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A Comparison of Boolean Ranks and Their Preservers

濟州大學校 大學院
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A Comparison of Boolean Ranks and Their Preservers

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CONTENTS

< **Abstract** >

A comparison of Boolean ranks and their preservers

Boolean rank, column rank, and maximal column rank over Boolean matrices have been studied and developed so far. And their preservers also have been characterized over Boolean matrices.

In this thesis, we compare Boolean rank and maximal column rank by the way of a function, β and we obtain the values of this function on the matrices over binary Boolean algebra, nonnegative integers, nonnegative reals, and general Boolean algebra, respectively. Finally we also characterize the linear operators that preserve maximal column rank over general Boolean matrices.

I. Introduction

There is much literature on the study of matrices over a finite Boolean algebra. But many results in Boolean matrix theory are stated only for binary Boolean matrices. This is due in part to a semiring isomorphism between the matrices over the Boolean algebra of subsets of a k element set and the k tuples of binary Boolean matrices. This isomorphism allows many questions concerning matrices over an arbitrary finite Boolean algebra to be answered using the binary Boolean case. However there are interesting results about the general (i.e. nonbinary)Boolean matrices that have not been mentioned and they differ somewhat from the binary case.

In many instances, the extension of results to the general case is not immediately obvious and an explicit version of the above mentioned isomorphism was not well known. In [5], Kirland and Pullman gave a way to derive results in the general Boolean algebra case by means of a canonical form derived from the isomorphism. In [7], Hwang, Kim and Song characterized the linear operators that preserve maximal column rank of Boolean matrices.

In [3], Beasley and Pullman compared semiring rank and column rank of the matrices over several semirings. The difference between semiring rank and column rank motivated Beasley and Song to investigate the column rank preservers of matrices over nonnegative integers [4] and over the binary Boolean algebra [6]. In [7], Hwang, Kim and Song compared column ranks with maximal column ranks over certain semirings.

In this paper, first we will show the extent of the difference between semiring rank and maximal column rank of matrices over a general Boolean algebra. Second, there are some unproved ones on semiring rank, column rank, and maximal column rank through the previous theses, and so we will give the solutions of them. Finally, we also obtain the characterizations of the linear operators that preserve maximal column ranks of general Boolean matrices.

In Chapter II, we present some definitions, and some properties which come from these definitions. In Chapter III, we introduce various ranks over Boolean matrices and compare them to see how much different they are in $\mathbb{B}_1$,$\mathbb{Z}^+$,$\mathbb{F}^+$ and $\mathbb{B}$. In detail, in section 3.1, a new function, β, between Boolean rank and Boolean maximal column rank over Boolean matrices is defined, and so some results are obtained. In section 3.2, we get the others of results in this Chapter. In Chapter IV, we characterize the linear operators that preserve the Boolean maximal column rank of general Boolean matrices.

II. Preliminaries

In this section, we give some known results and obtain basic results.

Let $\mathbb{B}$ be the *Boolean algebra* of the subsets of a k element set S_k and $\sigma_1, \sigma_2, \cdots, \sigma_k$ denote the singleton subsets of S_k. We write $+$ for union and denote intersection by juxtaposition;0 denotes the null set and 1 the set S_k. Under these two operations, $\mathbb{B}$ is a commutative, antinegative semiring(that is, only 0 has an additive inverse); all of its elements, except 0 and 1, are zero-divisors. Let $\mathbb{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{B}$. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well.

Definition [5] [the p-th constituent of A]. *Let A be an $m \times n$ matrix over $\mathbb{B}$. The p-th constituent of A, A_p is*

$$(a_p)_{st} = \begin{cases} 1 & \text{if } \sigma_p \subseteq a_{st}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that via the constituents, A can be written uniquely as $\sum \sigma_p A_p$, which is called the canonical form of A.

Example 2.1. Let $\mathbb{B} = \mathcal{P}(S_2) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ and $A = \begin{pmatrix} \emptyset & \{1\} & \{1,2\} \\ \{2\} & \{1,2\} & \emptyset \end{pmatrix}$. Then $A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Thus

$$\begin{aligned} A &= \sigma_1 A_1 + \sigma_2 A_2 \\ &= \{1\} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \{2\} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \emptyset & \{1\} & \{1\} \\ \emptyset & \{1\} & \emptyset \end{pmatrix} + \begin{pmatrix} \emptyset & \emptyset & \{2\} \\ \{2\} & \{2\} & \emptyset \end{pmatrix} \\ &= \begin{pmatrix} \emptyset & \{1\} & \{1,2\} \\ \{2\} & \{1,2\} & \emptyset \end{pmatrix}. \end{aligned}$$

□

This definition is one of the ways of transforming nonbinary Boolean algebra into binary Boolean algebra.

Remark 2.2. Let $\mathbb{B}$ be a Boolean algebra. Then we have

(1) the singletons are mutually orthogonal idempotent, for $\sigma_p \cdot \sigma_p = \sigma_p$ and $\sigma_p \cdot \sigma_q = \emptyset$ whenever $p \neq q$.

(2) from the fact that the decomposition is unique and (1), for all $m \times n$ matrices A , all $n \times r$ matrices B and C, and all $\alpha \in \mathbb{B}$,

(i) $(AB)_p = A_p B_p$,

(ii) $(B + C)_p = B_p + C_p$,

(iii) $(\alpha A)_p = \alpha_p A_p$ for all $1 \leq p \leq k$.

Proof. (2) For let $A = \sum \sigma_p A_p$, $B = \sum \sigma_p B_p$, and $C = \sum \sigma_p C_p$. Then $A \cdot B = (\sum \sigma_p A_p) \cdot (\sum \sigma_p B_p) = \sum \sigma_p A_p B_p$, and so

$$(AB)_p = A_p B_p.$$

But $(B+C) = (\sum \sigma_p B_p) + (\sum \sigma_p C_p) = \sum \sigma_p (B_p + C_p)$, and so

$$(B+C)_p = B_p + C_p.$$

Finally, $\alpha A = \alpha(\sum \sigma_p A_p) = \sum \alpha \sigma_p A_p = \alpha_p A_p$, and hence

$$(\alpha A)_p = \alpha_p A_p.$$

□

III. Comparisons of various ranks over Boolean matrices

3.1 Boolean rank versus Boolean maximal column rank

Definition 3.1 [2] [the Boolean rank , b(A)]. *Let* $A(\neq 0) \in \mathbb{M}_{m,n}(\mathbb{B})$. *The Boolean rank,* $b(A)$ *is the least index* r *such that* $A = B_{m\times r} \cdot C_{r\times n}$.

Note that $b(\emptyset) = 0$.

In the case that $\mathbb{B} = \mathbb{B}_1 = \{0,1\}$, we consider $b(A)$ as the binary Boolean rank, and denote it by $b_1(A)$.

For a binary Boolean matrix A , we have $b(A) = b_1(A)$ by the definition.

The following Lemma can be easily taken from the definition.

Lemma 3.1 [1]. *For any* $m \times n$ *, zero-one matrix* A *:*

(1) $b_1(A) = b_1(A^T)$,

(2) $b_1(AB) \leq \min(b_1(A), b_1(B))$ *for all* $n \times k$*, zero-one matrices* B*, and*

(3) $b_1(A) \leq \min(m,n)$.

Proof. (1) : Put $b_1(A) = k$. Then there exist $B_{m\times k}, C_{k\times n}$ such that $A = BC$. Since $A^T = (BC)^T = C^T_{n\times \mathbf{k}} \cdot B^T_{\mathbf{k}\times m}$, $b_1(A) = k \geq b_1(A^T)$ ——(i). Similarly, if $b(A^T) = k$, then there exist $B_{n\times k}, C_{k\times m}$ such that $A^T = BC$. $A = (A^T)^T = C^T_{m\times \mathbf{k}} \cdot B^T_{\mathbf{k}\times n}$, and so $b_1(A^T) = k \geq b_1(A)$ ——(ii). From (i) and (ii), $b_1(A) = b_1(A^T)$.

(2) : If $b_1(A) = h$, then there exist $A^1_{m\times h}, A^2_{h\times n}$ such that $A = A^1A^2$. So, $AB = (A^1A^2)B = A^1_{m\times \mathbf{h}} \cdot (A^2B)_{\mathbf{h}\times k}$.

Thus $b_1(AB) \leq h = b_1(A)$. Similarly, we can get $b_1(AB) \leq b_1(B)$. Hence $b_1(AB) \leq \min(b_1(A), b_1(B))$.

(3) : Note that $A_{m\times n} = A_{m\times n} \cdot I_{n\times n}$, and $A_{m\times n} = I_{m\times m} \cdot A_{m\times n}$.

So we can immedately obtain $b_1(A) \leq \min(m,n)$. □

Now we can generalize this Lemma as the following:

Lemma 3.2. *For any $m \times n$ matrix A :*

(1) $b(A) = b(A^T)$,

(2) $b(AB) \leq \min(b(A), b(B))$ for all $n \times k$ matrices B, and

(3) $b(A) \leq \min(m,n)$.

Proof. Similarly we prove this Lemma as the above. □

Examples 3.1.

(A) $b_1(I_2) = b\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2.$

(B) $b_1(I_3) = b\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 3.$

For we can easily obtain $b_1(I_3) \neq 0, 1$ and through checking case by case, also notice that it is impossible that $I_3 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_4 & \beta_5 & \beta_6 \end{pmatrix}$.

In fact , it is well-known that generally $b_1([I_n]) = n$. □

Let $\mathbb{V}$ be nonempty subset of $\mathbb{M}_{r,1}(\mathbb{B})$ such that it is closed under $+$ and $\cdot$ by scalars. Then $\mathbb{V}$ is called a *vector space* over $\mathbb{B}$.

We define "subspace" and "generating sets" as the things to coincide with familar definitions when $\mathbb{B}$ is a field. We think of $< F >$ as the subspace generated by the subset F of $\mathbb{V}$.

As with fields, a *basis* for a vector space $\mathbb{V}$ is a generating subset of the least cardinality. That cardinality is the *dimension*, $dim(\mathbb{V})$ of $\mathbb{V}$.

Definition 3.2 [2] [the Boolean column rank]. *The Boolean column rank, $c(A)$ of $A \in \mathbb{M}_{m\times n}(\mathbb{B})$ is the dimension of the space $< A >$ generated by the columns of A.*

In the binary Boolean algebra, we denote it by $c_1(A)$ for $A \in \mathbb{M}_{m\times n}(\mathbb{B}_1)$.

Definition 3.3 [7]. *A set G of vectors over $\mathbb{B}$ is linearly dependent if for some $g \in G, g \in < G\backslash\{g\} >$. Otherwise, G is linearly independent.*

Definition 3.4 [7] [the maximal column rank]. *The maximal column rank , $mc(A)$ of an $m \times n$ matrix A over $\mathbb{B}$ is the maximal number of the columns of A which are linearly independent over $\mathbb{B}$.*

When $\mathbb{B} = \mathbb{B}_1$, we denote it by $mc_1(A)$ for $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$.

Example 3.2. Let $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Then $b_1(A) \leq 3$ by Lemma 3.1. Clearly $b_1(A) \neq 0$.

If $b_1(A) = 1$, then $A = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \end{pmatrix}$.

From the first column and row of A , $y_1 = 1$ and $x_1 = 1$.

This is a contradiction of the fact that $x_1 \cdot y_1 = 0$.

Thus $b_1(A) \neq 1$.

If $b_1(A) = 2$, then

$$A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \\ x_5 & x_6 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ y_5 & y_6 & y_7 & y_8 \end{pmatrix}$$

$$= \begin{pmatrix} x_1y_1 + x_2y_5 & x_1y_2 + x_2y_6 & x_1y_3 + x_2y_7 & x_1y_4 + x_2y_8 \\ x_3y_1 + x_4y_5 & x_3y_2 + x_4y_6 & x_3y_3 + x_4y_7 & x_3y_4 + x_4y_8 \\ x_5y_1 + x_6y_5 & x_5y_2 + x_6y_6 & x_5y_3 + x_6y_7 & x_5y_4 + x_6y_8 \end{pmatrix}$$

So, suppose $x_1 = 0$ or $x_1 = 1$. Then we can lead $x_5 = 0$ and $x_6 = 0$ (;contradiction!!). Therefore $b_1(A) \neq 2$. Hence $b_1(A) = 3$.

Futhermore, since less than or equal to 3 columns of A cannot generate A, the four columns of A constitute a basis for the column space of A over $\mathbb{B}_1$.

So $c_1(A) = 4$.

And since all columns of A are linearly independent,

$mc_1(A) = 4$. Therefore we get

$$c_1(A) = mc_1(A) = 4.$$

□

We can immediately get the following :

$$0 \leq b(A) \leq c(A) \leq mc(A) \leq n$$

for all $m \times n$ matrices A over $\mathbb{B}$.

(3.1) For all $m \times n$ matrices A over $\mathbb{B}$,

$$0 \leq b(A) \leq mc(A) \leq n.([3])$$

Proof. Obviously, $0 \leq b(A)$ and $mc(A) \leq n$. We only show that $b(A) \leq mc(A)$. If $mc(A) = k$, then $\exists$ k linearly independent columns of A, say, $A_1, A_2, \cdots, A_k$. So, we obtain $A_{k+i} = \sum_{j=1}^{k} a_j^{(i)} \cdot A_j$, where i=1,2,$\cdots$,n-k. Let's rearrange the columns of A. Then we can write down A as $(A_1, A_2, \cdots, A_k, A_{k+1}, \cdots, A_n)$. Thus

$$\begin{aligned} A &= (A_1, A_2, \cdots, A_k, A_{k+1}, \cdots, A_n) \\ &= (A_1, A_2, \cdots, A_k)_{m \times k} \left(\begin{array}{c|cccc} & a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(n-k)} \\ I_k & \vdots & \vdots & \ddots & \vdots \\ & a_k^{(1)} & a_k^{(2)} & \cdots & a_k^{(n-k)} \end{array} \right)_{k \times n}. \end{aligned}$$

Therefore $b(A) \leq k$. □

(3.2) For any $p \times q$ matrix A over $\mathbb{B}$, the Boolean rank of $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ is $b(A)$ and its Boolean maximal column rank is $mc(A)$.

Proof.

From the definition, it is easily taken that $mc\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = mc(A)$.

$$\text{To show} \ : \ b\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = b(A).$$

$(\leq)$; If $b(A) = k$, then we obtain $A = M_{p \times k} N_{k \times q}$ for some M,N.

Note that

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} MN & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} M \\ 0 \end{bmatrix}_{m\times \mathbf{k}} [N \quad 0]_{\mathbf{k}\times n}.$$

Thus

$$b\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) \leq k.$$

$(\geq)$; Note that $A_{p\times q} = [I_p \quad 0]_{p\times m} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}_{m\times n} \begin{bmatrix} I_q \\ 0 \end{bmatrix}_{n\times q}.$

By Lemma 3.2(2),

$$b(A) \leq b\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right).$$

□

(3.3) The Boolean rank of a matrix is the maximum of the binary Boolean ranks of its constituents ([5]).

We will prove this later(Lemma 3.5 and Lemma 3.6).

Lemma 3.3. *For any binary Boolean matrix A, we have $mc(A) = mc_1(A)$.*

Proof. Assume $mc(A) = k$. Then $\exists$ k columns $\mathbb{X}_1, \mathbb{X}_2, \cdots, \mathbb{X}_k$ which are linearly independent over $\mathbb{B}_1$. Consider $(\mathbb{X}_1)_p, (\mathbb{X}_2)_p, \cdots, (\mathbb{X}_k)_p$. If $(\mathbb{X}_i)_p = \sum_{j\neq i}(\mathbb{X}_j)_p$, then

$$\mathbb{X}_i = (\mathbb{X}_i)_p = \sum_{j\neq i}(\mathbb{X}_j)_p = \sum_{j\neq i}\mathbb{X}_j.$$

This contradicts to the assumption.

Thus $mc_1(A) \geq k$.

Conversely, suppose $mc_1(A) = k$. Then $\exists$ k columns, $\mathbb{Y}_1, \mathbb{Y}_2, \cdots, \mathbb{Y}_k$ which are linearly independent over $\mathbb{B}_1$. If $\mathbb{Y}_i = \sum_{j \neq i}(\alpha_j)\mathbb{Y}_j$ where $\alpha_j \in \mathbb{B}_1$, then

$$\begin{aligned} \mathbb{Y}_i = (\mathbb{Y}_i)_p &= \sum_{j \neq i}(\alpha_i)_p(\mathbb{Y}_j)_p \\ &= \sum_{j \neq i}(\alpha_i)_p \mathbb{Y}_j. \end{aligned}$$

This is a contradiction. Therefore

$$mc(A) \geq k.$$

□

Definition 3.5 [3][$\mu(\mathbb{B}, m, n)$]. *$\mu(\mathbb{B}, m, n)$ is the largest integer r such that for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$, $b(A) = c(A)$ if $b(A) \leq r$.*

Definition 3.6 [7][$\alpha(\mathbb{B}, m, n)$]. *$\alpha(\mathbb{B}, m, n)$ is the largest integer r such that for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$, $c(A) = mc(A)$ if $c(A) \leq r$.*

Referring to the above, we may similarly define the following ;

Definition 3.7 [$\beta(\mathbb{B}, m, n)$]. *$\beta(\mathbb{B}, m, n)$ is the largest integer r such that for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$, $b(A) = mc(A)$ if $b(A) \leq r$.*

In general,

$$0 \leq \beta(\mathbb{B}, m, n) \leq n.$$

(3.4) Over any Boolean algebra $\mathbb{B}$, if $mc(A) > b(A)$ for some $p \times q$ matrix A, then for all $m \geq p$ and $n \geq q$, $\beta(\mathbb{B}, m, n) < b(A)$.

Proof. Since $mc(A) > b(A)$ for some $p \times q$ matrix A, we have $\beta(\mathbb{B}, p, q) < b(A)$ from the definition. Let $B = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ be an $m \times n$ matrix containing A as a submatrix. Then

$$b(B) = b(A) < mc(A) = mc(B).$$

So, $b(B) < mc(B)$. Hence

$$\beta(\mathbb{B}, m, n) < b(B),$$

for all $m \geq p$ and $n \geq q$. □

Lemma 3.4. *In* $\mathbb{B}_1$, $b_1(A) = 1$ *if and only if* $mc_1(A) = 1$.

Proof. $(\Longleftarrow)$; It is obvious.

$(\Longrightarrow)$; Suppose $b_1(A) = 1$. Then A can be split into two matrices, that is ,

$$A = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_m \end{pmatrix}_{m \times 1} \begin{pmatrix} n_1 & n_2 & \cdots & n_n \end{pmatrix}_{1 \times n}$$

$$= \begin{pmatrix} m_1 n_1 & \cdots & m_1 n_i & \cdots & m_1 n_n \\ m_2 n_1 & \cdots & m_2 n_i & \cdots & m_2 n_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ m_m n_1 & \cdots & m_m n_i & \cdots & m_m n_n \end{pmatrix}.$$

If $\exists\ n_i, n_j \neq 0\ (i \neq j)$, then $n_i = n_j = 1$ ($\because n_i, n_j \in \mathbb{B}_1 = \{0, 1\}$) So ith and jth columns of A are linearly dependent. Thus we get $mc_1(A) = 1$. □

Generally speaking, it is false that $b(A) = 1$ if and only if $mc(A) = 1$. Now, we suggest a counter-example.

Counter-example 3.3. Let $A = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$ where $\sigma_1, \sigma_2, \sigma_3$ are mutually distinct.

Then since $A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$,

$$b(A) = 1.$$

But it is easily obtained that $mc(A) = 3$. □

Theorem 3.1.

$$\beta(\mathbb{B}_1, m, n) = \begin{cases} 1 & \text{if } \min\{m, n\} = 1, \\ 3 & \text{if } m \geq 3 \text{ and } n = 3, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 3.4 , we have

$$\beta(\mathbb{B}_1, m, n) = 1,$$

whenever $\min\{m, n\}$=1. Let $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Then from Example 3.2,

$$mc_1(A) = 4 \quad \text{and} \quad b_1(A) = 3.$$

By (3.4),

$$\beta(\mathbb{B}_1, m, n) \leq 2,$$

for all $m \geq 3$ and $n \geq 4$.

Suppose $m \geq 2$ and $n \geq 2$. Then

$$b_1(A) = 2 \quad \text{iff} \quad mc_1(A) = 2 - - - - - - - - - (*).$$

For if $mc_1(A)$=2 ,then $b_1(A)$= 1 or 2.

So, $b_1(A)$=2 by Lemma 3.4.

Conversely, suppose $b_1(A)$=2. Then $\exists\ F_{m,2}, G_{2,n}$ such that $A = FG$. For some permutation P,

$$GP = \begin{pmatrix} 1 & 0 & x_3 & x_4 & \cdots & x_n \\ 0 & 1 & y_3 & y_4 & \cdots & y_n \end{pmatrix}, \quad \textit{or}$$
$$GP = \begin{pmatrix} 1 & 1 & x_3 & x_4 & \cdots & x_n \\ 0 & 1 & y_3 & y_4 & \cdots & y_n \end{pmatrix}, \quad \textit{or}$$
$$GP = \begin{pmatrix} 1 & 0 & x_3 & x_4 & \cdots & x_n \\ 1 & 1 & y_3 & y_4 & \cdots & y_n \end{pmatrix}, \quad \textit{with} \quad x_i, y_i \in \mathbb{B}_1.$$

If not, then $b_1(G)$=1 and hence $b_1(A)$=1. This is a contradiction. Hence certain two columns of F are maximal linearly independent columns of A. That is to say, $mc_1(A)$=2. Therefore we get

$$\beta(\mathbb{B}_1, m, n) \geq 2,$$

for all $\min\{m, n\} \geq 2$.

Finally we only show the case of $m \geq 3$ and n=3. Note that

$$b_1(A) = 3 \quad \text{iff} \quad mc_1(A) = 3,$$

whenever $A \in \mathbb{M}_{m,3}(\mathbb{B}_1)(m \geq 3)$.

For if $mc_1(A)$=3 ,then $b_1(A)$= 1 or 2 or 3.

But $b_1(A) \neq 1$ and 2 by Lemma 3.4 and $(*)$. Therefore

$$b_1(A) = 3.$$

Conversely , if $b_1(A)$=3 ,then $mc_1(A) \geq 3$ but $mc_1(A) \leq 3$. Therefore

$$mc_1(A) = 3.$$

Hence

$$\beta(\mathbb{B}_1, m, n) = \begin{cases} 1 & \text{if } \min\{m,n\} = 1, \\ 3 & \text{if } m \geq 3 \text{ and } n = 3, \\ 2 & \text{otherwise.} \end{cases}$$ □

Lemma 3.5. *If $b(A) = r$ and $\sum \sigma_p A_p$ is the canonical form of $A \in \mathbb{M}_{m,n}(\mathbb{B})$, then $\max\{b_1(A_p) \mid 1 \leq p \leq k\} \leq r$.*

Proof. Suppose $b(A) = r$. Then there exist $M_{m,r}, N_{r,n}$ such that $A = MN$. Therefore

$$A_p = (MN)_p = M_p \cdot N_p.$$

Hence

$$b_1(A_p) \leq r \ \ (1 \leq p \leq k),$$

because M_p, N_p are $m \times r, r \times n$ matrices respectively. □

Lemma 3.6. *If $\max\{b_1(A_p) \mid 1 \leq p \leq k\} = r$ and $\sum \sigma_p A_p$ is the canonical form of $A \in \mathbb{M}_{m,n}(\mathbb{B})$, then $b(A) \leq r$.*

Proof. Suppose $\max\{b_1(A_p) \mid 1 \leq p \leq k\} = r$. Put $b_1 A_p = r$, for some p $(1 \leq p \leq k)$.

$$\text{To show } : \ b(A) \geq b_1(A_p).$$

If $b(A) < b_1(A_p) = r$, then there exist a positive integer k such that $k < r$ and $A = B_{m\times k}\cdot C_{k\times n}$ for some B, C. Since $A_p = (BC)_p = B_pC_p$ and B_p, C_p are $m\times k, k\times n$ matrices respectively,

$$b_1(A_p) \le k.$$

This is a contradiction of the fact that $b_1(A_p) = r$. Therefore

$$b(A) \ge \max\{b_1(A_p) \mid 1 \le p \le k\}. \qquad \square$$

From Lemma 3.4 and Lemma 3.5, we prove (3.3).

Example 3.4. Let $A = \begin{bmatrix} \sigma_1 & \sigma_2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Then if $b(A) = 1$, then there exist $F_{2,1}, G_{1,3}$ such that $A = F\cdot G$. Put $F = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $G = (\, y_1 \quad y_2 \quad y_3\,)$. Then

$$A = \begin{bmatrix} \sigma_1 & \sigma_2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot [\, y_1 \quad y_2 \quad y_3\,]$$
$$= \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \end{bmatrix}.$$

Thus $x_1y_3 = x_2y_2 = x_2y_3 = 1$ and so $x_1 = x_2 = y_2 = y_3 = 1$. Hence we have $\begin{pmatrix} \sigma_1 & \sigma_2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} y_1 & 1 & 1 \\ y_1 & 1 & 1 \end{pmatrix}$, but it could not be happened. Therefore $b(A)$ must be 2. But

$$b_1(\sigma_1 A) = b_1\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = 2,$$
$$b_1(\sigma_2 A) = b_1\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = 1,$$
$$b_1(\sigma_p A) = b_1\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = 2 \quad (p \ge 3). \qquad \square$$

Lemma 3.7. *If* $mc(A) = r$ *and* $\sum \sigma_p A_p$ *is the canonical form of* $A \in \mathbb{M}_{m,n}(\mathbb{B})$*, then* $\max\{mc_1(A_p) \mid 1 \le p \le k\} \le r$.

Proof. Suppose $mc(A) = r$. If $mc_1(A_p) > r$ for some p , then there exist r+1 columns $(A_1)_p, (A_2)_p, \cdots, (A_{r+1})_p$ which are linearly independent over $\mathbb{B}$. Consider $A_1, A_2, \cdots, A_{r+1}$ of A. Since $mc(A) = r$, these are linearly dependent over $\mathbb{B}$. So $A_j = \sum_{i \ne j}^{r+1} \alpha_i A_i$, $(\alpha_i \in \mathbb{B})$. Thus $(A_j)_p = \left(\sum_{i \ne j}^{r+1} \alpha_i A_i\right)_p = \sum_{i \ne j}^{r+1} (\alpha_i)_p (A_i)_p$ which leads to a contradiction. Therefore

$$mc_1(A_p) \le r = mc(A), \quad \forall p.$$

□

We observe that we may say that the inequality in Lemma 3.7 would be strict for $r > 1$ as shown in the following Example.

That is to say, from Lemma 3.5 and Lemma 3.6 we can find out the Boolean rank from its p-th constituents but we can not exactly obtain the Boolean maximal column rank from its p-th constituents. We only know that $\max\{mc_1(A_p \mid 1 \le p \le k\} \le mc(A)$.

Example 3.5. Let $A = \begin{bmatrix} \sigma_1 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, where σ_1 is a singleton subset of S_k. Then $mc(A) = 3$. For if α,β is a singleton subset of S_k, then first $\alpha \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} \sigma_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha\sigma_1 + \beta\sigma_1 \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So $\beta = 1$ and hence $\alpha\sigma_1 + \beta\sigma_1 = \alpha\sigma_1 + \sigma_1 = \sigma_1 = 1$ (It is impossible !!). Second, $\alpha \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha\sigma_1 + \beta \\ \beta \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ 1 \end{bmatrix}$. So, $\beta = 1$ and hence $\alpha\sigma_1 + \beta = \alpha\sigma_1 + 1 = 1 = \sigma_1$ (It is impossible !!). Finally,

$\alpha \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha\sigma_1 + \beta \\ \alpha + \beta \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix}$. So, $\alpha + \beta = 0$ and hence $\alpha = \beta = 0$. Thus $\alpha\sigma_1 + \beta = 0 = \sigma_1$ (It is impossible !!).

But $mc_1(A_1) = mc_1\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = 2$ and we also have

$mc_1(A_p) = mc_1\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = 2$, for all p=2,3,$\cdots$,k. □

Lemma 3.8. *If $\mathbb{B}$ is a nonbinary Boolean algebra and $n \geq 2$,*

$$\beta(\mathbb{B}, m, n) = 0.$$

Proof. Let $A = (\sigma_1 \ \ \sigma_2)_{1\times 2}$, where σ_1 and σ_2 are distinct. Then $b(A) = 1$,but $mc(A) = 2$. So by (3.4), we have

$$\beta(\mathbb{B}, m, n) = 0,$$

for all $n \geq 2$. □

Theorem 3.2. *For a nonbinary Boolean algebra $\mathbb{B}$,*

$$\beta(\mathbb{B}, m, n) = \begin{cases} 0 & \text{if } n \geq 2, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Consider the case, n=1. Then for any $A \in \mathbb{M}_{m,1}$, $b(A) = 1$ and $mc(A) = 1$. Therefore

$$\beta(\mathbb{B}, m, 1) = 1.$$

If $n \geq 2$, then by Lemma 3.8

$$\beta(\mathbb{B}, m, n) = 0.$$

Hence we obtain the desired result. □

3.2 Comparisons of rank, column rank and maximal column rank over Boolean matrices

In this section, we shall now discuss some proofs which are related with μ,α and β in $\mathbb{B}_1$,$\mathbb{Z}^+$,$\mathbb{F}^+$, and $\mathbb{B}$. We will make a table to help figure out them with a look.

Theorem 3.3. *Let $\mathbb{Z}^+$ be a semiring of nonnegative integers. Then for $m \geq 1$,*

$$\beta(\mathbb{Z}^+, m, n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Proof. It is clear when n=1. Consider $A = (2\ \ 3)_{1\times 2}$ Then $b(A) = 1$ but $mc(A) = 2$. By (3.4),

$$\beta(\mathbb{Z}^+, m, n) = 0,$$

for all $n \geq 2$. Hence the proof is completed. □

Corollary.

$$\mu(\mathbb{Z}^+, m, n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Proof. Similarly, consider $A = (2\ \ 3)_{1\times 2}$. Then we can obtain the desired result from the fact that $b(A) = 1$ and $c(A) = 2$. □

Theorem 3.4. *Let $\mathbb{F}$ be a subfield of the reals, and $\mathbb{F}^+$ be the subset of $\mathbb{F}$ consisting of the nonnegative members of $\mathbb{F}$. Then*

$$\beta(\mathbb{F}^+, m, n) = \begin{cases} 1 & \text{if } \min\{m, n\} = 1, \\ 3 & \text{if } m \geq 3 \text{ and } n = 3, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. In $\mathbb{F}^+$,

$$r(A) = 1 \quad \text{iff} \quad mc(A) = 1 \quad --------(*).$$

The sufficient condition is obvious and so we only show that the necessary condition. Suppose $r(A) = 1$. Then there exist $F_{m\times 1}, G_{1\times n}$ such that $A = FG$. Put $F = (x_1, x_2, \cdots, x_m)^T$ and $G = (y_1, y_2, \cdots, y_n)$, where $x_i, y_i \in \mathbb{F}^+$. Then

$$\begin{aligned} A_{m\times n} &= F_{m\times 1}G_{1\times n} \\ &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} (\, y_1 \quad y_2 \quad \cdots \quad y_n \,) \\ &= \begin{pmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{pmatrix}. \end{aligned}$$

Since $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} y_i = (y_i/y_j) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \cdot y_j$, $mc(A) = 1$. Therefore

$$\beta(\mathbb{F}^+, m, n) \geq 1.$$

So if $\min\{m, n\} = 1$, then it is clear that $\beta(\mathbb{F}^+, m, n) = 1$.

In $\mathbb{F}^+$,

$$r(A) = 2 \quad \text{iff} \quad mc(A) = 2.$$

For if $mc(A) = 2$, then it is trivial that $r(A) = 2$ by $(*)$. Conversely, suppose $r(A) = 2$. Then $mc(A) \geq 2$. If $mc(A) > 2$, then there exist linearly independent columns, say $\mathbf{a_i}$,$\mathbf{a_j}$ and $\mathbf{a_k}$ of A over $\mathbb{F}^+$. Since the rank of A over the subfield $\mathbb{F}$ of the reals 2, there exist scalars α,β and γ, not all zero, such that $\alpha\mathbf{a_i} + \beta\mathbf{a_j} + \gamma\mathbf{a_k} = 0$. Since all the entries in A are nonnegative, at least one of α,β and γ is positive and one negative. We may assume that two are positive(or at least nonnegative) and one negative, say γ is negative. Then $(\alpha/-\gamma)\mathbf{a_i} + (\beta/-\gamma)\mathbf{a_j} = \mathbf{a_k}$. Thus $\mathbf{a_i}$,$\mathbf{a_j}$ and $\mathbf{a_k}$ are linear dependent over $\mathbb{F}^+$ which leads to a contradiction of the fact that they are linearly independent. Hence $mc(A) = 2$. Therefore we have

$$\beta(\mathbb{F}^+, m, n) \geq 2 \text{ ------------ } (**)$$

for $\min\{m, n\} \geq 2$.

If $A \in \mathbb{M}_{2,n}$ for $n \geq 2$, then $mc(A) = r(A) \leq 2$. Thus $(**)$ implies that

$$\beta(\mathbb{F}^+, 2, n) = 2 \text{ for } n \geq 2.$$

If $m \geq 3, n = 3$ and $A \in \mathbb{M}_{m,3}(F^+)$ with $mc(A) = 3$, then by $(*)$ and $(**)$, $r(A) = 3$. Therefore we obtain

$$\beta(\mathbb{F}^+, m, 3) = 3 \text{ for } m \geq 3.$$

Finally, if $m \geq 3$ and $n \geq 4$, then

$$\beta(\mathbb{F}^+, m, n) \leq 2.$$

For let $A = \begin{pmatrix} 0 & a_2 & a_3 & 0 \\ a_1 & 0 & 0 & a_5 \\ 0 & 0 & a_4 & a_6 \end{pmatrix}$ where $a_1, a_2, \cdots, a_6 \in \mathbb{F}^+$, then $mc(A) = 4$ and $b(A) \leq 3$. Therefore from (3.4) we get,

$$\beta(\mathbb{F}^+, m, n) \leq 2.$$

Hence from (**) we have

$$\beta(\mathbb{F}^+, m, n) = 2,$$

for $m \geq 3$ and $n \geq 4$. Hence the proof is completed. □

	μ	α	β
$\mathbb{B}_1$	1 if $\min\{m,n\}$=1 3 if$m \geq 3, n = 3$ 2 otherwise [3]	1 if $\min\{m,n\}$=1 3 if $m \geq 3, n = 3$ 4 if $m \geq 3, n = 4$ 2 otherwise [7]	1 if $\min\{m,n\}$=1 3 if $m \geq 3, n = 3$ 2 otherwise (Theorem 3.1)
$\mathbb{Z}^+$	1 if n=1 0 if $n \geq 2$ (Corollary 3.3)	1 if n=1 2 if n=2 0 if $n \geq 3$ [7]	1 if n=1 0 if $n \geq 2$ (Theorem 3.3)
$\mathbb{F}^+$	1 if $\min\{m,n\} = 1$ 3 if $m \geq 3, n = 3$ 2 otherwise [3]	1 if $\min\{m,n\} = 1$ 3 if $m \geq 3, n = 3$ 4 if $m \geq 3, n = 4$ 2 otherwise [7]	1 if $\min\{m,n\} = 1$ 3 if $m \geq 3, n = 3$ 2 otherwise (Theorem 3.4)
$\mathbb{B}$	2 if 2=n ≤ 3 1 otherwise [9]	1 if $m \geq 1, n = 1$ 0 otherwise [7]	1 if $n \geq 2$ 0 otherwise (Theorem 3.2)

< **Table 3.1** >

IV. Linear operators that preserve maximal column rank of the nonbinary Boolean matrices

In this section, we obtain the characterizations of the linear operators that preserve Boolean maximal column rank of the nonbinary Boolean matrices.

If $\mathbb{V}$ is a vector space over a Boolean algebra $\mathbb{B}$, a mapping $T : \mathbb{V} \to \mathbb{V}$ which preserves sums and 0 is said to be a *(Boolean) linear operator.*

Definition 4.1. *A linear operator T on $\mathbb{M}_{m,n}(\mathbb{B})$ is said to* **preserve Boolean maximal column rank** *if $mc(T(A)) = mc(A)$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$. In particular, T* **preserves Boolean maximal column rank r** *if $mc(T(A)) = r$ whenever $mc(A) = r$.*

Similarly we can define the terms, such as Boolean rank preserver and Boolean rank r preserver.

Definition 4.2 [9][T_p : p-th constituent]. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. For each $1 \le p \le k$, a map T_p* **is called its p-th constituent** *if $T_p(B) = (T(B))_p$ for every $B \in \mathbb{M}_{m,n}(\mathbb{B}_1)$.*

We notice that this definition is well-defined. For if $A = B$, then $T(A) = T(B)$ and so $(T(A))_p = (T(B))_p$. Hence $T_p(A) = T_p(B)$.

By the linearity of T, for any matrix $A \in \mathbb{M}_{m,n}(\mathbb{B})$,

$$\begin{aligned} T(A) &= T(\sum \sigma_p \cdot A_p) \\ &= \sum \sigma_p \cdot T(A_p) \\ &= \sum \sigma_p \cdot (\sum \sigma_q \cdot T(A_q)) \\ &= \sum \sigma_p \cdot T_p(A_p). \end{aligned}$$

Since $\mathbb{M}_{m,n}(\mathbb{B})$ is a semiring, we can consider the invertible members of its multiplicative monoid. The permutation matrices (obtained by permuting the columns of I_n, the identity matrix) are all invertible. Since 1 is the only invertible member of the multiplicative monoid of $\mathbb{B}$, the permutation matrices are the only invertible members of $\mathbb{M}_{m,n}(\mathbb{B})$.

Lemma 4.1 [9]. *If $A \in \mathbb{M}_{m,n}(\mathbb{B})$ and U,V are invertible matrices, then*

$$mc(A) = mc(UA) = mc(AV).$$

Proof. This follows from the fact that an invertible matrix is just a permutation matrix. □

Lemma 4.2. *Assume T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. If T preserves Boolean maximal column rank r, then each constituent T_p preserves Boolean maximal column rank r on $\mathbb{M}_{m,n}(\mathbb{B}_1)$.*

Proof. Suppose that $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$ with $mc_1(A) = r$. By Lemma 3.3, we have

$$mc(A)(= mc_1(A)) = r, \text{ and } mc(\sigma_p A) = r,$$

for each p=1,2,$\cdots$,k. since T preserves Boolean maximal column rank r, $mc(T(\sigma_p A)) = r$. But

$$\begin{aligned} r = mc(T(\sigma_p A)) &= mc(\sigma_p T(A)) \\ &= mc(\sigma_p \sum_i \sigma_i T_i(A_i)) \\ &= mc(\sigma_p T_p(A_p)) \\ &= mc(\sigma_p T_p(A)). \end{aligned}$$

Therefore $mc(\sigma_p T_p(A)) = r$ for each p=1,2,$\cdots$,k, and hence $mc_1(T_p(A)) = r$.

□

Lemma 4.3 [9]. *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. If each constituent T_p preserves binary Boolean rank r, then T preserves Boolean rank r.*

Proof. Let $b(A) = r$ for $A \in \mathbb{M}_{m,n}(\mathbb{B})$. Then there exists some p such that $b_1(A_p) = r$ and $b_1(A_p) \leq r$ for $1 \leq q \leq k$ by property (3.3). Thus $b_1(T_p(A_p)) = r$ and $b_1(T_q(A_q)) \leq r$ for $1 \leq q \leq k$. Since $b(T(A)) = max\{b_1(T_p(A_p)) \mid 1 \leq p \leq k\}$ by property (3.3), T preserves Boolean rank r. □

Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. Say that T is a

(i) *Congruence operator* if there exist invertible matrices $m \times m$ and $n \times n$ Boolean matrices U, V such that $T(A) = UAV$ for any A in $\mathbb{M}_{m,n}(\mathbb{B})$.

Let σ^* denote the complement of σ for each σ in $\mathbb{B}$.

(ii) the p-th *rotation operator*, $R^{(p)}$, on $\mathbb{M}_{m,n}(\mathbb{B})$ if

$$R^{(p)}(A) = \sigma_p A_p^t + \sigma_p^* A, \text{ for } 1 \le p \le k,$$

where A_p^t is the transpose matrix of A_p.

We see that $R^{(p)}$ has the effect of transposing A_p while leaving the remaining constituents unchanged. Each rotation operator is linear on the $n \times n$ matrices over $\mathbb{B}$ and their product is the transposition operator, $R : A \to A^t$.

Example 4.1. Let $A = \begin{pmatrix} 0 & 0 & 0 \\ \sigma_1 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ be a matrix over $\mathbb{B}$. Then $mc(A) = 3$ by Example 3.5 and property (3.2). But

$$\begin{aligned} R^{(1)}(A) &= \sigma_1 A_1^t + \sigma_1^* A \\ &= \sigma_1 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^t + \sigma_1^* \begin{pmatrix} 0 & 0 & 0 \\ \sigma_1 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \sigma_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \sigma_1^* \begin{pmatrix} 0 & 0 & 0 \\ \sigma_1 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma_1 & 0 \\ 0 & \sigma_1 & \sigma_1 \\ 0 & \sigma_1 & \sigma_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_1^* \\ 0 & \sigma_1^* & \sigma_1^* \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma_1 & 0 \\ 0 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= A^t, \end{aligned}$$

and so $R^{(1)}(A) = A^t$, the transpose matrix of A, has Boolean maximal column rank 2. Consider $B = A \oplus 0_{n-3,n-3}$ for $n \ge 3$. By property (3.2), the rotation operator does not preserve Boolean maximal column rank 3 on $\mathbb{M}_{m,n}(\mathbb{B})$. □

Lemma 4.4 [5]. *If T is a linear operator on the $m \times n$ matrices $(m, n \geq 1)$ over a general Boolean algebra $\mathbb{B}$, then the followings are equivalent.*

(1) T preserves Boolean ranks 1 and 2.

(2) T is in the group of operators generated by the congruence (if m=n , also the rotation) operators.

Theorem 4.1. *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ for $m \geq 2$ and $n \geq 1$. Then the following are equivalent.*

(1) T preserves Boolean maximal column rank.

(2) T preserves Boolean maximal column ranks 1,2 and 3.

(3) T is a congruence operator.

Proof. Clearly (1) implies (2). Now we show that (2) implies (3). Assume T preserves Boolean maximal column rank 1,2 and 3. Then by Lemma 4.2, each constituent T_p preserves binary Boolean maximal column ranks 1,2 and 3. For $A \in \mathbb{M}_{m,n}(\mathbb{B})$, Theorem 3.1 implies $b_1(A) = mc_1(A)$ for $b_1(A) \leq 2$. Thus T_p preserves binary Boolean ranks 1 and 2, and hence T preserves Boolean ranks 1 and 2 by Lemma 4.3. So T is in the group of operators generated by the congruence(if m=n, also the rotation) operators by Lemma 4.4. But the rotation operator does not preserve Boolean maximal column ranks 3 by Example 4.1. Hence T is a congruence operator since T preserves Boolean maximal column rank 3. That is (2) implies (3). Finally, assume that T is a congruence operator of the form $T(A) = UAV$, where U and V are invertible $m \times m$ and $n \times n$ Boolean matrices respectively. Then T preserves Boolean maximal column rank by Lemma 4.1. Hence (3) implies (1). □

If $m \leq 2$, then the linear operators that preserve maximal column rank on $\mathbb{M}_{m,n}(\mathbb{B})$ are the same as the Boolean rank-preservers, which were characterized in [5].

Thus we have characterizations of the linear operators that preserve the Boolean maximal column rank of general Boolean matrices.

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< 국문초록 >

부울계수들에 대한 비교와 이들의 선형 연산자

부울 행렬들상에서 부울계수, 열계수, 극대열계수가 이제까지 많이 연구, 개발되어 왔었고, 이들을 보존하는 선형 연산자들에 관한 특성이 부울 행렬들상에서 밝혀졌다.

이 논문에서는 부울계수와 극대열계수의 관계를 β 함수를 이용하여 비교함으로써, 각각 이항 부울대수와 양의 정수, 양의 실수, 부울대수상의 행렬들에 대한 β 함수값을 얻었다. 그리고 부울 행렬들상에서 극대열계수를 보존하는 선형 연산자의 형태를 규명하고 특성들을 밝혔다.

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항상 내 곁을 맴도는 자그마한 저의 꿈들이더군요.

탈고의 기쁨을 모든 사람들과 나누고 싶습니다. 공부에 참 맛을 느끼게 해 주시고 넓은 사랑으로 감싸 주신 송석준 교수님께 깊이 감사를 드립니다. 저의 학문에 폭이 넓혀지게끔 아낌없는 강의를 해 주신 양영오, 정승달, 윤용식, 고봉수, 양성호, 고윤희, 박진원 교수님들께 깊은 감사를 드립니다. 부족한 저에게 늘 관심을 갖고 대해 주신 방은숙, 현진오, 김도현, 김철수 교수님께도 고마움을 전해 드립니다. 교수님들, 대단히 고맙습니다.

초등학교시절 의를 맺고 지내온 동우야, 바오로야 우린 모두 해낸거지!! 재현, 명철, 상현, 영주 언제나 소중한 나의 친구들, 충범, 재석, 미숙, 은주, 예은 항상 반가운 나의 벗들, 량규, 창부, 은숙누나, 경태선배, 지순이 늘 든든했던 나의 동기와 선배님과 김영미 선생님, 논문을 읽고 쓰느라 같이 땀을 흘린 재환형, 사회를 잘 모르는 저에게 격려를 해 주시던 김영고시학원의 원장님 이하 강, 고, 양, 박, 이진순 선생님들, 도현, 주현, 경영, 용범 나의 멋진 제자들, 캐나다 여행중 독특한 인연으로 만나게 되었던 Mica, N. J, Jeff, 내가 힘들때나 기쁠때나 언제나 정신적 버팀목이 되어 주었고 지금도 그런 권형, 윤정누나, 동우형 그리고 Maxine, 올 한해 나와 함께 고생을 많이 한 태일이와 부모님, 기타 나를 아는 모든 이들에게 고마움을 드립니다. 마지막으로 어린 막내의 뒷바라지를 해 주신 부모님, 형, 누나들과 이 기쁨을 나눕니다. 부모님께 여태껏 해 보지 못한 말을 이 지면을 빌어 해봅니다.

엄마, 아빠 사랑해요.

1998년 12월