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Boolean Rank Preserving Linear Operator

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Boolean Rank Preserving Linear Operator

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부울 階數를 保存하는 線形演算子

本 論文에서는 부울 階數를 保存하는 線形演算子를 研究하여, 부울 階數를 保存하는 線形演算子의 몇 가지 特性들을 찾아서 기존에 알려진 特性들과 同値가 됨을 證明하였다. 곧 線形演算子가 모든 부울 階數를 保存할 必要充分條件으로서 그 演算子가 부울 階數 1을 강하게 保存하는 것과, 임의의 부울 階數 하나를 保存하는 가역함수가 되는 것임을 밝혔다. 이것은 L. B. Beasley 와 N. J. Pullman [1]이 알아낸 부울 階數를 保存하는 線形演算子의 모습을 一般化한 결과이다.

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I. Introduction

Partly because of their association with nonnegative real matrices, Boolean matrices [(0,1)-matrices with the usual arithmetic except 1+1=1] have been the subject of research by many authors. Recently Kim [8] has published a compendium of results on the theory and applications of Boolean matrices. Often, parallels are sought for results known for field-valued matrices or other rings. See de Caen and Gregory [3], Rao and Rao [12,13], Richman and Schneider [15], Beasley and Song [2], Song [14] and Hwang, Kim and Song [6].

Let $\mathcal{M}_{m,n}(\mathbb{S})$ be the set of $m \times n$ matrices and $T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S})$ be a linear operator over algebraic structure $\mathbb{S}$. Say that T is a

(1) *(U, V)-operator* if there exist invertible matrices U and V such that $T(A) = UAV$ for all $A \in \mathcal{M}_{m,n}(\mathbb{S})$, or $T(A) = UA^tV$ for $m = n$.

(2) *rank preserver* if $\text{rank}(T(A)) = \text{rank}(A)$ for all $A \in \mathcal{M}_{m,n}(\mathbb{S})$.

(3) *rank 1 preserver* if $\text{rank}(T(A)) = 1$ whenever $\text{rank}(A) = 1$ for all $A \in \mathcal{M}_{m,n}$.

The fact that (1), (2) and (3) are equivalent for a field $\mathbb{S}$ was established in the work of Marcus and Moyls [10,11], Westwick [16] and Lautemann [9]. In [1], Beasley and Pullman obtained nearly analogous results for "linear operators" on Boolean matrices (operators that fix 0 and preserve sums), employing a particular definition of Boolean rank.

Section II contains all definitions and other preliminaries.

Section III concerns Boolean rank 1 preservers. Marcus and Moyls [11] and Westwick [16] have shown that

(1.1) Over $\mathbb{F}$, T preserves rank 1 if and only if T is a (U, V)-operator.

Beasley and Pullman [1] showed that over $\mathbb{B}$, although all (U, V)-operators are rank 1 preserver, the converse is false. In analogy with terminology introduced in [11], we call a family of Boolean matrices consisting of 0 and some Boolean rank 1 matrices a "Boolean rank 1 space" if it is closed under addition.

Section IV concerns Boolean rank k preservers. In this section, we obtain the following theorem.

Theorem. *Let T be a Boolean linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$. Then T is an invertible rank k preserver, $k \geq 1$ if and only if T is a (U, V)-operator (Theorem 4.10).*

Section V concerns Boolean rank preserver. Marcus and Moyls [10] and Lautemann [9] have shown that

(1.2) Over $\mathbb{F}$, T is a rank preserver if and only if T is a (U, V)-operator.

It follows from (1.1), (1.2) that

(1.3) Over $\mathbb{F}$, T is a rank preserver if and only if T is a rank 1 preserver.

In this thesis, we obtain analogue results for $m \times n$ Boolean matrix as follows.

Theorem. *Let T be a Boolean linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$. T is a (U, V)-operator if and only if T strongly preserves rank 1 (Theorem 5.6).*

II. Notations, Definitions and Other Preliminaries

We let $\mathcal{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{B} = \{0, 1\}$, the two-element Boolean algebra. Arithmetic in $\mathbb{B}$ follows the usual rules except that 1+1=1. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well. Throughout this thesis we shall adopt the convention that $m \leq n$, unless otherwise specified. Also lowercase, boldface letters will represent vectors, all vectors $\mathbf{u}$ are column vectors ($\mathbf{u}^t$ is a row vector), and $J_{m,n}$ denotes the matrix in $\mathcal{M}_{m,n}(\mathbb{B})$ all of whose entries are 1.

2.1 Rank of Boolean Matrix

There are several notions of rank for Boolean matrices. We have found the following definition useful for our purposes. It appears in [1], where it is ascribed to B. M. Schein. If A is a nonzero $m \times n$ Boolean matrix, its *Boolean rank*, $b(A)$, is the least integer k for which there exist $m \times k$ and $k \times n$ Boolean matrices B and C with $A = BC$. The Boolean rank of 0 is 0.

It is well known([8]) that $b(A)$ is the least k such that A is the sum of k matrices of Boolean rank 1. Although Boolean rank enjoys many properties of the rank of field-valued matrices ($b(A) = b(A^t)$, $b(AB) \leq \min(b(A), b(B))$), there are others which it fails to enjoy. For example, even though $b(A) = r$, A may contain no $r \times r$ submatrix of Boolean rank r ([4]).

2.2. Singularity and Invertibility of Boolean Matrices

We say that a Boolean matrix A is *singular* if $A\mathbf{x} = 0$ for some vector $\mathbf{x} \neq 0$ ($\mathbf{x} \in \mathcal{M}_{n,1}(\mathbb{B})$). Note that having full Boolean rank (i.e., $b(A) = m$) is a sufficient, but not a neccessary condition for nonsingularity when A is $m \times n$ and that the nonsingularity of a squre matrix does not guarantee the nonsingularity of its transpose, A^t. For any A in $M_{m,n}(\mathbb{B})$, A is nonsingular if and only if A has no zero column.

An $n \times n$ Boolean matrix A is said to be *invertible* if for some X, $AX = XA = I_n$, where I_n is the $n \times n$ identity matrix. This matrix X is necessarily unique when it exists. It is then denoted A^{-1}. It is well known that the permutation matrices are the only invertible Boolean matrices and therefore $A^{-1} = A^t$ when A is invertible. The characterization of nonsingularity given above shows that nonsingularity does not imply invertibility. Of course, invertible matrices are nonsingular.

2.3. Boolean Vector Subspaces, Bases and Dimension

For our purpose, we can define a *Boolean vector space* to be any subset of $\mathbb{B}^m (= \mathcal{M}_{m,1})$ containing 0 which is closed under addition. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{B}^m$, we say $\mathbf{x}$ *absorbs* $\mathbf{y}$, written $\mathbf{x} \geq \mathbf{y}$, if $x_i = 0$ only when $y_i = 0$, for all $1 \leq i \leq m$. If $\mathbb{V}$, W are vector spaces with $\mathbb{V} \subseteq \mathbb{W}$, then $\mathbb{V}$ is called a *subspace* of $\mathbb{W}$. We identity $M_{m,n}(\mathbb{B})$ with $\mathbb{B}^{mn}$ in the usual way when we discuss it as a Boolean vector space and consider its subspaces.

Let $\mathbb{V}$ be a Boolean vector space. If S is a subset of V, then $\langle S \rangle$ denotes the intersection of all subspaces of $\mathbb{V}$ containing S. This is a subspace of $\mathbb{V}$ too, called the subspace *generated* by S. If $S = \{s_1, s_2, \cdots, s_p\}$, then $\langle S \rangle = \{\sum_{i=1}^{p} x_i s_i : x_i \in \mathbb{B}\}$, the set of *linear combinations of* S. Note that

$\langle\phi\rangle = \{0\}$. Define the *dimension* of $\mathbb{V}$, written $\dim(\mathbb{V})$, to be the minimum of the cardinalities of all subsets S of $\mathbb{V}$ generating $\mathbb{V}$. We call a generating set of cadinality equal to $\dim(\mathbb{V})$ a *basis* of $\mathbb{V}$. A subset of V is called *independent* if none of its members is a linear combination of the others. Evidently every basis is independent.

2.4. Linear Transformations, Operators and Boolean Matrix Representation

If $\mathbb{V}$, $\mathbb{W}$ are Boolean vector spaces, a mapping $T : \mathbb{V} \to \mathbb{W}$ which preserves sums and 0 is said to be a (Boolean) *linear transformation*. If $\mathbb{V} = \mathbb{W}$, the word *operator* is used instead of "*transformation.*" Evidently, when T is linear its behavior on V's basis determines its behavior completely. As with transformations of vector spaces over fields, by ordering the basis of $\mathbb{V}$ and $\mathbb{W}$ we can represent T by an $m \times n$ matrix $[t_{ij}]$ in an analogous way. But the t_{ij} are not usually uniquely defined by Boolean T, so T may have several matrix representations for the same bases orderings.

A matrix $A \in \mathcal{M}_{m,n}(\mathbb{B})$ determines a linear transformation T_A of $\mathbb{B}^n$ into $\mathbb{B}^m$ by $T_A(X) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{B}^n$. The image of $\mathbb{V}$ in $\mathbb{W}$, $T(\mathbb{V})$, is generated by the image, $T(\mathfrak{B})$, of the basis $\mathfrak{B}$ of $\mathbb{V}$. This proves that

Lemma 2.4.1. *For every linear Boolean transformation T, $dim(T(\mathbb{V})) \le dim(\mathbb{V})$.*

Proof. Let $T : \mathbb{V} \to \mathbb{W}$ be a linear Boolean transformation T. For any $\mathbf{x} \in V$, $\mathbf{x} = \sum \alpha_i \mathbf{x}_i$ where $\mathbf{x}_i \in \mathfrak{B}$ is the basis of $\mathbb{V}$ and $\alpha_i \in \mathbb{B} = \{0, 1\}$. Then $T(\mathbf{x}) = T(\sum \alpha_i \mathbf{x}_i) = \sum \alpha_i T(\mathbf{x}_i)$. Thus $T(\mathfrak{B})$ generates $T(\mathbb{V})$. Therefore

$dim(T(\mathbb{V})) \leq dim(\mathbb{V})$. □

Lemma 2.4.2. *If the Boolean linear transformation $T : \mathbb{V} \to \mathbb{W}$ is injective then $dim(T(\mathbb{V})) = dim(\mathbb{V})$ and T maps the basis of $\mathbb{V}$ onto the basis of $T(\mathbb{V})$.*

Proof. From Lemma 2.4.1, $dim(T(\mathbb{V})) \leq dim(\mathbb{V})$. Suppose $dim(T(\mathbb{V})) < dim(\mathbb{V})$. Let $\mathfrak{B} = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ be a basis of $\mathbb{V}$. Then there exist $\mathbf{x}_i \in \mathfrak{B}$ such that

$$T(x_i) = \sum_{j \neq i} \alpha_j T(x_j) \text{ for } \alpha_j \in \mathbb{B} = \{0, 1\}$$
$$= T(\sum \alpha_j x_j)$$

for $i \neq j$. Since T is injective, $\mathbf{x}_i = \sum \alpha_j \mathbf{x}_j$. This is a contradiction that $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ is linearly independent. Therefore $dim(T(\mathbb{V})) = dim(\mathbb{V})$. Since $T(\mathfrak{B})$ generates $T(\mathbb{V})$ and it is linearly independent, $T(\mathfrak{B})$ is the basis of $T(\mathbb{V})$. □

2.5. Invertibility of Boolean Transformations

A transformation $T : \mathbb{V} \to \mathbb{W}$ is invertible if and only if T is injective and $T(\mathbb{V}) = \mathbb{W}$. As with vector spaces over fields, the inverse, T^{-1}, of a Boolean linear transformation T is also linear. Let $T^{-1}(\mathbf{x}) = \mathbf{a}$ and $T^{-1}(\mathbf{y}) = \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{V}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{W}$. Then $\mathbf{x} + \mathbf{y} = T(\mathbf{a}) + T(\mathbf{b}) = T(\mathbf{a} + \mathbf{b})$. Thus $T^{-1}(\mathbf{x} + \mathbf{y}) = \mathbf{a} + \mathbf{b} = T^{-1}(\mathbf{x}) + T^{-1}(\mathbf{y})$.

Lemma 2.5.1. *If $T : \mathbb{V} \to \mathbb{W}$ is a surjective Boolean linear transformation, then T is invertible if and only if T preserves the dimension of every subspace of $\mathbb{V}$.*

Proof. Suppose T is not invertible, that is, T is not injective. Then for some $\mathbf{x} \neq \mathbf{y}$, T reduces the dimension of $\langle \mathbf{x}, \mathbf{y} \rangle$. That is, there exist $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x} \neq \mathbf{y}$ and $T(\mathbf{x}) = T(\mathbf{y})$. Thus $T(\langle \mathbf{x}, \mathbf{y} \rangle) = \langle T(\mathbf{x}) \rangle$. This is a contradiction that T preserves the dimension of every subspace of $\mathbb{V}$. Conversely, if T is invertible, then the conclusion follows by Lemma 2.4.2. □

The finiteness of $|\mathbb{V}|$ and the previous lemma give us that

Corollary 2.5.2. *If T is a linear Boolean operator on V, then the following statements are equivalent.*

(a) T is invertible.

(b) T is injective.

(c) T is surjective.

(d) T permutes the basis of $\mathbb{V}$.

(e) T preserves the dimension of every subspace of $\mathbb{V}$.

Proof. By the definition of invertible, (a) $\Longleftrightarrow$ (b). Since $|\mathbb{V}|$ is finite, T is injective if and only if T is surjective, i.e., (b) $\Longleftrightarrow$ (c). In the following, we prove that (d)$\rightarrow$ (e)$\rightarrow$ (a)$\rightarrow$ (d). Suppose T permutes the basis $\mathfrak{B}$ of $\mathbb{U}$ subspace of $\mathbb{V}$. Then $T(\mathfrak{B})$ is the basis of $T(\mathbb{U})$. Thus $dim(\mathbb{U}) = dim(T(\mathbb{U}))$. Let T preserve the dimension of every subspace of $\mathbb{V}$. By Lemma 2.5.1, (a) holds. By Lemma 2.4.2, if T is invertible then T maps the basis $\mathfrak{B}$ onto the basis $T(\mathfrak{B})$. Since T is injective and $dim(\mathfrak{B})$ is finite, T permutes $\mathfrak{B}$. □

We note that T_A is invertible if and only if A is invertible. Suppose T_A is invertible. Then $\mathbf{x} = T_A^{-1}(T_A(\mathbf{x})) = T_A^{-1}(A\mathbf{x})$. Put $T_A^{-1} = A^{-1}$. Then A is invertible. Suppose A is invertible. Put $(T_A)^{-1} = T_{A^{-1}} = A^{-1}$. Then $T_{A^{-1}}(T_A(\mathbf{x})) = T_{A^{-1}}(A\mathbf{x}) = A^{-1}A\mathbf{x} = \mathbf{x}$. That is, T is invertible. By Corollary 2.5.2(d), T_A is invertible if and only if T_A permutes the basis of every subspace of $\mathbb{V}$. Therefore the invertible Boolean matrix A is the permutation matrix.

The main application of the ideas in section 2.5 in this thesis is to the linear operators on the space $\mathcal{M}_{m,n}(B)$ of all $m \times n$ Boolean matrices. Let $\triangle_{m,n} = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$, $E_{i,j}^{m,n}$ be the $m \times n$ matrix whose (i, j)-th entry is 1 and whose other entries are all 0 and $\mathcal{E}_{m,n} = \{E_{i,j}^{m,n} : (i, j) \in \triangle_{m,n}\}$.

Corollary 2.5.3. *The linear operator T on $\mathcal{M}_{m,n}(\mathbb{B})$ is invertible if and only if T permutes $\mathcal{E}_{m,n}$ if and only if T preserves the dimension of every subspace of $\mathcal{M}_{m,n}(\mathbb{B})$.*

Proof. Suppose T is invertible. Then T permutes the basis $\mathcal{E}$ of $\mathcal{M}_{m,n}(B)$, by Corollary 2.5.2(d). Let T permute $\mathcal{E}$. Then $dim(\mathcal{E}) = dim(T(\mathcal{E}))$. Let T preserve the dimension of every subspace of $\mathcal{M}_{m,n}(\mathbb{B})$. By Corollary 2.5.2, T is invertible. □

We can describe any operator T on $\mathcal{M}_{m,n}(\mathbb{B})$ by expressing $(T(X))_{i,j}$ as a scalar-valued function of X for all $(i, j) \in \triangle_{m,n}$. The operator T will be linear if and only if each component function $t_{ij} : X \to (T(X))_{ij}$ is a linear transformation of $\mathcal{M}_{m,n}(\mathbb{B})$ into $\mathbb{B}$. $(T(\mathbf{x}+\mathbf{y}))_{ij} = T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x})+T(\mathbf{y}) = (T(\mathbf{x}))_{ij} + (T(\mathbf{y}))_{ij}$. Applying Corollary 2.5.3, we see that the operator T on $\mathcal{M}_{m,n}(\mathbb{B})$ is invertible if and only if there exists a permutation τ of $\triangle_{m,n}$ such that $T([x_{ij}]) = [x_{\tau(i,j)}]$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$.

2.6 Boolean Rank-1 Matrices and Rank-1 Spaces

It is easy to verify that (just as with field-valued matrices) the Boolean rank of A is 1 if and only if there exist nonzero (Boolean) vectors $\mathbf{x}$ and $\mathbf{y}$ [$\mathbf{x} \in \mathcal{M}_{m,1}(B)$ and $\mathbf{y} \in \mathcal{M}_{n,1}(B)$] such that $A = \mathbf{x}\mathbf{y}^t$. Unlike the corresponding situation for field-valued matrices, these vectors x and y are uniquely determined by A. Let $A = \mathbf{x}\mathbf{y}^t = \mathbf{x}_*\mathbf{y}_*^t$ where $\mathbf{x} = [x_1, x_2, \cdots, x_m]$, $\mathbf{x}_* = [x'_1, x'_2, \cdots, x'_m]$, $\mathbf{y} = [y_1, y_2, \cdots, y_n]$, $\mathbf{y}_* = [y'_1, y'_2, \cdots, y'_n]$. Note that

$$\begin{aligned} \mathbf{x}\mathbf{y}^t &= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots x_1y_n & \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \cdots & & & \\ x_my_1 & x_my_2 & \cdots x_my_n & \end{bmatrix} \\ &= \begin{bmatrix} x'_1y'_1 & x'_1y'_2 & \cdots & x'_1y'_n \\ x'_2y'_1 & x'_2y'_2 & \cdots & x'_2y'_n \\ \cdots & & & \\ x'_my'_1 & x'_my'_2 & \cdots x'_my'_n & \end{bmatrix} \\ &= \mathbf{x}_*\mathbf{y}_*^t. \end{aligned}$$

Suppose $\mathbf{x} \neq \mathbf{x}_*$, then there exist i such that $x_i \neq x'_i$. Let $x_i = 1$ and $x'_i = 0$. Then for some $y_k \neq 0$, $x_iy_k \neq 0$ and $x'_iy'_k = 0$, a contradiction. Similarly, it holds for $\mathbf{y} \neq \mathbf{y}_*$. Therefore there are exactly $(2^m - 1)(2^n - 1)$ rank 1 $m \times n$ matrices. We use the notation $A \leq B$ to mean $b_{ij} = 0$ only if $a_{ij} = 0$. Equivalently, $A \leq B$ if and only if $A + B = B$. For any vector $\mathbf{x}$, let $|\mathbf{x}|$ be the number of nonzero entries in $\mathbf{x}$ and when $A = \mathbf{a}\mathbf{b}^t$ is not zero, define the perimeter of A, $p(A)$, as $|\mathbf{a}| + |\mathbf{b}|$.

Lemma 2.6.1. *If $A \leq B$ and $b(A) = b(B) = 1$, then $p(A) < p(B)$ unless $A = B$.*

Proof. Since $b(A) = b(B) = 1$, we can write $A = \mathbf{a}\mathbf{b}^t$, $B = \mathbf{c}\mathbf{d}^t$, where $\mathbf{a}, \mathbf{c} \in \mathcal{M}_{m,1}(\mathbb{B})$ and $\mathbf{b}, \mathbf{d} \in \mathcal{M}_{n,1}(\mathbb{B})$. Since $A \leq B$, $b_{ij} = 0$ only if $a_{ij} = a_ib_j = 0$. Thus $c_j = 0$ or $d_j = 0$ implies $a_i = 0$ or $b_j = 0$. Therefore $p(A) = |\mathbf{a}| + |\mathbf{b}| \leq |\mathbf{c}| + |\mathbf{d}| = p(B)$. Hence $p(A) < p(B)$ unless $A = B$. □

Analogously with [13, 22], we define a subspace of $\mathcal{M}_{m,n}(\mathbb{B})$ whose nonzero members have Boolean rank 1 as a rank 1 space. If $A = \mathbf{a}\mathbf{x}^t$ is a rank 1 matrix, then $\mathbf{a}$ and $\mathbf{x}$ are uniquely determined by A. We call $\mathbf{a}$ the left factor and $\mathbf{x}$ the right factor of A.

Lemma 2.6.2. *If A, B and $A + B$ are rank 1 matrices and neither $A \leq B$ nor $b \leq A$, then A, B and $A + B$ have a common factor.*

Proof. Let $A = \mathbf{a}\mathbf{x}^t$, $B = \mathbf{b}\mathbf{y}^t$ and $C = A + B = \mathbf{c}\mathbf{z}^t$ be the factorization of A, B and C. We have for all i, j, $a_i\mathbf{x} + b_i\mathbf{y} = c_i\mathbf{z}$ and $x_j\mathbf{a} + y_j\mathbf{b} = z_j\mathbf{c}$ where a_i, b_i and c_i are i-th entry of all $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, and x_j, y_j and z_j are j-th entry of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, respectively. Since $A \nleq B$ and $B \nleq A$, if $\mathbf{a} \nleq \mathbf{b}$ and $\mathbf{b} \nleq \mathbf{a}$ then for some i, j, $a_i = 1$, $b_i = 0, a_j = 0$ and $b_j = 1$. Then $\mathbf{x} = c_i\mathbf{z}$ and $\mathbf{y} = c_j\mathbf{z}$. But $\mathbf{x} \neq 0$ and $\mathbf{y} \neq 0$, so $\mathbf{x} = \mathbf{y} = \mathbf{z}$. Thus A, B and C have a common right factor. If $\mathbf{a} \leq \mathbf{b}$, then $\mathbf{x} \nleq \mathbf{y}$ (as $A \nleq B$). Thus $x_i = 1$, $y_i = 0$ and $\mathbf{a} = z_i\mathbf{c}$ for some i, and $x_j = 0$, $y_j = 1$ and $x_j\mathbf{a} + y_j\mathbf{b} = \mathbf{b} = z_j\mathbf{c}$ for some j. Since $\mathbf{a} \neq 0$ and $\mathbf{b} \neq 0$, $\mathbf{a} = \mathbf{b} = \mathbf{c}$. Thus A, B and C have a common left factor. A parallel argument if $\mathbf{b} \leq \mathbf{a}$. □

Convention: Since we can write 0 as $0\mathbf{x}^t$ or $\mathbf{a}0^t$ for all $\mathbf{a}$ and $\mathbf{x}$, let us agree to say that 0 and A have a common left factor and a common right factor any rank 1 matrix A.

III. Boolean rank 1 Preserving Operators

As was mentioned in section 1, Marcus, Moyls and Westwick showed that if T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{F})$ ($\mathbb{F}$ algebraically closed) and T maps rank 1 matrices to rank 1 matrices(i.e., T preserves rank 1 matrices) then (and only then) T is a (U, V)-operator. This result does not hold in the Boolean case. The following example shows that not rank 1 preserving operators T are of the form $T(X) = UXV$ for some nonsingular U, V^t, contrary to the situation for algebraically closed fields. Since invertible Boolean matrices are nonsingular, it also shows that not all rank 1 preserving operators T are (U, V)-operators.

Example 3.1. Let

$$T\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = (b+e+c+f)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 & d \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, T is a linear operator and $b(T(X)) = 1$ whenever $b(X) = 1$ (in fact whenever $X \neq 0$). That is, T is a rank 1 preserver. Suppose T is (U, V)-operator. Then there exist nonsingular matrices $U \in \mathcal{M}_{2,2}(\mathbb{B})$ and $V^t \in \mathcal{M}_{3,3}(\mathbb{B})$ such that $T(X) = UXV$ for all $X \in \mathcal{M}_{2,3}(\mathbb{B})$ and for $j = 1, 2, 3$, we have $T(E_{1,j}) = \mathbf{u}\mathbf{v}_j^t$ where $\mathbf{u}$ is the first column of U and $\mathbf{v}_j$ is the j-th column of V. But since

$$T(E_{1,1}) = T\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \mathbf{u}\mathbf{v}_1^t$$

and

$$T(E_{1,2}) = T\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \mathbf{u}\mathbf{v}_2^t,$$

hence $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, this is a contradiction that there exist nonsingular matrices U and V such that $T(X) = UXV$.

Suppose U and V^t are nonsingular members of $\mathcal{M}_{m,m}(\mathbb{B})$ and $\mathcal{M}_{n,n}(\mathbb{B})$ respectively and T is the operator on $M_{m,n}(\mathbb{B})$ defined by $T(X) = UXV$ for all X. Clearly T is linear. Moreover $T(X)$ has rank 1 whenever X has rank 1. Suppose X has rank 1, so that $X = \mathbf{a}\mathbf{b}^t$ where $\mathbf{a} \neq 0$, $\mathbf{b} \neq 0$. Then $T(X) = U\mathbf{a}\mathbf{b}^t V = (U\mathbf{a})(V^t\mathbf{b})^t$ and since U and V^t are nonsingular, neither $U\mathbf{a}$ or $V^t\mathbf{b}$ is 0, so $T(X)$ has rank 1. It follows that all Boolean (U, V)-operators are rank 1 preservers.

Example 3.2. Suppose C is a fixed rank 1 member of $\mathcal{M}_{m,n}(\mathbb{B})$, and T is the operator defined by $T(X) = C$ if $X \neq 0$ and $T(0) = 0$.

Example 3.2 shows that for each k $(1 \leq k \leq n)$ there exist a linear operator T_k that preserves the Boolean rank of every rank k $m \times n$ matrix but is not a (U, V)-operator when $k \geq 1$ (just take C to be a fixed rank k matrix; recall that (U, V)-operators preserve rank 1). Beasley [1] showed that over F, for most $k \leq n$, each operator on field-valued matrices preserves the rank of rank k matrices if and only if it is a (U, V)-operator. We were unable to find a condition necessary and sufficient for a Boolean operator to preserve the rank of all rank 1 matrices, so the Boolean analogue of the work of Marcus, Moyls and Westwick mentioned in section I, charaterizing the rank 1 preservers, remains to be discorvered.

Lemma 3.3. *If T is a rank 1 preserving operator on $\mathcal{M}_{m,n}(\mathbb{B})$ that preserves the dimension of all rank 1 spaces, then the restriction of T to the rank 1 matrices is injective or T reduces the rank of some rank 2 matrix to*

1.

Proof. Let $\mathcal{M}^1 = \{A \in \mathcal{M}_{m,n}(\mathbb{B}) : b(A) = 1\}$ and $\mathbf{W} \equiv \{0\} \cup \{X \in \mathcal{M}^1 : T(X) = T(B)\}$ for each $B \in \mathcal{M}^1$. If $\mathbb{W}$ is a rank 1 space then $dim(\mathbf{W}) = dim(T(X)) = 1$, so $W = \langle B \rangle$. Thus $T|_{\mathbb{W}}$ is injective. Otherwise there are X, Y in $\mathbf{W}$ such that $b(X+Y) = 2$. □

Corollary 3.4. *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ that*

(i) preserves the rank of all rank 1 and rank 2 matrices and

(ii) preserves the dimension of all rank 1 spaces,

then

(a) T is invertible and

(b) T^{-1} satisfies (i) and (ii).

Proof. Part (a) : Let $\mathcal{E}$ be the basis of $\mathcal{M}_{m,n}(\mathbb{B})$. T is invertible if it permutes $\mathcal{E}$. Lemma 3.3 implies that T permutes $\mathcal{M}^1$. But $\mathcal{M}^1 \supseteq \mathcal{E}$, so it suffices to show that $T(\mathcal{E}) \subseteq \mathcal{E}$. Let $E \in \mathcal{E}$, then $E = T(C)$ for some $C \in \mathcal{M}^1$. Since $C \neq 0$, we have $C \geq F$ for some F in the basis $\mathcal{E}$. Therefore $E \geq T(F)$. Then $E = T(F)$ by Lemma 2.6.1 (because $p(T(F)) = p(E) = 2 > p(F) = 1$, a contradiction). Part (b) follows directly. □

The idea of a permutation of $\triangle_{m,n} = (i,j) : 1 \leq i \leq m,\ 1 \leq j \leq n$ representing an invertible operator was introduced in section 2.5.

Lemma 3.5. *If T is an invertible linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ that preserves the rank of every rank 1 matrix and τ is the permutation of $\triangle_{m,n}$ representing T, then there exist permutations α, β of $\{1, 2, \cdots, m\}$ and $\{1, 2, \cdots, n\}$ respectively such that*

(a) $\tau(i,j) = (\alpha(i), \beta(j))$ for all $(i,j) \in \triangle_{m,n}$ or

(b) $m = n$ *and* $\tau(i,j) = (\beta(j), \alpha(i))$ *for all* $(i,j) \in \triangle_{m,n}$.

Proof. We will denote the abscissa of $\tau(i,j)$ by u_{ij} and its ordinate by v_{ij}. So $\tau(i,j) = (u_{ij}, v_{ij})$. Let $[\tau]$ be the $m \times n$ array whose (i,j)-th entry is (u_{ij}, v_{ij}). Any two entries in the same row (or column) of $[\tau]$ have a common abscissa or common ordinate. This is because $T(\langle E_{ij}, E_{ik}\rangle)$ and $T(\langle E_{ji}, E_{ki}\rangle)$ are rank 1 spaces. It follows that if $u_{i1} = u_{i2}$ (respectively $v_{i1} = v_{i2}$) then u_{i1} is the abscissa (respectively ordinate) of each entry in the i-th row of $[\tau]$. Let $\beta_i(j) = v_{ij}$(respectively u_{ij}). Then for all i, β_i permutes $\{1, 2, \cdots, n\}$. If x were a common abscissa for one row and y were a common ordinate for another then (x, y) would belong to both rows (because $m \leq n$ and each β_i is a permutation), contradicting the injectivity of τ. Therefore either

(1) for all $(i,j) \in \triangle_{m,n}$, $u_{ij} = u_{i1}$ or

(2) for all $(i,j) \in \triangle_{m,n}$, $v_{ij} = v_{i1}$.

Suppose (1) holds. Define $\alpha(i) = u_{i1}$ for all $i, 1 \leq i \leq m$. For some j, $v_{ij} = u_{i1}$ because β_i is a permutation. It follows that (u_{i1}, u_{i1}) occurs in the i-th row of $[\tau]$ and in no other. Thus α permutes $\{1, 2, \cdots, m\}$. If $i \neq 1$, then $T(\langle E_{1j}, E_{ij}\rangle) = \langle E_{ux}, E_{vy}\rangle$ is a rank 1 space with $u = \alpha(1)$, $v = \alpha(i)$ and $\beta_1(j) = x$, $\beta_i(j) = y$. But $\alpha(1) \neq \alpha(i)$, so $x = y$. Therefore $\beta_i = \beta_1$ for all i, $1 \leq i \leq m$. Let $\beta = \beta_1$, then $\tau(i,j) = (\alpha(i), \beta(j))$ for all $(i,j) \in \triangle_{m,n}$. If (2) holds then $m = n$. Let $\tau'(i,j) = (v_{ij}, u_{ij})$ for all $(i,j) \in \triangle$, and apply (1) to τ' to complete the proof of the lemma. □

Lemma 3.6. *If* τ *satisfies the conclusion of Lemma 3.5, then* T *is a* (U, V)*-operator.*

Proof. Let π be any permutation of $\{1, 2, \cdots, k\}$. Let $E_{i,j}^{m,n}$ denote an $m \times n$

matrix of the form E_{ij} defined in section II. Let $P_k(\pi) = \sum_{l=1}^{k} E_{l,\pi(l)}^{k,k}$. Then $P_k(\pi)$ is a permutation matrix. But $E_{i,j}^{m,n}E_{u,v}^{n,r} = \delta_{u,j}E_{i,v}^{m,r}$ (where $\delta_{u,j}$ is the Kronecker delta). Thus $E_{i,j}^{m,n}P_n(\pi) = E_{i,\pi(j)}^{m,n}$, and therefore $P_m(\alpha^{-1})E_{i,j}^{m,n}P_n(\beta) = E_{\alpha(i),\beta(j)}^{m,n}$. If (a) holds in Lemma 3.5 then we define $U = P_m(\alpha^{-1})$ and $V = P_n(\beta)$. If A is any $m\times n$ Boolean matrix, we have $A = \sum\{E_{i,j} : a_{ij} = 1\}$ and hence $T(A) = \sum\{E_{\tau(i,j)} : a_{ij} = 1\} = \sum\{UE_{ij}V : a_{ij} = 1\} = UAV$. If (b) holds in Lemma 3.5, define $U = P_n(\beta^{-1})$ and $V = P_n(\alpha)$. Let T' be the operator on $\mathcal{M}_{m,n}(\mathbb{B})$ defined by $T'(A) = [T(A)]^t$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$. Then $T'(E_{ij}) = E_{\alpha(i),\beta(j)}$, so $T'(A) = P_n(\alpha^{-1})AP_n(\beta)$ by the result for conclusion (a). Hence $T(A) = UA^tV$. □

Theorem 3.7. *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$, then the followings are equivalent.*

(a) T is invertible and preserves the rank of all rank 1 matrices.

(b) T preserves the ranks of all rank 1 matrices and rank 2 matrices and preserves the dimension of all rank 1 spaces.

(c) T is a (U,V)-operator.

Proof. In the following, we prove that (a) $\Rightarrow$ (c). Let T be invertible and preserves the rank of all rank 1 matrices. By Lemma 3.5, there exist permutations α, β of $\{1, 2, \cdots, m\}$ and $\{1, 2, \cdots, m\}$ respectively such that

(a) $\tau(i, j) = (\alpha(i), \beta(j))$ for all $(i, j) \in \triangle_{m,n}$ or

(b) $m = n$ and $\tau(i, j) = (\beta(j), \alpha(i))$ for all $(i, j) \in \triangle_{m,n}$.

Let π be any permutation of $\{1, 2, \cdots, k\}$. Let $E_{i,j}^{m,n}$ denote an $m \times n$ matrix of the form E_{ij} defined in section II. Let $P_k(\pi) = \sum_{l=1}^{k} E_{l,\pi(l)}^{k,k}$. Then $P_k(\pi)$ is a permutation matrix. But $E_{i,j}^{m,n}E_{u,v}^{n,r} = \delta_{u,j}E_{i,v}^{m,r}$ (where $\delta_{u,j}$ is the Kronecker delta). Thus $E_{i,j}^{m,n}P_n(\pi) = E_{i,\pi(j)}^{m,n}$ and therefore $P_m(\alpha^{-1})E_{i,j}^{m,n}$

$P_n(\beta) = E^{m,n}_{\alpha(i),\beta(j)}$. If (a) holds in Lemma 3.5 then we define $U = P_m(\alpha^{-1})$ and $V = P_n(\beta)$. If A is any $m \times n$ Boolean matrix, we have $A = \sum\{E_{i,j} : a_{ij} = 1\}$ and hence $T(A) = \sum\{E_{\tau(i,j)} : a_{ij} = 1\} = \sum\{UE_{ij}V : a_{ij} = 1\} = UAV$. If (b) holds in Lemma 3.5, define $U = P_n(\beta^{-1})$ and $V = P_n(\alpha)$. Let T' be the operator on $\mathcal{M}_{m,n}(\mathbb{B})$ defined by $T'(A) = [T(A)]^t$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$. Then $T'(E_{ij}) = E_{\alpha(i),\beta(j)}$, so $T'(A) = P_n(\alpha^{-1})AP_n(\beta)$ by the result for conclusion (a). Therefore $T(A) = UA^tV$. By Corollary 3.4, (b) implies (a). So it suffices to show that (c) implies (b). Any operator T that satisfies (c) is invertible. In fact, $T^{-1}(A) = U^{-1}AV^{-1}$ or $T^{-1}(A) = U^{-1}A^tV^{-1}$. Let $b(X) = 1$. Then $X = \mathbf{a}\mathbf{b}^t$ where $\mathbf{a}$ is a $m \times 1$ Boolean matrix and $\mathbf{b}$ is a $n \times 1$ Boolean matrix. Thus $T(X) = UXV = U\mathbf{a}\mathbf{b}^tV = (U\mathbf{a})(\mathbf{b}^tV)$ where $U\mathbf{a} \in \mathcal{M}_{m,1}(B)$ and $V^t\mathbf{b} \in \mathcal{M}_{n,1}(B)$. By the definition of Boolean rank, $b(T(X)) = 1$. Let $b(X) = 2$. Then $X = AB$ where A, B are $m \times 2$ Boolean matrix and $2 \times n$ Boolean matrix, respectively. Thus $T(X) = UXV = UABV = (UA)(BV)$ where UA is a $m \times 2$ matrix and BV is a $2 \times n$ matrix. Thus $b(T(X)) \leq 2$. Suppose $b(T(X)) = 1$. Then $T(X) = UXV = CD$, where $C \in \mathcal{M}_{m,1}(\mathbb{B})$, $D \in \mathcal{M}_{1,n}(\mathbb{B})$. Therefore $X = U^{-1}CDV^{-1}$ and $b(X) = 1$, a contradiction to $b(X) = 2$. That is, T is rank 2 preserver. Since T is invertible, we conclude that T preserves the dimension of all rank 1 spaces by Lemma 2.4.2. $\square$

IV. Boolean Rank-k Preserving Operators

We study the extent to which known properties of Boolean linear operators preserving the ranks of Boolean matrices. For our purpose, we can define a Boolean linear operator preserving rank k matrices.

We say that an operator T *strongly* preserves Boolean rank k provided that $b(T(X)) = k$ if and only if $b(X) = k$.

Lemma 4.1. *If T is an invertible linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ that preserves the rank of every rank k matrix then T strongly preserves rank k, $1 \leq k \leq n$.*

Proof. Since T is a rank k preserver, we will show that if $b(T(X)) = k$ then $b(X) = k$. Consider the restriction of T to rank k matrices. Let $\mathbb{V} = \{X \in \mathcal{M}_{m,n}(\mathbb{B}) : b(X) = k\}$. Then $|\mathbb{V}|$ is finite. Since T is rank k preserver, $T(V) \subset V$. Thus we can write $T|_V : V \to V$. Suppose $T|_V$ is not injective. Then $T(X) = T(Y)$ if $X \neq Y$, for some $X, Y \in V$. This is a contradiction to injectivity of T. Thus $T|_V$ is injective. Let $b(T(X)) = k$, for $b(X) \neq k$, $X \in \mathcal{M}_{m,n}(\mathbb{B})$. But since $T|_V$ is injective and $|V|$ is finite, we can choose $B \in \mathcal{M}_{m,n}(\mathbb{B})$ such that $b(B) = k$ and $T(B) = T(X)$. Then $X \neq B$ if $T(X) = T(B)$. This is a contrandiction to injectivity of T. Therefore $b(X) = k$, that is, T strongly preserves rank k. □

Lemma 4.2. *If T is a Boolean linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ that preserves the rank of every rank 1 matrix then $T(\langle E_{ij}, E_{is}\rangle)$ and $T(\langle E_{ij}, E_{rj}\rangle)$ are rank 1 space for any i, j, $1 \leq i \leq m$, $1 \leq j \leq n$.*

Proof. Since T is a rank 1 preserver, $T(E_{ij} + E_{is}) = T(E_{ij}) + T(E_{is})$ and $b(E_{ij} + E_{is}) = 1$. Thus $T(\langle E_{ij}, E_{is}\rangle)$ is a rank 1 space. Similarly, $b(T(E_{ij} +$

$E_{rj}\rangle) = 1$, that is, $T(\langle E_{ij},\ E_{rj}\rangle)$ is rank 1 space. □

Lemma 4.3. *If T is an invertible Boolean linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ that preserves the rank of every rank 2 matrix then $T(\langle E_{ij},\ E_{is}\rangle)$ and $T(\langle E_{ij},\ E_{rj}\rangle)$ are rank 1 space for any $i,\ j,\ 1 \leq i \leq m,\ 1 \leq j \leq n$.*

Proof. Since T is invertible, T permutes $\mathcal{E}_{m,n}$, by Corollary 2.5.3. And since T permutes $\mathcal{E}_{m,n}$, $b(T(E_{ij}+E_{is})) = 1$ or 2. Suppose $T(\langle E_{ij},\ E_{is}\rangle)$ is not rank 1 space. For any $E_{ij},\ E_{is} \in \mathcal{E}_{m,n}$, $T(E_{ij}+E_{is}) = T(E_{ij})+T(E_{is}) = E_{xy}+E_{uv}$ with $x \neq u,\ y \neq v$. Thus $b(T(E_{ij} + E_{is})) = b(E_{xy} + E_{uv}) = 2$. Since T is an invertible and rank 2 preserver, T strongly preserves rank 2 by Lemma 4.1. This is a contradiction to assumption. Therefore $T(\langle E_{ij},\ E_{is}\rangle)$ is rank 1 space. Suppose $T(\langle E_{ij},\ E_{rj}\rangle)$ is not rank 1 space. For any $E_{ij},\ E_{rj} \in \mathcal{E}_{m,n}$, $T(E_{ij} + E_{rj}) = T(E_{ij}) + T(E_{rj}) = E_{x^*y^*} + E_{u^*v^*}$ with $x^* \neq u^*,\ y^* \neq v^*$. Thus $b(T(E_{ij}+E_{rj})) = b(E_{x^*y^*}+E_{u^*v^*}) = 2$. Since T is invertible and rank 2 preserver, T strongly preserves rank 2 by Lemma 4.1. This is a contradiction to assumption. Therefore $T(\langle E_{ij},\ E_{rj}\rangle)$ is rank 1 space. □

Lemma 4.4. *If T is an invertible Boolean linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ that preserves the rank of every rank k matrix with $3 \leq k \leq n$ then $T(\langle E_{ij},\ E_{is}\rangle)$ and $T(\langle E_{ij},\ E_{rj}\rangle)$ are rank 1 space for any $i,\ j,\ 1 \leq i \leq m,\ 1 \leq j \leq n$.*

Proof. Since T is an invertible linear operator, T permutes $\mathcal{E}_{m,n}$, by Corollary 2.5.3. And since T permutes $\mathcal{E}_{m,n}$, $b(T(E_{ij},\ E_{is})) = b(T(E_{ij}) + T(E_{is})) =$ 1 or 2. Suppose $T(\langle E_{ij},\ E_{is}\rangle)$ is not rank 1 space. For any $E_{ij},\ E_{is} \in \mathcal{E}_{m,n}$, $T(E_{ij} + E_{is}) = T(E_{ij}) + T(E_{is}) = E_{xy} + E_{uv}$ with $x \neq u,\ y \neq v$. Since T is an invertible, T is bijective by Corollary 2.5.2. We can choose $E_{k_il_i}$ such that $b(\sum_{i=1}^{k-2} E_{k_il_i}) = k-2$ for $k_i \neq x,\ k_i \neq u$ and $l_i \neq y,\ l_i \neq v$. Then

$b(\sum_{i=1}^{k-2} E_{k_i l_i} + E_{xy} + E_{uv}) = k$ and

$$\begin{aligned} T^{-1}(\sum_{i=1}^{k-2} E_{k_i l_i} + E_{xy} + E_{uv}) &= \sum_{i=1}^{k-2} T^{-1}(E_{k_i l_i}) + T^{-1}(E_{xy}) + T^{-1}(E_{uv}) \\ &= \sum_{i=1}^{k-2} T^{-1}(E_{k_i l_i}) + E_{ij} + E_{is}. \end{aligned}$$

Thus $b(\sum_{i=1}^{k-2} T^{-1}(E_{k_i l_i}) + E_{ij} + Eis) < k$. This is a contradiction that T strongly preserves rank k by Lemma 4.1. Therefore $T(\langle E_{ij}, E_{is}\rangle)$ is rank 1 space. Similarly, $T(\langle E_{ij}, E_{rj})$ is rank 1 space. □

From Lemma 4.2 , Lemma 4.3 and Lemma 4.4, we obtain the following.

Theorem 4.5. *If T is an invertible Boolean linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ that preserves the rank of every rank k matrix with $1 \leq k \leq n$ then $T(\langle E_{ij}, E_{is}\rangle)$ and $T(\langle E_{ij}, E_{rj}\rangle)$ are rank 1 space for any i, j, $1 \leq i \leq m$, $1 \leq j \leq n$.*

Lemma 4.6. *If T is an invertible linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ that preserves the rank of every rank 2 matrix and τ is the permutation of $\triangle_{m,n}$ representing T, then there exist permutations α, β of $\{1, 2, \cdots, m\}$ and $\{1, 2, \cdots, n\}$ respectively such that*

(a) $\tau(i, j) = (\alpha(i), \beta(j))$ for all $(i, j) \in \triangle_{m,n}$ or

(b) $m = n$ and $\tau(i, j) = (\beta(j), \alpha(i))$ for all $(i, j) \in \triangle_{m,n}$.

Proof. We will denote the abscissa of $\tau(i, j)$ by u_{ij} and its ordinate by v_{ij}. So $\tau(i, j) = (u_{ij}, v_{ij})$. Let $[\tau]$ be the $m \times n$ array whose (i, j)-th entry is (u_{ij}, v_{ij}). Since T is rank 2 preserver and invertible, $T(\langle E_{ij}, E_{ik}\rangle)$ and $T(\langle E_{ij}, E_{ki}\rangle)$ are rank 1 spaces, by Lemma 4.3. Thus any two entries in the same row (or culumn) of $[\tau]$ have a common abscissa or common ordinate. This is because $T(\langle E_{ij}, E_{ik}\rangle)$ and $T(\langle E_{ji}, E_{ki}\rangle)$ are rank 1 spaces. It follows that if $u_{i1} = u_{i2}$ (respectively $v_{i1} = v_{i2}$) then u_{i1} is the abscissa (respectively ordinate) of each entry in the i-th row of $[\tau]$. Let $\beta_i(j) = v_{ij}$(respectively u_{ij}). Then for all i, β_i permutes $\{1, 2, \cdots, n\}$. If x were a common abscissa for one row and y were a common ordinate for another, then (x, y) would belong to both rows (because $m \leq n$ and each β_i is a permutation), contradicting the injectivity of τ. Therefore either

(1) for all $(i, j) \in \triangle_{m,n}$, $u_{ij} = u_{i1}$ or

(2) for all $(i, j) \in \triangle_{m,n}$, $v_{ij} = v_{i1}$.

Suppose (1) holds. Define $\alpha(i) = u_{i1}$ for all i, $1 \leq i \leq m$. For some j, $v_{ij} = u_{i1}$ because β_i is a permutation. It follows that (u_{i1}, u_{i1}) occurs in the i-th row of $[\tau]$ and in no other. Thus α permutes $\{1, 2, \cdots, m\}$. Since T is invertible and rank 2 preserver, for $i \neq 1$, $T(\langle E_{1j}, E_{ij}\rangle) = \langle E_{ux}, E_{vy}\rangle$ is a rank 1 space with $u = \alpha(1)$, $v = \alpha(i)$ and $\beta_1(j) = x$, $\beta_i(j) = y$ by Lemma 4.3. But $\alpha(1) \neq \alpha(i)$, so $x = y$. Therefore $\beta_i = \beta_1$ for all i, $1 \leq i \leq m$. Let $\beta = \beta_1$, then $\tau(i, j) = (\alpha(i), \beta(j))$ for all $(i, j) \in \triangle_{m,n}$. If (2) holds then $m = n$. Let $\tau'(i,j) = (v_{ij}, u_{ij})$ for all $(i, j) \in \triangle_{m,n}$ and apply (1) to τ' to complete the proof of the lemma. □

Lemma 4.7. *If τ satisfies the conclusion of Lemma 4.6, then T is a (U, V)-operator.*

Proof. Let π be any permutation of $\{1, 2, \cdots, k\}$. Let $E_{i,j}^{m,n}$ denote an $m \times n$ matrix of the form E_{ij} defined in section II. Let $P_k(\pi) = \sum_{l=1}^{k} E_{l,\pi(l)}^{k,k}$. Then $P_k(\pi)$ is a permutation matrix. But $E_{i,j}^{m,n} E_{u,v}^{n,r} = \delta_{u,j} E_{i,v}^{m,r}$ (where $\delta_{u,j}$ is the Kronecker delta). Thus $E_{i,j}^{m,n} P_n(\pi) = E_{i,\pi(j)}^{m,n}$ and therefore

$P_m(\alpha^{-1})E^{m,n}_{i,j}P_n(\beta) = E^{m,n}_{\alpha(i),\,\beta(j)}$. If (a) holds in Lemma 4.6 then we define $U = P_m(\alpha^{-1})$ and $V = P_n(\beta)$. If A is any $m \times n$ Boolean matrix, we have $A = \sum\{E_{i,j} : a_{ij} = 1\}$ and thus $T(A) = \sum\{E_{\tau(i,\,j)} : a_{ij} = 1\} = \sum\{UE_{ij}V : a_{ij} = 1\} = UAV$. If (b) holds in Lemma 4.6, define $U = P_n(\beta^{-1})$ and $V = P_n(\alpha)$. Let T' be the operator on $\mathcal{M}_{m,n}(\mathbb{B})$ defined by $T'(A) = [T(A)]^t$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$. Then $T'(E_{ij}) = E_{\alpha(i),\,\beta(j)}$, so $T'(A) = P_n(\alpha^{-1})AP_n(\beta)$ by the result for conclusion (a). Hence $T(A) = UA^tV$. □

Lemma 4.8. *If T is an invertible linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ that preserves the rank of every rank k matrix and τ is the permutation of $\triangle_{m,n}$ representing T, then there exist permutations α, β of $\{1, 2, \cdots, m\}$ and $\{1, 2, \cdots, n\}$ respectively such that*

(a) $\tau(i,\,j) = (\alpha(i),\,\beta(j))$ for all $(i,\,j) \in \triangle_{m,n}$ or

(b) $m = n$ and $\tau(i,\,j) = (\beta(j),\,\alpha(i))$ for all $(i,\,j) \in \triangle_{m,n}$.

Proof. We will denote the abscissa of $\tau(i,\,j)$ by u_{ij} and its ordinate by v_{ij}. So $\tau(i,\,j) = (u_{ij},\,v_{ij})$. Let $[\tau]$ be the $m \times n$ array whose $(i,\,j)$-th entry is $(u_{ij},\,v_{ij})$. Since T is rank k preserver and invertible, $T(\langle E_{ij},\,E_{ik}\rangle)$ and $T(\langle E_{ij},\,E_{ki}\rangle)$ are rank 1 spaces, by Theorem 4.5. Thus any two entries in the same row (or column) of $[\tau]$ have a common abscissa or common ordinate. This is because $T(\langle E_{ij},\,E_{ik}\rangle)$ and $T(\langle E_{ji},\,E_{ki}\rangle)$ are rank 1 spaces. It follows that if $u_{i1} = u_{i2}$ (respectively $v_{i1} = v_{i2}$) then u_{i1} is the abscissa (respectively ordinate) of each entry in the i-th row of $[\tau]$. Let $\beta_i(j) = v_{ij}$(respectively u_{ij}). Then for all i, β_i permutes $\{1, 2, \cdots, n\}$. If x were a common abscissa for one row and y were a common ordinate for another then $(x,\,y)$ would belong to both rows (because $m \le n$ and each β_i is a permutation), contradicting the injectivity of τ. Therefore either

(1) for all $(i, j) \in \triangle_{m,n}$, $u_{ij} = u_{i1}$ or

(2) for all $(i, j) \in \triangle_{m,n}$, $v_{ij} = v_{i1}$.

Suppose (1) holds. Define $\alpha(i) = u_{i1}$ for all i, $1 \le i \le m$. For some j, $v_{ij} = u_{i1}$ because β_i is a permutation. It follows that (u_{i1}, u_{i1}) occurs in the i-th row of $[\tau]$ and in no other. Thus α permutes $\{1, 2, \cdots, m\}$. Since T is invertible and rank k preserver, for $i \ne 1$, $T(\langle E_{1j}, E_{ij}\rangle) = \langle E_{ux}, E_{vy}\rangle$ is a rank 1 space with $u = \alpha(1)$, $v = \alpha(i)$ and $\beta_1(j) = x$, $\beta_i(j) = y$, by Theorem 4.5. But $\alpha(1) \ne \alpha(i)$, so $x = y$. Therefore $\beta_i = \beta_1$ for all i, $1 \le i \le m$. Let $\beta = \beta_1$, then $\tau(i, j) = (\alpha(i), \beta(j))$ for all $(i, j) \in \triangle_{m,n}$. If (2) holds, then $m = n$. Let $\tau'(i,j) = (v_{ij}, u_{ij})$ for all $(i, j) \in \triangle_{m,n}$, and apply (1) to τ' to complete the proof of the lemma. □

Lemma 4.9. *If τ satisfies the conclusion of Lemma 4.8, then T is a (U, V)-operator.*

Proof. Let π be any permutation of $\{1, 2, \cdots, k\}$. Let $E^{m,n}_{i,j}$ denote an $m \times n$ matrix of the form E_{ij} defined in section II. Let $P_k(\pi) = \sum_{l=1}^{k} E^{k,k}_{l,\pi(l)}$. Then $P_k(\pi)$ is a permutation matrix. But $E^{m,n}_{i,j} E^{n,r}_{u,v} = \delta_{u,j} E^{m,r}_{i,v}$ (where $\delta_{u,j}$ is the Kronecker delta). Thus $E^{m,n}_{i,j} P_n(\pi) = E^{m,n}_{i,\pi(j)}$ and therefore $P_m(\alpha^{-1}) E^{m,n}_{i,j} P_n(\beta) = E^{m,n}_{\alpha(i),\beta(j)}$. If (a) holds in Lemma 4.8 then we define $U = P_m(\alpha^{-1})$ and $V = P_n(\beta)$. If A is any $m \times n$ Boolean matrix, we have $A = \sum\{E_{i,j} : a_{ij} = 1\}$ and thus $T(A) = \sum\{E_{\tau(i,j)} : a_{ij} = 1\} = \sum\{U E_{ij} V : a_{ij} = 1\} = UAV$. If (b) holds in Lemma 4.8, define $U = P_n(\beta^{-1})$ and $V = P_n(\alpha)$. Let T' be the operator on $\mathcal{M}_{m,n}(\mathbb{B})$ defined by $T'(A) = [T(A)]^t$ for all A in $\mathcal{M}_{m,n}(\mathbb{B})$. Then $T'(E_{ij}) = E_{\alpha(i),\beta(j)}$, so $T'(A) = P_n(\alpha^{-1}) A P_n(\beta)$ by the result for conclusion (a). Hence $T(A) = UA^tV$. □

Theorem 4.10. *Let T be a Boolean linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$. Then T is invertible and rank k preserver, $k \geq 1$ if and only if T is a (U, V)-operator.*

Proof. Suppose T is a (U, V)-operator. Then T is a invertible by Theorem 3.7. And let $b(X) = k$. Then $X = AB$, $A \in \mathcal{M}_{m,k}(\mathbb{B})$ and $B \in \mathcal{M}_{k,n}(\mathbb{B})$. Thus for $X \in M_{m.n}(\mathbb{B})$, $T(X) = UXV = UABV = (UA)(BV)$ where $UA \in M_{m,k}(\mathbb{B})$ and $BV \in M_{k,n}(\mathbb{B})$. Thus $b(T(X)) \leq k$. Suppose $b(T(X)) = l$, $l < k$. Then $T(X) = UXV = CD$, where $C \in \mathcal{M}_{m,l}(\mathbb{B}), D \in \mathcal{M}_{l,n}(\mathbb{B})$. That is, $X = U^{-1}CDV^{-1}$ and $b(X) \leq l$, a contradiction to $b(X) = k$. Thus T is a rank k preserver. Therefore T is invertible and rank k preserver. Conversely, let T be an invertible rank k preserver. Hence T is a (U, V)-operator, by Lemma 4.9. □

Corollary 4.11. *Let T be a Boolean linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$. The followings are equivalent.*

(a) T is invertible and rank 1 preserver.

(b) T preserves the ranks of all rank 1 matrices and rank 2 matrices and preserves the dimension of all rank 1 spaces.

(c) T is invertible and rank k preserver, $k \geq 1$.

(d) T is a (U, V)-operator.

Proof. From Theorem 3.7, (a) $\iff$ (b) $\iff$ (d). By Theorem 4.10, (c) $\iff$ (d). □

V. Boolean Rank preserver

The following is true for operator T on $m \times n$ matrices over an algebraically closed field $\mathbb{F}$. It characterizes the rank-preserving operators (see Marcus and Moyls [10] or Lautemann [9]).

(5.1) The rank of $T(A)$ equals the rank of A for all A if and only if T is a (U, V)-operator.

In this case, the characterization carries over completely to Boolean operators (Theorem 5.3). The theorem quoted at the outset of section III implies that

(5.2) Over $\mathbb{F}$, T is rank-preserving if and only if T perserves the rank of each rank 1 matrix.

Theorem 5.4 below gives a nearly exact analogue.

Lemma 5.1. *If A, B are in $\mathcal{M}_{m,n}(\mathbb{B})$, $A \neq B$, $p(A) \geq p(B)$, $m > 1$ and $b(A) = b(B) = 1$, then there exists C in $\mathcal{M}_{m,n}(\mathbb{B})$ such that $b(A + C) = 1$ and $b(B + C) = 2$.*

Proof. If $b(A + B) = 2$, then the conclusion is obtained by letting $C = A$. So we may assume that $b(A + B) = 1$. Define E_{pq} as in section 2.5. Factoring A, B and E_{pq}, we have $A = \mathbf{a}\mathbf{x}^t$, $B = \mathbf{b}\mathbf{y}^t$ and $E_{pq} = \mathbf{e}_p\mathbf{f}_q^t$. By our hypotheses and Lemma 2.6.1, $A \not\leq B$. Therefore Lemma 2.6.2 implies that (1) $\mathbf{a} = \mathbf{b}$ and $\mathbf{x} \neq \mathbf{y}$, or (2) $\mathbf{x} = \mathbf{y}$ and $\mathbf{a} \neq \mathbf{b}$, or (3) $\mathbf{b} \leq \mathbf{a}$, $\mathbf{b} \neq \mathbf{a}$, $\mathbf{y} \leq \mathbf{x}$ and $\mathbf{y} \neq \mathbf{x}$.

Case (1). We have $\mathbf{x} \not\leq \mathbf{y}$ because $\mathbf{a} = \mathbf{b}$ and $A \not\leq B$. So we can select $j, l \leq n$ so that $x_j = 1$, $y_j = 0$ and $x_l = 0$, $y_l = 1$. Since $A \neq 0$, there exists $i \leq m$ such that $a_i = 1$. If $a_i = 1$ and others are 0 then $B = E_{il}$. Put

$C = E_{kj}$ with $k \neq i$. Then $b(A+C) = 1$ and $b(B+C) = b(E_{il} + E_{kj}) = 2$. If there exists s so that $s \neq i$, $a_s = 1$ then $b(B + E_{ij}) = 2$. On the other hand, $A \geq E_{ij}$, $b(A + E_{ij}) = b(A) = 1$. Thus the conclusion is obtained by letting $C = E_{ij}$.

Case (2). We have $\mathbf{a} \not\leq \mathbf{b}$ because $\mathbf{x} = \mathbf{y}$ and $A \not\leq B$. So we can select $j, l \leq n$ so that $a_j = 1$, $b_j = 0$ and $a_l = 0$, $b_l = 1$. Since $A \neq 0$, there exists $i \leq m$ such that $x_i = 1$. If $x_i = 1$ and others are 0 then $B = E_{li}$. Put $C = E_{jk}$ with $k \neq i$. Then $b(A + C) = 1$ and $b(B + C) = b(E_{li} + E_{jk}) = 2$. If there exists s so that $s \neq i$, $x_s = 1$ then $b(B + E_{ji}) = 2$. On the other hand, $A \geq E_{ji}$, $b(A + E_{ji}) = b(A) = 1$. Thus the conclusion is obtained by letting $C = E_{ji}$.

Case (3). Since $\mathbf{b} \leq \mathbf{a}$, $\mathbf{b} \neq \mathbf{a}$, $\mathbf{y} \leq \mathbf{x}$ and $\mathbf{y} \neq \mathbf{x}$, $a_i = 1$, $b_i = 0$ and $x_j = 1$, $y_j = 0$, $a_s = 1$, $b_s = 1$, and $x_l = 1$, $y_l = 1$, for some i, j, s, l. Then $b(B + E_{ij}) = 2$. On the other hand, $A \geq E_{ij}$, $b(A + E_{ij}) = b(A) = 1$.Thus the conclusion is obtained by letting $C = E_{ij}$. □

Lemma 5.2. *If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ with $m > 1$ and T is not invertible but preserves the rank of rank 1 matrices, then T decrease the rank of some rank 2 matrix.*

Proof. By the proof of Corollary 3.3, T is not injective on $\mathcal{M}_1$ so $T(X) = T(Y)$ for some X, Y in $\mathcal{M}_1$ with $X \neq Y$. Without loss of generality, we may suppose that $p(X) \geq p(Y)$. By Lemma 5.1, there is some matrix D such that $b(X + D) = 2$ while $b(Y + D) = 1$. However, $T(X + D) = T(X) + T(D) = T(Y) + T(D) = T(Y + D)$. □

Theorem 5.3. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ with $m > 1$.*

Then T is a rank preserver if and only if T is a (U, V)-operator.

Proof. Suppose T is a rank preserver. Then T preserves the rank 1 and rank 2 matrices. By the contraposition of Lemma 5.2, T is invertible. By Theorem 3.7, T is a (U, V)-operator on $\mathcal{M}_{m,n}(\mathbb{B})$. Conversely, let T be a (U, V)-operator and for $X \in \mathcal{M}_{m,n}(\mathbb{B})$, let $b(X) = k$, with $1 \leq k \leq m$. Then $X = AB$ where $A \in \mathcal{M}_{m,k}(\mathbb{B}), B \in \mathcal{M}_{k,n}(\mathbb{B})$. Thus $T(X) = UXV = UABV = (UA)(BV)$ where $U \in \mathcal{M}_{m,m}(\mathbb{B}), V \in \mathcal{M}_{n,n}(\mathbb{B})$. Since $UA \in \mathcal{M}_{m,k}(\mathbb{B})$ and $BV \in \mathcal{M}_{k,n}(\mathbb{B}), b(T(X)) = k$, by the definition of Boolean rank. Hence T is a rank-preserver. □

Theorem 5.4. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$. Then T is a rank preserver if and only if T preserves the rank of all rank 1 and rank 2 matrices.*

Proof. Obviously, if T is a rank preserver then T preserves the rank of all rank 1 and rank 2 matrices. Conversely, let T preserve the rank of all rank 1 and rank 2 matrices. Then T is invertible, by Lemma 5.2. And since T is invertible and rank 1 preserver, T is a (U, V)-operator, by Theorem 3.7. Hence T is a rank preserver, by Theorem 5.3. □

Lemma 5.5. *Let T is a rank 1 preserving linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$. Then $b(X) \geq b(T(X))$.*

Proof. Let $b(X) = k$. Then $X = X_1 + X_2 + \cdots + X_k$, where $b(X_i) = 1$, $1 \leq i \leq k$. $T(X) = T(X_1 + X_2 + \cdots + X_k) = T(X_1) + T(X_2) + \cdots + T(X_k)$. Since T is a rank 1 preserver, $T(X_i)$ has Boolean rank 1. By the property of Boolean rank, $b(T(X)) \leq k$. □

Using the previous lemma, we obtain the following.

Theorem 5.6. *Let T be a Boolean linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$. T is a (U, V)-operator if and only if T strongly preserves rank 1.*

Proof. Suppose T strongly preserves rank 1. Obviously, T is rank 1 preserver. Let $b(X) = 2$. By Lemma 5.5, $b(T(X)) = 1$ or $b(T(X)) = 2$. If $b(T(X)) = 1$ then $b(X) = 1$ since T strongly preserves rank 1. This is a contradiction. Therefore T is rank 1 and rank 2 preserver. Thus T is a (U, V)-operator, by Theorem 5.3 and Theorem 5.4. Conversely, suppose T is a (U, V)-operator. We will show that if $b(T(X)) = 1$ then $b(X) = 1$. Since T is (U, V)-operator, $T(X) = UXV$ for nonsingular matrices $U \in \mathcal{M}_{m,m}(\mathbb{B})$ and $V \in \mathcal{M}_{n,n}(\mathbb{B})$. Then $X = U^{-1}T(X)V^{-1}$. Since $b(T(X)) = 1$, we can write $T(X) = \mathbf{ab}$, where $\mathbf{a}$ and $\mathbf{b}^t$ are $m \times 1$ and $n \times 1$ Boolean matrices. Therefore $X = U^{-1}T(X)V^{-1} = U^{-1}\mathbf{ab}^tV^{-1}$ where $U^{-1}\mathbf{a} \in \mathcal{M}_{m,1}(\mathbb{B})$ and $\mathbf{b}^tV^{-1} \in \mathcal{M}_{1,n}(\mathbb{B})$. By the definition of Boolean rank, $b(X) = 1$. Hence T strongly preserves rank 1. □

Corollary 5.7. *Let T be a Boolean linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$. The followings are equivalent.*

(a) *T is invertible and rank k preserver, with $1 \leq k \leq m$.*

(b) *T is rank 1 and rank 2 preserver.*

(c) *T is a (U, V)-operator.*

(d) *T is a rank preserver.*

(e) *T strongly preserves rank 1.*

Proof. By Theorem 5.3 and Theorem 5.4, (b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d). (a) $\Longleftrightarrow$ (c), by Theorem 4.10. (c) $\Longleftrightarrow$ (e), by Theorem 5.6. □

Example 5.8. Let $T : M_{3,3}(\mathbb{B}) \to M_{3,3}(\mathbb{B})$ be a Boolean linear operator given by $T((a_{ij})) = (a_{\delta(i)\tau(j)})$ where δ, $\tau \in S_3$ are permutations corresponding

to $U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, respectively.

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Then $T(A) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Since $UAV = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = T(A)$,

T is a (U, V)-operator.

Let $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Then $b(B) = 1$ and $T(B) = UBV = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, that is, $b(T(B)) = 1$.

Let $C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Then $b(C) = 2$ and $T(C) = UCV = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,

that is, $b(T(C)) = 2$.

Let $D = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then $b(D) = 3$ and $T(D) = UDV = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$,

that is, $b(T(D)) = 3$.

Example 5.8 shows that T satisfies the equivalent conditions of Corollary 5.7

Thinking of a Boolean matrix M as a zero-one matrix over a field, the (field) rank of M is 1 if and only if $b(M) = 1$ and the field rank of M is 2 only if $b(M) = 2$. Suppose the field rank of M is 1. Then the column rank of M is 1 and $M = [\mathbf{a}, \mathbf{a}, \cdots, \mathbf{a}]$ where $\mathbf{a}$ is $m \times 1$ zero-one matrix. Then $M = \mathbf{a}[1, 1, \cdots, 1]$. Thus $b(M) = 1$. Let $b(M) = 1$. Then $M = \mathbf{ab}^t = [\mathbf{a}b_1, \mathbf{a}b_2, \cdots, \mathbf{a}b_n]$ where $b_i = 0$ or 1, $1 \leq i \leq n$. Thus field rank of M is 1. Suppose the field rank of M is 2. Since M is zero-one matrix and has column rank 2, $M = X_1 + X_2$ where X_1 and X_2 are rank 1 zero-one matrices over field. By above X_1 and X_2 are Boolean rank 1 matrices. Therefore $b(M) = 2$.

Corollary 5.9. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$. Then T preserves the Boolean rank of all matrices if it preserves the field rank 1 and 2 zero-one $m \times n$ matrices.*

Proof. Suppose M is zero-one matrix over field. If the field rank of M is 1 then M has the Boolean rank 1, by above. If the field rank of M is 2 then M has the Boolean rank 2, by above. By Corollary 5.7, T preserves the Boolean rank of all matrices. □

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< Abstract >

Boolean Rank Preserving Linear Operator

In this thesis, we study Boolean rank preserving linear operator and found several equivalent conditions of Boolean rank preserving linear operator. That is, a linear operator preserves all rank if and only if either it strongly preserves rank 1 or it is invertible and rank k preserver for any k. This equivalent conditions generalize Boolean rank preserving linear operator which were obtained by L. B. Beasley and N. J. Pullman in [1].

감 사 의 글

먼저 하나님께 감사와 영광을 돌려드립니다. 바쁘신 가운데서도 깊은 관심과 배려로 지도해 주신 송석준교수님께 깊은 감사를 드립니다. 대학원 4학기 동안 훌륭한 강의를 해주시고 여러가지로 도움을 주신 고봉수, 양영오, 고윤희, 정승달 교수님들께 감사를 드립니다. 그리고 저를 항상 아껴주신 방은숙 교수님과 섬세하게 논문을 검토해주시고 도움을 주신 양성호 교수님 그리고 논문편집 과정에서 아낌없이 시간을 내어 도와주신 윤용식 교수님께 감사의 마음을 드립니다. 저를 변함없이 사랑하여 주시고 지금까지 저의 뒷바라지를 아끼지 아니하신 부모님과 언제나 밝은 웃음으로 저를 응원해준 사랑하는 딸 은애 그리고 연구에 전념할 수 있도록 희생적으로 저를 섬겨준 사랑하는 아내와 이 기쁨을 나누고 싶습니다. 4학기 동안 기쁠때 같이 웃고 힘들때 서로 격려하며 같이 생활해온 이진아, 김희선, 김인영 동기들과 저를 위해 끊임없이 기도해주신 형제자매님들께 감사를 드립니다.

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