Psi}) = e^{-\frac{q+1}{2}u}\overline{D_{tr}\Psi} \tag{3.26}$$

*for any spinor field $\Psi \in S(\mathcal{F})$.*

From Proposition 3.12, if $D_{tr}\Psi = 0$, then $\bar{D}_{tr}\bar{\Phi} = 0$, where $\Phi = e^{-\frac{q-1}{2}u}\Psi$, and conversely. So we have the following corollary.



**Corollary 3.13** *On the transverse spin foliation $\mathcal{F}$, the dimension of the space of the foliated harmonic spinors is a transversally conformal invariant.*

Let the mean curvature form $\kappa$ of $\mathcal{F}$ be basic- harmonic, i.e., $\kappa \in \Omega_B^1(\mathcal{F})$ and $\delta_B\kappa = 0$. Then by direct calculation, we have the Lichnerowicz type formula.

**Theorem 3.14** *On the transverse spin foliation with the basic harmonic mean curvature form $\kappa$, we have on $\bar{S}(\mathcal{F})$*

$$\bar{D}_{tr}^2\bar{\Psi} = \bar{\nabla}_{tr}^*\bar{\nabla}_{tr}\bar{\Psi} + \mathcal{R}^{\bar{\nabla}}(\bar{\Psi}) + K^{\bar{\nabla}}\bar{\Psi}, \tag{3.27}$$

*where*

$$\bar{\nabla}^*_{tr}\bar{\nabla}_{tr}\bar{\Psi} = -\sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\sum \bar{\nabla}_{\bar{E}_a}\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \tag{3.28}$$

$$K^{\bar{\nabla}} = \frac{1}{2}(q-2)\kappa_{\bar{g}}(u) + \frac{1}{4}|\bar{\kappa}|^2, \tag{3.29}$$

$$\mathcal{R}^{\bar{\nabla}}(\bar{\Psi}) = \sum_{a<b} \bar{E}_a \cdot \bar{E}_b \cdot \bar{R}^S(\bar{E}_a, \bar{E}_b)\bar{\Psi}. \tag{3.30}$$

**Proof.** Fix $x \in M$ and choose a local orthonormal basic frame $\{E_a\}$ satisfying $(\nabla E_a)_x = 0$ at $x \in M$. Then by definition,

$$\begin{aligned}
\bar{D}^2_{tr}\bar{\Psi} &= \bar{D}_{tr}\{\sum_a \bar{E}_a \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi} - \frac{1}{2}\kappa_{\bar{g}} \bar{\cdot} \bar{\Psi}\} \\
&= -\sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \sum_{a<b} \bar{E}_a \bar{\cdot} \bar{E}_b \bar{\cdot} \bar{R}^S(\bar{E}_a, \bar{E}_b)\bar{\Psi} \\
&\quad + \sum_{a<b} \bar{E}_a \bar{\cdot} \bar{E}_b \bar{\cdot} \bar{\nabla}_{[\bar{E}_a, \bar{E}_b]}\bar{\Psi} + \sum_{a,b} \bar{E}_b \bar{\cdot} \bar{\nabla}_{\bar{E}_b}\bar{E}_a \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi} \\
&\quad - \frac{1}{2}\sum_b \bar{E}_b \bar{\cdot} (\bar{\nabla}_{\bar{E}_b}\kappa_{\bar{g}}) \bar{\cdot} \bar{\Psi} + \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi} + \frac{1}{4}\kappa_{\bar{g}} \bar{\cdot} \kappa_{\bar{g}} \bar{\cdot} \bar{\Psi}.
\end{aligned}$$



From Proposition 3.10, we have

$$\bar{\nabla}_{\bar{E}_a}\bar{E}_b = e^{-2u}\{E_b(u)E_a - \delta_{ab} grad_{\nabla}(u)\}. \tag{3.31}$$

Hence we have

$$\sum_{a<b} \bar{E}_a \bar{\cdot} \bar{E}_b \bar{\cdot} \bar{\nabla}_{[\bar{E}_a, \bar{E}_b]}\bar{\Psi} = e^{-u}\{\sum_a \overline{E_a \cdot grad_{\nabla}(u)} \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\overline{grad_{\nabla}(u)}}\bar{\Psi}\},$$

$$\sum_{a,b} \bar{E}_b \bar{\cdot} \bar{\nabla}_{\bar{E}_b}\bar{E}_a \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi} = -e^{-u}\{q\bar{\nabla}_{\overline{grad_{\nabla}(u)}}\bar{\Psi} + \sum_a \bar{E}_a \bar{\cdot} \overline{grad_{\nabla}(u)} \bar{\cdot} \bar{\nabla}_{\bar{E}_a}\bar{\Psi}\},$$

$$\sum_a \bar{E}_a \bar{\cdot} (\bar{\nabla}_{\bar{E}_a}\kappa_{\bar{g}}) \bar{\cdot} \bar{\Psi} = e^{-2u}\{\sum_a \overline{E_a \cdot \nabla_{E_a}\kappa \cdot \Psi} + (2-q)\kappa(u)\bar{\Psi}\}.$$

From the above equations, we have

$$
\begin{aligned}
\bar{D}_{tr}^2\bar{\Psi} &= -\sum_a \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\sum_a \bar{\nabla}_{\bar{E}_a}\bar{E}_a}\bar{\Psi} + \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi} \\
&\quad + \sum_{a<b} \bar{E}_a \bar{\cdot} \bar{E}_b \bar{\cdot} \bar{R}^S(\bar{E}_a, \bar{E}_b)\bar{\Psi} + \frac{1}{2}(q-2)\kappa_{\bar{g}}(u)\Psi + \frac{1}{4}|\bar{\kappa}|^2\bar{\Psi}.
\end{aligned}
$$

This completes the proof. □

**Lemma 3.15** ([12]) *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric $g_M$ with respect to $\mathcal{F}$. Then*

$$
\ll \bar{\nabla}_{tr}^*\bar{\nabla}_{tr}\bar{\Psi}, \bar{\Phi} \gg_{\bar{g}_Q} = \ll \bar{\nabla}_{tr}\bar{\Psi}, \bar{\nabla}_{tr}\bar{\Phi} \gg_{\bar{g}_Q} \tag{3.32}
$$

*for all $\Phi, \Psi \in S(\mathcal{F})$, where $< \bar{\nabla}_{tr}\bar{\Psi}, \bar{\nabla}_{tr}\bar{\Phi} >_{\bar{g}_Q} = \sum_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Phi} >_{\bar{g}_Q}$.*

**Proof.** Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$ for all $a$. Then we have that at $x$



$$
\bar{\nabla}_{\bar{E}_a}\bar{E}_b = e^{-2u}\{E_b(u)E_a - \delta_{ab}grad_{\nabla}(u)\}. \tag{3.33}
$$

Hence we have

$$
\begin{aligned}
< \bar{\nabla}_{tr}^*\bar{\nabla}_{tr}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} &= -\sum_a < \bar{\nabla}_{\bar{E}_a}\bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} + < \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} \\
&\quad + (1-q)e^{-2u} < \bar{\nabla}_{grad_{\nabla}(u)}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} \\
&= -\sum_a \bar{E}_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} + < \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} \\
&\quad + (1-q)e^{-2u} < \bar{\nabla}_{grad_{\nabla}(u)}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} \\
&\quad + \sum_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Phi} >_{\bar{g}_Q} \\
&= -div_{\bar{\nabla}}(V) + \sum_a < \bar{\nabla}_{\bar{E}_a}\bar{\Psi}, \bar{\nabla}_{\bar{E}_a}\bar{\Phi} >_{\bar{g}_Q} \\
&\quad + < \bar{\nabla}_{\kappa_{\bar{g}}}\bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q},
\end{aligned}
$$

where $V \in \Gamma Q \otimes \mathbb{C}$ are defined by $\bar{g}_Q(V, Z) =< \bar{\nabla}_Z \bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q}$ for all $Z \in \Gamma Q$. The last line is proved as follows: At $x \in M$,

$$\begin{aligned} div_{\bar{\nabla}}(V) &= \sum_a \bar{g}_Q(\bar{\nabla}_{\bar{E}_a} V, \bar{E}_a) = \sum_a \bar{E}_a \bar{g}_Q(V, \bar{E}_a) - \bar{g}_Q(V, \sum_a \bar{\nabla}_{\bar{E}_a} \bar{E}_a) \\ &= \sum_a \bar{E}_a < \bar{\nabla}_{\bar{E}_a} \bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} -(1-q)e^{-2u} < \bar{\nabla}_{grad_{\nabla}(u)} \bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} . \end{aligned}$$

By the transversal divergence theorem on the foliated Riemannian manifold([19,23])

$$\int_M div_{\bar{\nabla}}(V) v_{\bar{g}} = \int_M \bar{g}_Q(\kappa_{\bar{g}}, V) v_{\bar{g}} = \int_M < \bar{\nabla}_{\kappa_{\bar{g}}} \bar{\Psi}, \bar{\Phi} >_{\bar{g}_Q} v_{\bar{g}},$$

where $v_{\bar{g}}$ is the volume form associated to the metric $\bar{g}_M = g_L + \bar{g}_Q$. By integrating, we obtain our result. □

# 4 Eigenvalue estimate of the basic Dirac operator

## 4.1 Eigenvalue estimate I

Let $(M, \tilde{g}_M, \mathcal{F}, S(\mathcal{F}))$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 2$. Let $\tilde{g}_M$ be the bundle-like metric for which the mean curvature $\kappa$ is basic-harmonic, i.e., $\kappa \in \Omega^1_B(\mathcal{F})$ and $\delta_B \kappa = 0$.

Now, we introduce a new connection $\overset{l}{\nabla}$ on $S(\mathcal{F})$ as followings.

**Definition 4.1** Let $l$ be a linear symmetric endomorphism of $Q$. For any tangent vector field $X$ and any spinor field $\Psi$, we define the *modified connection* $\overset{l}{\nabla}$ on $S(\mathcal{F})$ by



$$\overset{l}{\nabla}_X \Psi = \nabla_X \Psi + l(\pi(X)) \cdot \Psi \tag{4.1}$$

where $\pi : TM \to Q$.

**Proposition 4.2** *The connection* $\overset{l}{\nabla}$ *is a metric connection. For any tangent vector field* $X$*, and any spinor fields* $\Psi$ *and* $\Phi$*, one has*

$$X g_Q(\Psi, \Phi) = g_Q(\overset{l}{\nabla}_X \Psi, \Phi) + g_Q(\Psi, \overset{l}{\nabla}_X \Phi).$$

**Proof.** By the direct calculation, we have

$$\begin{aligned} X g_Q(\Psi, \Phi) &= g_Q(\nabla_X \Psi, \Phi) + g_Q(\Psi, \nabla_X \Phi) \\ &= g_Q(\overset{l}{\nabla}_X \Psi - l(\pi(X)) \cdot \Psi, \Phi) + g_Q(\Psi, \overset{l}{\nabla}_X \Phi - l(\pi(X)) \cdot \Phi) \\ &= g_Q(\overset{l}{\nabla}_X \Psi, \Phi) + g_Q(\Psi, \overset{l}{\nabla}_X \Phi). \qquad \square \end{aligned}$$

We now define $\overset{l}{\nabla}{}^*{}_{tr}\overset{l}{\nabla}_{tr}$: $\Gamma S(\mathcal{F}) \to \Gamma S(\mathcal{F})$ as

$$\overset{l}{\nabla}{}^*{}_{tr}\overset{l}{\nabla}_{tr}\ \Psi = -\sum_a \overset{l}{\nabla}{}^2{}_{Ea,Ea}\ \Psi + \overset{l}{\nabla}_\kappa\ \Psi \tag{4.2}$$

where $\overset{l}{\nabla}{}^2{}_{v,w} = \overset{l}{\nabla}_v\overset{l}{\nabla}_w - \overset{l}{\nabla}_{\nabla_v w}$ for any $v, w \in TM$. Then we have the following lemma.

**Lemma 4.3** *Let $(M, \tilde{g}_M, \mathcal{F})$ be a compact Riemanian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric $\tilde{g}_M$ with respect to $\mathcal{F}$. Then*

$$\ll \overset{l}{\nabla}{}^*{}_{tr}\overset{l}{\nabla}_{tr}\ \Phi, \Psi \gg = \ll \overset{l}{\nabla}_{tr}\ \Phi, \overset{l}{\nabla}_{tr}\ \Psi \gg \tag{4.3}$$

*for all $\Phi, \Psi \in \Gamma S(\mathcal{F})$, where $\ll \Phi, \Psi \gg = \int_M < \Phi, \Psi >$ is the (Complex Hermitian) inner product on $S(\mathcal{F})$.*



**Proof.** Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$ for all $a$. Then we have at the point $x$ that for any $\Phi, \Psi$,

$$\begin{aligned}
< \overset{l}{\nabla}{}^*{}_{tr}\ \overset{l}{\nabla}_{tr}\ \Phi, \Psi > &= -\sum_a <\overset{l}{\nabla}_{E_a}\overset{l}{\nabla}_{E_a}\ \Phi, \Psi> + <\overset{l}{\nabla}_\kappa\ \Phi, \Psi> \\
&= -\sum_a E_a <\overset{l}{\nabla}_{E_a}\ \Phi, \Psi> + \sum_a <\overset{l}{\nabla}_{E_a}\ \Phi, \overset{l}{\nabla}_{E_a}\ \Psi> \\
&\quad + < \nabla_\kappa \Phi, \Psi > + < l(\kappa)\cdot\Phi, \Psi > \\
&= -\sum_a E_a < \nabla_{E_a}\Phi, \Psi > - \sum_a E_a < l(E_a)\cdot\Phi, \Psi > \\
&\quad + < \nabla_\kappa \Phi, \Psi > + < l(\kappa)\cdot\Phi, \Psi > \\
&\quad + \sum_a <\overset{l}{\nabla}_{E_a}\ \Phi, \overset{l}{\nabla}_{E_a}\ \Psi> .
\end{aligned}$$

So,

$$\begin{aligned} < \overset{l}{\nabla}{}^*{}_{tr} \overset{l}{\nabla}_{tr} \Phi, \Psi > &= -div_{\nabla}(V) - div_{\nabla}(W) + \sum_a < \overset{l}{\nabla}_{E_a} \Phi, \overset{l}{\nabla}_{E_a} \Psi > \\ &\quad + < \nabla_\kappa \Phi, \Psi > + < l(\kappa) \cdot \Phi, \Psi >, \end{aligned}$$

where $V, W \in \Gamma Q \otimes \mathbb{C}$ are defined by the condition that $g_Q(V, Z) =< \nabla_Z \Phi, \Psi >$ and $g_Q(W, Z) =< l(Z) \cdot \Phi, \Psi >$ for all $Z \in \Gamma Q$. The last line is proved as followings: At $x \in M$,

$$\begin{aligned} div_{\nabla}(V) &= \sum_a g_Q(\nabla_{E_a} V, E_a) = \sum_a E_a g_Q(V, E_a) \\ &= \sum_a E_a < \nabla_{E_a} \Phi, \Psi > . \end{aligned} \tag{4.4}$$

Similarly, we have

$$\begin{aligned} div_{\nabla}(W) &= \sum_a g_Q(\nabla_{E_a} W, E_a) = \sum_a E_a g_Q(W, E_a) \\ &= \sum_a E_a < l(E_a) \cdot \Phi, \Psi > . \end{aligned} \tag{4.5}$$



By the transversal divergence theorem([19,23]) on the foliated Riemannian manifold, we have

$$\int_M div_{\nabla}(V) =\ll \kappa, V \gg=\ll \nabla_\kappa \Phi, \Psi \gg,$$

and

$$\int_M div_{\nabla}(W) =\ll l(\kappa) \cdot \Phi, \Psi \gg .$$

By integrating,

$$\begin{aligned} \int_M < \overset{l}{\nabla}{}^*{}_{tr} \overset{l}{\nabla}_{tr} \Phi, \Psi > &= -\int_M div_{\nabla}(V) - \int_M div_{\nabla}(W) \\ &\quad + \int_M < \nabla_\kappa \Phi, \Psi > + \int_M < l(\kappa) \cdot \Phi, \Psi > \\ &\quad + \int_M \sum_a < \overset{l}{\nabla}_{E_a} \Phi, \overset{l}{\nabla}_{E_a} \Psi > . \end{aligned}$$

Hence, we obtain our result. □

**Proposition 4.4** *For any linear symmetric endomorphism $l$ of $Q$, and for any spinor field $\Psi$ , the following identity holds:*

$$\begin{aligned} |\overset{l}{\nabla}_{tr}\Psi|^2 &:= \sum_a <\overset{l}{\nabla}_{E_a}\Psi, \overset{l}{\nabla}_{E_a}\Psi> \\ &= |\nabla_{tr}\Psi|^2 - 2Re\sum_a < l(E_a)\cdot\nabla_{E_a}\Psi, \Psi> + |l|^2|\Psi|^2. \end{aligned} \tag{4.6}$$

**Proof.** Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$ for all $a$. Then we have at the point $x$ that for any $\Psi$,

$$\begin{aligned} |\overset{l}{\nabla}_{tr}\Psi|^2 &:= \sum_a <\overset{l}{\nabla}_{E_a}\Psi, \overset{l}{\nabla}_{E_a}\Psi> \\ &= \sum_a < \nabla_{E_a}\Psi + l(\pi(E_a))\cdot\Psi, \nabla_{E_a}\Psi + l(\pi(E_a))\cdot\Psi> \\ &= \sum_a < \nabla_{E_a}\Psi, \nabla_{E_a}\Psi> + \sum_a < \nabla_{E_a}\Psi, l(E_a)\cdot\Psi> \\ &\quad + \sum_a < l(E_a)\cdot\Psi, \nabla_{E_a}\Psi> + \sum_a < l(E_a)\cdot\Psi, l(E_a)\cdot\Psi> \\ &= |\nabla_{tr}\Psi|^2 - \sum_a\{< l(E_a)\nabla_{E_a}\Psi, \Psi> + <\Psi, l(E_a)\nabla_{E_a}\Psi>\} \\ &\quad + |l|^2|\Psi|^2 \\ &= |\nabla_{tr}\Psi|^2 - 2Re\sum_a < l(E_a)\cdot\nabla_{E_a}\Psi, \Psi> + |l|^2|\Psi|^2. \qquad \square \end{aligned}$$

We now show that for an appropriate choice of the symmetric endomorphism $l$, one gets a sharp estimate of the first eigenvalue of the basic Dirac operator on compact foliated Riemannian manifolds. For this, we need the following:

**Definition 4.5** On the complement of the set of zeroes of a spinor field $\Psi \in \Gamma_B S(\mathcal{F})$, define for any tangent vector fields $X$ and $Y$, the *symmetric* *bilinear tensor* $F_\Psi$ by

$$F_\Psi(X,Y) = \frac{1}{2} Re < \pi(X) \cdot \nabla_Y \Psi + \pi(Y) \cdot \nabla_X \Psi, \Psi/|\Psi|^2 >, \tag{4.7}$$

where $\pi : TM \to Q$.

Let $l_\Psi$ be a symmetric linear map associated to $F_\Psi$. Namely, it follows that for any $X, Y \in \Gamma Q$

$$F_\Psi(X,Y) = g_Q(l_\Psi(X), Y). \tag{4.8}$$

Since $< \kappa \cdot \Psi, \Psi >$ is pure imaginary, we have

$$\begin{aligned} \text{tr } l_\Psi &= \sum_a Re < E_a \cdot \nabla_{E_a} \Psi, \Psi/|\Psi|^2 > \\ &= Re < D_b \Psi, \Psi/|\Psi|^2 > . \end{aligned}$$

It is trivial that if $\Psi$ satisfies $D_b \Psi = \lambda \Psi$, then



$$\text{tr } l_\Psi = \lambda. \tag{4.9}$$

On the other hand, since $l$ is linear symmetric endomorphism, we have

$$\begin{aligned} Re \sum_a < l(E_a) \cdot \nabla_{E_a} \Psi, \Psi > &= \sum_a < l(E_a), l_\Psi(E_a) > |\Psi|^2 \\ &= < l, l_\Psi > |\Psi|^2. \end{aligned} \tag{4.10}$$

From (4.6) and (4.10), we have the following equation;

$$| \overset{l}{\nabla}_{tr} \Psi|^2 = |\nabla_{tr} \Psi|^2 - f(l)|\Psi|^2, \tag{4.11}$$

where $f(l) = 2 < l, l_\Psi > -|l|^2$. Note that $f(l)$ has maximum value $|l_\Psi|^2$ at $l = l_\Psi$ because $l$ is linear endomorphism. From (3.14) and (4.11), we have that for any eigenspinor $\Psi$ corresponding to an eigenvalue $\lambda$

$$\int_M | \overset{l_\Psi}{\nabla}_{tr} \Psi|^2 = \int_M \{\lambda^2 - (\frac{1}{4} K^\sigma + |l_\Psi|^2)\} |\Psi|^2. \tag{4.12}$$

Hence we have the following theorem.

**Theorem 4.6** *Let $(M, \tilde{g}_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric $\tilde{g}_M$. Then any eigenvalue $\lambda$ of the basic Dirac operator $D_b$ corresponding to the eigenspinor $\Psi \in \Gamma S(\mathcal{F})$ satisfies*

$$\lambda^2 \geq \inf_M (\frac{1}{4} K^\sigma + |l_\Psi|^2), \tag{4.13}$$

*where $K^\sigma = \sigma^\nabla + |\kappa|^2$.*

Moreover, by the Cauchy-Schwarz inequality we have

$$\begin{aligned} \frac{1}{q}(\mathrm{tr} l_\Psi)^2 &= \frac{1}{q}(\sum_a < l_\Psi(E_a), E_a >)^2 \\ &\leq \frac{1}{q} \cdot q \sum_a |l_\Psi(E_a)|^2 |E_a|^2 \\ &= \sum_a |l_\Psi(E_a)|^2 = |l_\Psi|^2. \end{aligned}$$



Hence from (4.9), we have

$$|l_\Psi|^2 \geq \frac{\lambda^2}{q}. \tag{4.14}$$

Hence we have the following corollary.

**Corollary 4.7** (cf.[11]) *Under the same conditions as in Theorem 4.6, one has*

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M K^\sigma. \tag{4.15}$$

## 4.2 Eigenvalue estimate II

Now, we introduce a connection $\overset{l}{\bar{\nabla}}$ on $\bar{S}(\mathcal{F})$, as

$$\overset{l}{\bar{\nabla}}_X \bar{\Psi} = \bar{\nabla}_X \bar{\Psi} + l(\pi(X)) \,\bar{\cdot}\, \bar{\Psi} \quad \text{for } X \in TM, \tag{4.16}$$

where $l$ is a linear symmetric endomorphism on $Q$. Trivially, the connection $\overset{l}{\bar{\nabla}}$ is a metric connection and $\overset{l}{\bar{\nabla}}{}^*{}_{tr}\overset{l}{\bar{\nabla}}_{tr}$ is positive definite.

**Lemma 4.8** *On the foliated spinor bundle $\bar{S}(\mathcal{F})$, we have*

$$\ll \overset{l}{\bar{\nabla}}{}^*{}_{tr}\overset{l}{\bar{\nabla}}_{tr}\ \bar{\Psi}, \bar{\Phi} \gg_{\bar{g}_Q} = \ll \overset{l}{\bar{\nabla}}_{tr}\ \bar{\Psi}, \overset{l}{\bar{\nabla}}_{tr}\ \bar{\Phi} \gg_{\bar{g}_Q}$$

*for all $\Psi, \Phi \in \Gamma S(\mathcal{F})$.*

On the other hand, we obtain the following lemma.

**Lemma 4.9** *On the spinor bundle $\bar{S}(\mathcal{F})$ associated with the metric $\bar{g}_Q = e^{2u}g_Q$*

$$F_{\bar{\Phi}}(\bar{X}, \bar{Y}) = e^{-u}F_{\Phi}(X, Y) = e^{-u}F_{\Psi}(X, Y) \tag{4.17}$$

*for any $X, Y \in \Gamma Q$, where $\Phi = e^{-\frac{q-1}{2}u}\Psi$.*



**Proof.** By definition, we have

$$\begin{aligned}
F_{\bar{\Phi}}(\bar{X}, \bar{Y}) &= \frac{1}{2}Re < \bar{X}\bar{\cdot}\bar{\nabla}_{\bar{Y}}\bar{\Phi} + \bar{Y}\bar{\cdot}\bar{\nabla}_{\bar{X}}\bar{\Phi}, \bar{\Phi}/|\bar{\Phi}|^2_{\bar{g}_Q} >_{\bar{g}_Q} \\
&= \frac{1}{2}e^{-u}Re < \bar{X}\bar{\cdot}\{\overline{\nabla_Y\Phi} - \frac{1}{2}\overline{Y \cdot grad_{\nabla}(u) \cdot \Phi} \\
&\quad -\frac{1}{2}g_Q(grad_{\nabla}(u), Y)\bar{\Phi}\} \\
&\quad +\bar{Y}\bar{\cdot}\{\overline{\nabla_X\Phi} - \frac{1}{2}\overline{X \cdot grad_{\nabla}(u) \cdot \Phi} \\
&\quad -\frac{1}{2}g_Q(grad_{\nabla}(u), X)\bar{\Phi}\}, \bar{\Phi}/|\bar{\Phi}|^2_{\bar{g}_Q} >_{\bar{g}_Q} .
\end{aligned}$$

From (3.16) and (3.17), we have

$$\begin{aligned}
F_{\bar{\Phi}}(\bar{X}, \bar{Y}) &= \frac{1}{2}e^{-u}Re < \overline{X \cdot \nabla_Y\Phi + Y \cdot \nabla_X\Phi} \\
&\quad \overline{g_Q(X, Y)grad_{\nabla}(u) \cdot \Phi} - \frac{1}{2}g_Q(grad_{\nabla}(u), Y)\overline{X \cdot \Phi} \\
&\quad -\frac{1}{2}g_Q(grad_{\nabla}(u), X)\overline{Y \cdot \Phi}, \bar{\Phi}/|\bar{\Phi}|^2_{\bar{g}_Q} >_{\bar{g}_Q} .
\end{aligned}$$

Since $\tilde{I}_u$ is isometry and $< X \cdot \Psi, \Psi >$ is pure imaginary, it follows that

$$\begin{aligned} F_{\bar{\Phi}}(\bar{X}, \bar{Y}) &= \frac{1}{2} e^{-u} Re < X \cdot \nabla_Y \Phi + Y \cdot \nabla_X \Phi, \Phi / |\Phi|^2 > \\ &= e^{-u} F_{\Phi}(X, Y) \end{aligned}$$

On the other hand, since $\Phi = e^{-(q-1)u/2}\Psi$ by direct calculation we have

$$F_{\Phi}(X, Y) = F_{\Psi}(X, Y).$$

Hence the proof is completed. □

From (4.17), we have the following identity

$$l_{\bar{\Phi}} = e^{-u} l_{\Phi} = e^{-u} l_{\Psi}. \tag{4.18}$$

From (4.18), we have the following proposition (see [10] for the details).

**Proposition 4.10** *The following relations hold:*

$$|l_{\bar{\Phi}}|^2_{\bar{g}_Q} = e^{-2u} |l_{\Phi}|^2 = e^{-2u} |l_{\Psi}|^2, \tag{4.19}$$



*where* $\Phi = e^{-\frac{q-1}{2}u}\Psi$.

By using the connection $\overset{l_{\bar{\Phi}}}{\bar{\nabla}}$, we can obtain the following equation

$$\begin{aligned} |\overset{l_{\bar{\Phi}}}{\bar{\nabla}}_{tr} \bar{\Phi}|^2_{\bar{g}_Q} &= \sum_a < \overset{l_{\bar{\Phi}}}{\bar{\nabla}}_{\bar{E}_a} \bar{\Phi}, \overset{l_{\bar{\Phi}}}{\bar{\nabla}}_{\bar{E}_a} \bar{\Phi} >_{\bar{g}_Q} \\ &= |\bar{\nabla}_{tr} \bar{\Phi}|^2 - 2Re \sum_a < l_{\bar{\Phi}}(\bar{E}_a) \bar{\cdot} \bar{\nabla}_{\bar{E}_a} \bar{\Phi}, \bar{\Phi} >_{\bar{g}_Q} + |l_{\bar{\Phi}}|^2|_{\bar{g}_Q} |\bar{\Phi}|_{\bar{g}_Q}. \end{aligned}$$

Since

$$\begin{aligned} &Re \sum_a < l_{\bar{\Phi}}(\bar{E}_a) \bar{\cdot} \bar{\nabla}_{\bar{E}_a} \bar{\Phi}, \bar{\Phi} >_{\bar{g}_Q} \\ &= \frac{1}{2} Re \sum_a < \bar{E}_a \bar{\cdot} \bar{\nabla}_{l_{\bar{\Phi}}(\bar{E}_a)} \bar{\Phi} + l_{\bar{\Phi}}(\bar{E}_a) \bar{\cdot} \bar{\nabla}_{\bar{E}_a} \bar{\Phi}, \bar{\Phi} >_{\bar{g}_Q} \\ &= \sum_a F_{\bar{\Phi}}(\bar{E}_a, l_{\bar{\Phi}}(\bar{E}_a)) |\bar{\Phi}|^2_{\bar{g}_Q} = \sum_a < l_{\bar{\Phi}}(\bar{E}_a), l_{\bar{\Phi}}(\bar{E}_a) >_{\bar{g}_Q} |\bar{\Phi}|^2_{\bar{g}_Q} \\ &= |l_{\bar{\Phi}}|^2_{\bar{g}_Q} |\bar{\Phi}|^2_{\bar{g}_Q}. \end{aligned}$$

Summing up the above equations, we have

$$| \overset{l_{\bar\Phi}}{\bar\nabla}_{tr} \bar\Phi|^2_{\bar g_Q} = |\bar\nabla_{tr}\bar\Phi|^2 - |l_{\bar\Phi}|^2_{\bar g_Q}|\bar\Phi|^2_{\bar g_Q}. \tag{4.20}$$

Hence we have from (3.27), (3.32) and (4.20)

$$\int_M | \overset{l_{\bar\Phi}}{\bar\nabla}_{tr} \bar\Phi|^2_{\bar g_Q} = \int_M \{< \bar D^2_{tr}\bar\Phi, \bar\Phi >_{\bar g_Q} - \frac{1}{4} < K^{\bar\nabla}_\sigma \bar\Phi, \bar\Phi >_{\bar g_Q} - |l_{\bar\Phi}|^2_{\bar g_Q}|\bar\Phi|^2_{\bar g_Q}\}. \tag{4.21}$$

where $K^{\bar\nabla}_\sigma = \sigma^{\bar\nabla} + 4K^{\bar\nabla}$.

Let $D_b\Psi = \lambda\Psi (\Psi \neq 0)$. From Proposition 3.12, we have

$$\bar D_b \bar\Phi = \lambda e^{-u}\bar\Phi, \tag{4.22}$$

where $\Phi = e^{-\frac{q-1}{2}u}\Psi$. From (4.19) and (4.22), we have that for any eigenspinor $\Psi$ corresponding to the eigenvalue $\lambda$



$$\begin{aligned}\int | \overset{l_{\bar\Phi}}{\bar\nabla}_{tr} \bar\Phi|^2_{\bar g_Q} &= \int e^{-2u}\{\lambda^2 - \frac{1}{4}(e^{2u}K^{\bar\nabla}_\sigma + 4|l_\Psi|^2)\}|\bar\Phi|^2_{\bar g_Q} \\ &\leq \int e^{-2u}\{\lambda^2 - \frac{1}{4}\inf_M(e^{2u}K^{\bar\nabla}_\sigma + 4|l_\Psi|^2)\}|\bar\Phi|^2_{\bar g_Q}\end{aligned} \tag{4.23}$$

where $\Phi = e^{-\frac{q-1}{2}u}\Psi$. From (4.23), we have that $\lambda^2 \geq \frac{1}{4}\inf_M(e^{2u}K^{\bar\nabla}_\sigma + 4|l_\Psi|^2)$. Hence we have the following theorem.

**Theorem 4.11** *Let $(M, \tilde g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 2$ and bundle-like metric $\tilde g_M$. Assume that $K^{\bar\nabla}_\sigma \geq 0$ for some transversally conformal metric $\bar g_Q = e^{2u}g_Q$. Then any eigenvalue $\lambda$ of the basic Dirac operator $D_b$ corresponding to the eigenspinor $\Psi \in \Gamma S(\mathcal{F})$ satisfies*

$$\lambda^2 \geq \frac{1}{4}\sup_u \inf_M (e^{2u}K^{\bar\nabla}_\sigma + 4|l_\Psi|^2), \tag{4.24}$$

*where $K^{\bar\nabla}_\sigma = \sigma^{\bar\nabla} + |\bar\kappa|^2 + 2(q-2)\kappa_{\bar g}(u)$.*

From (4.14), we have the following corollary.

**Corollary 4.12** (cf.[12]) *Under the same assumption as in Theorem 4.11, we have*

$$\lambda^2 \geq \frac{q}{4(q-1)} \sup_u \inf_M (e^{2u} K_\sigma^{\bar\nabla}). \tag{4.25}$$

The transversal Ricci curvature $\rho^{\bar\nabla}$ of $\bar g_Q = e^{2u} g_Q$ and the transversal scalar curvature $\sigma^{\bar\nabla}$ of $\bar g_Q$ are related to the transversal Ricci curvature $\rho^{\nabla}$ of $g_Q$ and the transversal scalar curvature $\sigma^{\nabla}$ of $g_Q$ by the following lemma.

**Lemma 4.13** *On a Riemannian foliation $\mathcal{F}$, we have that for any $X \in Q$,*

$$\begin{aligned} e^{2u}\rho^{\bar\nabla}(X) =& \rho^{\nabla}(X) + (2-q)\nabla_X grad_\nabla(u) + (2-q)|grad_\nabla(u)|^2 X \\ &+ (q-2)X(u)grad_\nabla(u) + \{\Delta_B u - \kappa(u)\}X. \end{aligned} \tag{4.26}$$

$$e^{2u}\sigma^{\bar\nabla} = \sigma^{\nabla} + (q-1)(2-q)|grad_\nabla(u)|^2 + 2(q-1)\{\Delta_B u - \kappa(u)\}. \tag{4.27}$$

**Proof.** Let $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ with the property that $(\nabla E_a)_x = 0$ for all $a$. Then

$$\begin{aligned} \rho^{\bar\nabla}(X) &= \sum_a R^{\bar\nabla}(X, \bar E_a)\bar E_a \\ &= \sum_a \bar\nabla_X \bar\nabla_{E_a} \bar E_a - \sum_a \bar\nabla_{\bar E_a} \bar\nabla_X \bar E_a - \sum_a \bar\nabla_{[X,\bar E_a]} \bar E_a. \end{aligned}$$

By direct calculation, we have

$$\begin{aligned} e^{2u}\sum_a \bar\nabla_X \bar\nabla_{\bar E_a} \bar E_a &= (1-q)\{\nabla_X grad_\nabla(u) + |grad_\nabla(u)|^2 X \\ &\quad -2X(u)grad_\nabla(u)\} + \sum_a \nabla_X \nabla_{E_a} E_a. \end{aligned}$$

Similarly,

$$\begin{aligned} e^{2u}\sum_a \bar\nabla_{\bar E_a}\bar\nabla_X \bar E_a &= \sum_a \nabla_{E_a}\nabla_X E_a + \sum_a E_aE_a(u)X \\ &+\nabla_{grad_\nabla(u)}X - \sum_a g(\nabla_{E_a}X, E_a)grad_\nabla(u) \\ &-\nabla_X grad_\nabla(u) - |grad_\nabla(u)|^2X - X(u)grad_\nabla(u). \end{aligned}$$

and

$$\begin{aligned} e^{2u}\sum_a \bar\nabla_{[X,\bar E_a]}\bar E_a &= \sum_a \nabla_{[X,E_a]}E_a + X(u)(q-1)grad_\nabla(u) \\ &-\nabla_{grad_\nabla(u)}X + \sum_a g(\nabla_{E_a}X, E_a)grad_\nabla(u). \end{aligned}$$

Since $\Delta_B u = \delta_B d_B u = -\sum_a E_aE_a(u) + i(\kappa)d_B u$, the above equations give (4.26).

On the other hand,

$$\sigma^{\bar\nabla} = \sum_a \bar g_Q(\rho^{\bar\nabla}(\bar E_a), \bar E_a) = \sum_a g_Q(\rho^{\bar\nabla}(E_a), E_a).$$

From (4.26) we have

$$\begin{aligned} e^{2u}\sigma^{\bar\nabla} &= \sum_a g_Q(e^{2u}\rho^{\bar\nabla}(E_a), E_a) \\ &= \sigma^{\nabla} + (2-q)\sum_a g_Q(\nabla_{E_a}grad_\nabla(u), E_a) \\ &\quad +(q-1)(2-q)|grad_\nabla(u)|^2 + q\{\Delta_B u - \kappa(u)\}. \end{aligned}$$

Since $\sum_a g_Q(\nabla_{E_a}grad_\nabla(u), E_a) = \sum_a E_aE_a(u) = -\Delta_B u + \kappa(u)$, we have

$$e^{2u}\sigma^{\bar\nabla} = \sigma^{\nabla} + (q-1)(2-q)|grad_\nabla(u)|^2 + 2(q-1)\{\Delta_B u - \kappa(u)\},$$

which proves (4.27). □

Since $K_\sigma^{\bar\nabla} = \sigma^{\bar\nabla} + |\bar\kappa|^2 + 2(q-2)\kappa_{\bar g}(u)$, from (4.27), we have

$$e^{2u}K_\sigma^{\bar\nabla} = \sigma^\nabla + |\kappa|^2 + 2(q-1)\Delta_B u + (q-1)(2-q)|grad_\nabla(u)|^2 - 2\kappa(u). \tag{4.28}$$

On the other hand, for $q \geq 3$, if we choose the positive function $h$ by $u = \frac{2}{q-2}\ln h$, then we have

$$\Delta_B u = \frac{2}{q-2}\{h^{-2}|grad_\nabla(h)|^2 + h^{-1}\Delta_B h\}, \tag{4.29}$$

$$|grad_\nabla(u)|^2 = (\frac{2}{q-2})^2 h^{-2}|grad_\nabla(h)|^2. \tag{4.30}$$

From (4.28), (4.29) and (4.30), we have

$$e^{2u}K_\sigma^{\bar\nabla} = h^{\frac{4}{q-2}}K_\sigma^{\bar\nabla} = h^{-1}Y_b h + |\kappa|^2 - \frac{4}{q-2}h^{-1}\kappa(h), \tag{4.31}$$

where

$$Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^\nabla, \tag{4.32}$$



which is called a *basic Yamabe operator* of $\mathcal{F}$.

Now we put $\mathcal{K}_u = \{u \in \Omega_B^0(\mathcal{F}) | \kappa(u) = 0\}$. If we choose $u \in \mathcal{K}_u$, then $\kappa(h) = 0 = \kappa(u)$. From (4.28) and (4.31), we have

$$e^{2u}K_\sigma^{\bar\nabla} = K^\sigma + 2(q-1)\Delta_B u + (q-1)(2-q)|grad_\nabla(u)|^2 = h^{-1}Y_b h + |\kappa|^2, \tag{4.33}$$

where $K^\sigma = \sigma^\nabla + |\kappa|^2$. Assume that the transversal scalar curvature $\sigma^\nabla$ is non-negative. Then the eigenfunction $h_1$ associated to the first eigenvalue $\mu_1$ of $Y_b$ can be chosen to be positive and then $\mu_1$ is non-negative. Thus

$$h_1^{-1}Y_b h_1 = \mu_1. \tag{4.34}$$

Since $\sup\inf\{h^{-1}Y_b h\} \geq \mu_1$, we have from (4.24) in Theorem 4.11 the following Theorem.

**Theorem 4.14** *Let $(M, \tilde{g}_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric $\tilde{g}_M$ . Then, any eigenvalue $\lambda$ of the basic Dirac operator corresponding to the eigenspinor $\Psi \in \Gamma S(\mathcal{F})$ satisfies*

$$\lambda^2 \geq \frac{1}{4}(\mu_1 + \inf_M |\kappa|^2) + \inf_M |l_\Psi|^2 \tag{4.35}$$

*where $\mu_1$ is the first eigenvalue of the basic Yamabe operator of $\mathcal{F}$*

$$Y_b = 4\frac{q-1}{q-2}\Delta_B + \sigma^\nabla \tag{4.36}$$

*acting on functions, and $l_\Psi$ is the field of symmetric endomorphism associated with $F_\Psi$.*

From (4.14), we have the following corollary.



**Corollary 4.15** (cf.[12]) *Let $(M, \tilde{g}_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric $\tilde{g}_M$. If the transversal scalar curvature satisfies $\sigma^\nabla \geq 0$, then any eigenvalue $\lambda$ of $D_b$ satisfies*

$$\lambda^2 \geq \frac{q}{4(q-1)}(\mu_1 + \inf |\kappa|^2). \tag{4.37}$$

**Remark.** Since $\mu_1 \geq \inf \sigma^\nabla$, the inequality (4.37) is a sharper estimate than the previous one (1.4). Moreover, Corollary 4.15 is a specialization of the result on an ordinary manifold by O. Hijazi ([10]) to the case of Riemannian foliations.

# 5 The limiting cases

In this chapter, we study the limiting foliations of (4.13) and (4.35).

## 5.1 The limiting case I

For any linear symmetric endomorphism $l$ of $Q$, we define $Ric^l_\nabla : \Gamma Q \otimes S(\mathcal{F}) \to S(\mathcal{F})$ ([10,11]) by

$$Ric^l_\nabla(X \otimes \Psi) = \sum_a E_a \cdot R^l(X, E_a)\Psi, \tag{5.1}$$

where $R^l$ is the curvature tensor with respect to $\overset{l}{\nabla}$ defined by (4.1).

**Lemma 5.1** *On the transverse spin foliation $\mathcal{F}$, we have that for $X \in \Gamma Q$ and $\Psi \in \Gamma S(\mathcal{F})$*



$$Ric^l_\nabla(X \otimes \Psi) = -\frac{1}{2}\rho^\nabla(X)\cdot\Psi + \sum_a \{E_a \cdot dl(X, E_a) + E_a \cdot [l(X), l(E_a)]\} \cdot \Psi, \tag{5.2}$$

*where $[X, Y] = X \cdot Y - Y \cdot X$ and $dl(X, Y) = (\nabla_X l)(Y) - (\nabla_Y l)(X)$.*

**Proof.** Fix $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ such that $(\nabla E_a)_x = 0$ for all $a$. For $X \in \Gamma Q$, we have

$$\begin{aligned}
\overset{l}{\nabla}_X \overset{l}{\nabla}_{E_a} \Psi &= \overset{l}{\nabla}_X \{\nabla_{E_a}\Psi + l(E_a)\cdot\Psi\} \\
&= \nabla_X\nabla_{E_a}\Psi + \nabla_X l(E_a)\cdot\Psi + l(E_a)\cdot\nabla_X\Psi + l(X)\cdot\nabla_{E_a}\Psi \\
&\quad + l(X)\cdot l(E_a)\cdot\Psi.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\overset{l}{\nabla}_{E_a} \overset{l}{\nabla}_X \Psi &= \nabla_{E_a}\nabla_X\Psi + \nabla_{E_a} l(X)\cdot\Psi + l(X)\cdot\nabla_{E_a}\Psi + l(E_a)\cdot\nabla_X\Psi \\
&\quad + l(E_a)\cdot l(X)\cdot\Psi.
\end{aligned}$$

From the above equations, we have

$$
\begin{aligned}
Ric^l_\nabla(X\otimes\Psi) &= \sum_a E_a\cdot R^l(X,E_a)\Psi\\
&= \sum_a E_a\cdot\{R^S(X,E_a)\Psi + ((\nabla_X l)E_a - (\nabla_{E_a} l)X)\cdot\Psi\\
&\quad +(l(X)\cdot l(E_a) - l(E_a)\cdot l(X))\cdot\Psi\}\\
&= \sum_a E_a\cdot R^S(X,E_a)\Psi + \sum_a E_a\cdot dl(X,E_a)\cdot\Psi\\
&\quad +\sum_a E_a\cdot[l(X),l(E_a)]\cdot\Psi.
\end{aligned}
$$

Hence Theorem 3.7 gives our proof. □

By the definition of Clifford multiplication, we have

$$\sum_a E_a\cdot dl(X,E_a) = \sum_a E_a\wedge dl(X,E_a) - \{X(\mathrm{tr}\ l) - (div_\nabla l)(X)\}, \quad (5.3)$$

where $(div_\nabla l)(X) = \sum_a g_Q((\nabla_{E_a} l)(X), E_a)$. Since $l$ is symmetric,



$$\sum_a E_a\cdot l(E_a) = \sum_a l(E_a)\cdot E_a, \quad (5.4)$$

and then

$$\sum_a E_a\cdot l(E_a) = -\mathrm{tr}\ l. \quad (5.5)$$

From (5.4) and (5.5), we have

$$\sum_a E_a\cdot[l(X),l(E_a)]\cdot\Psi = 2(\mathrm{tr}\ l)l(X)\cdot\Psi - 2l^2(X)\cdot\Psi. \quad (5.6)$$

From (5.2), (5.3) and (5.6), we have the following proposition.

**Proposition 5.2** *On the transverse spin foliation, we have*

$$
\begin{aligned}
Ric^l_\nabla(X\otimes\Psi) = &-\frac{1}{2}\rho^\nabla(X)\cdot\Psi + \sum_a(E_a\wedge dl(X,E_a))\cdot\Psi - 2l^2(X)\cdot\Psi\\
&-\{X(\mathrm{tr}\ l) - (div_\nabla l)(X)\}\cdot\Psi + 2(\mathrm{tr}\ l)l(X)\cdot\Psi.
\end{aligned}
\quad (5.7)
$$

From (5.7), we have the following facts.

**Proposition 5.3** *If $M$ admits a non-zero spinor field $\Psi \in \Gamma S(\mathcal{F})$ with $\overset{l}{\nabla} \Psi = 0$, then $|\Psi|^2$ is constant and*

$$grad_\nabla(\text{tr } l) = div_\nabla l, \tag{5.8}$$

$$(\text{tr } l)^2 = \frac{1}{4}\sigma^\nabla + |l|^2. \tag{5.9}$$

**Proof.** Since $\overset{l}{\nabla}$ is a metric connection, the condition $\overset{l}{\nabla} \Psi = 0$ imply that $|\Psi|^2$ is constant. If $\overset{l}{\nabla}_X \Psi = 0$ for any $X \in \Gamma Q$ , then $Ric^l_\nabla = 0$. Hence from (5.7), we have

$$\begin{aligned}\{X(\text{tr } l) - (div_\nabla l)(X)\}\Psi = -\frac{1}{2}\rho^\nabla(X)\cdot\Psi + \sum_a (E_a \wedge dl(X, E_a))\cdot\Psi \\ + 2(\text{tr } l)l(X)\cdot\Psi - 2l^2(X)\cdot\Psi\end{aligned} \tag{5.10}$$



Then we have that for any $X \in \Gamma Q$,

$$\begin{aligned}&< \{X(\text{tr } l) - (\text{div}_\nabla l)(X)\}\cdot\Psi, \Psi > \\ &=< \{-\frac{1}{2}\rho^\nabla(X) + \sum_a (E_a \wedge dl(X, E_a)) + 2(\text{tr } l)l(X) - 2l^2(X)\}\cdot\Psi, \Psi > .\end{aligned}$$

The left-hand side is real, but the right-hand side is pure imaginary because $\sum_a < E_a \wedge dl(X, E_a)\cdot\Psi, \Psi >$ is pure imaginary. Therefore, both sides are zeros. Hence we get that for any $X \in \Gamma Q$,

$$X(\text{tr } l) = (\text{div}_\nabla l)(X), \tag{5.11}$$

$$\frac{1}{2}\rho^\nabla(X)\cdot\Psi = \{\sum_a (E_a \wedge dl(X, E_a)) + 2(\text{tr } l)l(X) - 2l^2(X)\}\cdot\Psi. \tag{5.12}$$

Hence from (5.11) the equation (5.8) is proved. For the equation (5.9) , Clifford multiplication of (5.12) with $E_b$ and for $X = E_b$, gives

$$\begin{aligned}\frac{1}{2}\sum_b E_b \cdot \rho^{\nabla}(E_b) \cdot \Psi \;&=\; \sum_{a,b} E_b \cdot (E_a \wedge dl(E_b, E_a)) \cdot \Psi \\ &+2(\text{tr } l)\sum_b E_b \cdot l(E_b) \cdot \Psi - 2\sum_b E_b \cdot l^2(E_b) \cdot \Psi.\end{aligned}$$

From Theorem 3.7 and (5.5), we have

$$-\frac{1}{2}\sigma^{\nabla}\Psi = \sum_{a,b} E_a \cdot (E_b \wedge dl(E_a, E_b)) \cdot \Psi - 2(\text{tr } l)^2\Psi + 2|l|^2\Psi. \quad (5.13)$$

On the other hand,

$$\begin{aligned}\sum_{a,b} E_a \cdot (E_b \wedge dl(E_a, E_b)) \cdot \Psi =& (\sum_{a,b} E_a \wedge E_b \wedge dl(E_a, E_b)) \cdot \Psi \\ & - \sum_{a,b} i(E_a)(E_b \wedge dl(E_a, E_b)) \cdot \Psi.\end{aligned} \quad (5.14)$$



The first term of the right-hand side of (5.14) is zero since $l$ is symmetric, and the last term is the Clifford multiplication of $\Psi$ with a vector field, which gives an imaginary function when taking its scalar product with $\Psi$. Thus we have

$$-\frac{1}{2}\sigma^{\nabla}|\Psi|^2 = -2(\text{tr } l)^2|\Psi|^2 + 2|l|^2|\Psi|^2.$$

Hence the proof of (5.9) is completed. □

Let $\Psi_1$ be the eigenspinor corresponding to the eigenvalue $\lambda_1^2 = \frac{1}{4}\inf_M(K^\sigma + 4|l_{\Psi_1}|^2)$. From (4.12), we have

$$\overset{l_{\Psi_1}}{\nabla} \Psi_1 = 0, \quad K^\sigma = \text{constant}, \quad |l_{\Psi_1}| = \text{constant}. \quad (5.15)$$

From Proposition 5.3, we know that $|\Psi_1|$ is constant and

$$\lambda_1^2 = \frac{1}{4}\sigma^{\nabla} + |l_{\Psi_1}|^2. \quad (5.16)$$

From (5.16), the transversal scalar curvature $\sigma^{\nabla}$ is constant and we have

$$\inf |\kappa|^2 = 0. \tag{5.17}$$

Since $\sigma^{\nabla}$ and $K^{\sigma} = \sigma^{\nabla} + |\kappa|^2$ are constant, $|\kappa|$ is constant and then $|\kappa| = \inf_M |\kappa| = 0$. This implies that $\mathcal{F}$ is minimal. Hence we have the following theorem.

**Theorem 5.4** *Under the same assumption as in Theorem 4.6, if there exists an eigenspinor $\Psi(\neq 0)$ of the basic Dirac operator $D_b$ for the eigenvalue $\lambda^2 = \frac{1}{4}\inf_M(K^{\sigma} + 4|l_{\Psi}|^2)$, then $|\Psi|$ is constant and $\mathcal{F}$ is minimal with the constant transversal scalar curvature $\sigma^{\nabla} = 4(\lambda^2 - |l_{\Psi}|^2)$.*

## 5.2 The limiting case II



Next, we study the limiting foliation of (4.35). Similarly, we have that for any $X \in \Gamma Q$,

$$\begin{aligned} Ric_{\bar{\nabla}}^{l_{\bar\Phi}}(X \otimes \bar\Phi) = &-\frac{1}{2}\rho^{\bar\nabla}(X)\,\bar\cdot\,\bar\Phi + \sum(\bar E_a \wedge dl_{\bar\Phi}(X, \bar E_a))\,\bar\cdot\,\bar\Phi - 2l_{\bar\Phi}^2(X)\,\bar\cdot\,\bar\Phi \\ &+ 2(\mathrm{tr}\ l_{\bar\Phi})l_{\bar\Phi}(X)\,\bar\cdot\,\bar\Phi - \{X(\mathrm{tr}l_{\bar\Phi}) - (div_{\bar\nabla}l_{\bar\Phi})(X)\}\,\bar\cdot\,\bar\Phi. \end{aligned} \tag{5.18}$$

From (5.18), we have that for any $X \in TM$,

$$\frac{1}{2}\rho^{\bar\nabla}(\pi(X)) = \sum \bar E_a \wedge dl_{\bar\Phi}(\pi(X), \bar E_a) - 2l_{\bar\Phi}^2(\pi(X)) + 2(\mathrm{tr}\ l_{\bar\Phi})l_{\bar\Phi}(\pi(X)). \tag{5.19}$$

Hence we have the following proposition.

**Proposition 5.5** *If $M$ admits a non-zero spinor $\Psi$ with $\overset{l_{\bar\Phi}}{\bar\nabla}\bar\Phi = 0$, where*

$\Phi = e^{-\frac{q-1}{2}u}\Psi$, *then* $|\Phi|$ *is constant and for any vector field* $X$

$$\nabla_X \Psi = \frac{1}{2}\pi(X)\cdot grad_{\nabla}(u)\cdot\Psi + \frac{q}{2}X(u)\Psi - l_{\Psi}(\pi(X))\cdot\Psi, \qquad (5.20)$$

$$X(tr\ l_{\bar\Phi}) = (div_{\bar\nabla} l_{\bar\Phi})(X), \quad (tr\ l_{\bar\Phi})^2 = \frac{1}{4}\sigma^{\bar\nabla} + |l_{\bar\Phi}|^2. \qquad (5.21)$$

**Proof.** Since $\overset{l_{\bar\Phi}}{\bar\nabla}$ is metrical, $|\Phi|$ is constant. Moreover, $\overset{l_{\bar\Phi}}{\bar\nabla}\ \bar\Phi = 0$ is equivalent to

$$\bar\nabla_X\bar\Phi + l_{\bar\Phi}(\pi(X))\bar\cdot\bar\Phi = 0. \qquad (5.22)$$

From Proposition 3.11 and (5.22), we have that for $\Phi = e^{-\frac{q-1}{2}u}\Psi$

$$\nabla_X \Phi = \frac{1}{2}\pi(X)\cdot grad_{\nabla}(u)\cdot\Phi + \frac{1}{2}X(u)\Phi - l_{\Psi}(\pi(X))\cdot\Phi, \qquad (5.23)$$

which gives (5.20). The proof of (5.21) is similar to the one in Proposition 5.3. □

By direct calculation together with (3.18), we obtain the following lemma.



**Lemma 5.6** *For any vector field* $X \in \Gamma Q$ *and any isomorphism* $l$, *we have*

$$(div_{\bar\nabla} l)(X) = (div_{\nabla} l)(X) + q\ g_Q(l(X), grad_{\nabla}(u)) - X(u)\ \mathrm{tr}\ l, \qquad (5.24)$$

*where* $\bar\nabla$ *is a Levi-Civita connection with respect to* $\bar g_Q = e^{2u}g_Q$.

On the other hand, we have that for any $\Phi = e^{-\frac{q-1}{2}u}\Psi$ and any $X \in \Gamma Q$

$$(div_{\nabla} l_{\bar\Phi})(X) = e^{-u}\{(div_{\nabla} l_{\Psi})(X) - g_Q(l_{\Psi}(X), grad_{\nabla}(u))\}. \qquad (5.25)$$

From (5.24) and (5.25), we have

$$\begin{aligned}(div_{\bar\nabla} l_{\bar\Phi})(X) =& e^{-u}\{(div_{\nabla} l_{\Psi})(X) + (q-1)g_Q(l_{\Psi}(X), grad_{\nabla}(u))\} \\ &- e^{-u}X(u)\ \mathrm{tr}\ l_{\Psi}.\end{aligned} \qquad (5.26)$$

Comparing with (5.21) and (5.26), we have the following corollary.

**Corollary 5.7** *If $M$ admits a non-zero spinor $\Psi$ with $\overset{l_{\bar\Phi}}{\bar\nabla}\,\bar\Phi = 0$, where $\Phi = e^{-\frac{q-1}{2}u}\Psi$, then for any $X \in \Gamma Q$*

$$X(tr\, l_\Psi) = (div_\nabla l_\Psi)(X) + (q-1)g_Q(l_\Psi(X), grad_\nabla(u)).$$

Let $D_b\Psi = \lambda\Psi$ with $\lambda^2 = \frac{1}{4}(\mu_1 + \inf_M|\kappa|^2) + \inf_M|l_\Psi|^2$. From (4.24) and (4.35), we have that for $\Phi = e^{-\frac{q-1}{2}u}\Psi$

$$\begin{aligned}\mu_1 + \inf_M|\kappa|^2 + 4\inf_M|l_\Psi|^2 &= \sup_u \inf_M (e^{2u}K_\sigma^{\bar\nabla} + 4|l_\Psi|^2) \\ &= \sup_{u\in\mathcal{K}_u} \inf_M (e^{2u}K_\sigma^{\bar\nabla} + 4|l_\Psi|^2) \qquad (5.27) \\ &= \inf_M (e^{2u}K_\sigma^{\bar\nabla} + 4|l_\Psi|^2).\end{aligned}$$

By (4.23),

$$\overset{l_{\bar\Phi}}{\bar\nabla}\,\bar\Phi = 0, \quad \inf_M(e^{2u}K_\sigma^{\bar\nabla}) = e^{2u}K_\sigma^{\bar\nabla}, \quad |l_\Psi| = \text{constant}. \qquad (5.28)$$



From (5.28), we have

$$\sup_{u\in\mathcal{K}_u} \inf_M (e^{2u}K_\sigma^{\bar\nabla}) = \sup_{u\in\mathcal{K}_u} (e^{2u}K_\sigma^{\bar\nabla}). \qquad (5.29)$$

From (5.29), we have

$$\sup_{u\in\mathcal{K}_u} \inf_M \{h^{-1}Y_b h\} + \inf_M|\kappa|^2 = \sup_{u\in\mathcal{K}_u} \{h^{-1}Y_b h\} + |\kappa|^2. \qquad (5.30)$$

From (5.27) and (5.30), we have

$$\mu_1 = \sup_{u\in\mathcal{K}_u} \inf_M \{h^{-1}Y_b h\}, \quad |\kappa| = \text{constant}. \qquad (5.31)$$

From (4.19) and (5.21), we have

$$\mu_1 + \inf_M|\kappa|^2 = e^{2u}\sigma^{\bar\nabla}. \qquad (5.32)$$

From Lemma 4.13, we have that for $u \in \mathcal{K}_u$

$$e^{2u}\sigma^{\bar{\nabla}} = \sigma^{\nabla} + 2(q-1)\Delta_B u + (q-1)(2-q)|grad_{\nabla}(u)|^2. \tag{5.33}$$

From (4.33), (5.32) and (5.33), we have

$$\mu_1 + \inf_M |\kappa|^2 = \sup_{u\in\mathcal{K}_u} \inf_M (e^{2u}\sigma^{\bar{\nabla}}) = \sup_{u\in\mathcal{K}_u} \inf_M \{h^{-1}Y_b h\} = \mu_1. \tag{5.34}$$

From (5.34) we get

$$\inf_M |\kappa|^2 = 0. \tag{5.35}$$

This implies that $|\kappa| = 0$, i.e. $\mathcal{F}$ is minimal. Hence we have the following theorem.

**Theorem 5.8** *Let $(M, \tilde{g}_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 3$ and bundle-like metric $\tilde{g}_M$. Assume that an eigenvalue $\lambda$ of $D_b$ corresponding to the eigenspinor $\Psi$ satisfies*

$$\lambda^2 = \frac{1}{4}(\mu_1 + \inf_M |\kappa|^2) + \inf_M |l_\Psi|^2.$$

*Then $|l_\Psi|$ is constant and the foliation $\mathcal{F}$ is minimal. Moreover*

$$(div_{\nabla} l_\Psi)(X) = (1-q) g_Q(l_\Psi(X), grad_{\nabla}(u)) \tag{5.36}$$

*for any $X \in \Gamma Q$.*

**Proof.** The equation (5.36) is trivial from corollary 5.7. □

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〈국 문 초 록〉

## 엽층구조를 가지는 리만 다양체에서의 basic Dirac 연산자의 고유치 계산

엽층적 구조를 가지는 리만다양체의 횡단적 구조가 spin 구조를 가질 때 basic Dirac 연산자의 고유치의 제곱값은 엽층구조 $F$ 의 횡단적 곡률과 평균곡률 및 법속 $Q$ 의 적당한 자기준동형 $l_{\Psi}$ 의 길이에 종속되어진다는 것을 보인다. 여기서 $\Psi$ 는 고유 spinor이다. 또한 이 고유치의 하한은 기존의 알려진 많은 결과들을 모두 유도한다는 것을 보인다. 더구나, 등식을 만족하는 경우의 엽층의 성질을 조사하여 실제로 엽층들은 모두 극소부분공간임을 알 수 있고, 가장 작은 고유치에 대응하는 고유 spinor에 의해 결정되어진 자기준동형사상의 길이는 상수라는 것을 밝혔다.

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