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Chance Constrained Programming Using the Confidence Interval of Linear Regression

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Contents

Ⅰ. Introduction

This paper considers stochastic linear programming when the coefficients of the left hand side of a constraint function are estimated from the sample data using multiple linear regression. Considerations of model stability for stochastic programming in this case are virtually unexplored in the literature despite the wide spread use of regression to determine components of linear systems and despite the intensive attention focused on chance constrained programming. The probabilistic model with an estimated constraint is converted to an equivalent deterministic model using approximate confidence intervals. Consequently, the stochastic constraint admits some violations in a manner analogous to chance constrained programming.

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Ⅱ. Background

Many of the earlier applications of operations research "involved fairly straightforward use of linear programming coupled with technical parameters based on regression analysis." (See Cook 1964). In other words, estimates were considered to be true coefficients, and probabilistic constraints were treated as if deterministic. Caveats were issued, but generally ignored, regarding this practice. Beer and Ackoff (1969) advised" … more and more use is being made of models based on large regression analyses… regression ought to be used only as a filter for the selection of variables whose effect on system performance the researcher should seek to understand. Computers have obscured this fact."

The pioneering work in stochastic programming began with works by Dantzig (1955), Beale (1955), Tintner (1955) and Charnes and Cooper (1959) among others. According to Hansotia (1980), in the twenty year period from 1955 to 1975 more than 700 articles related to stochastic programming were published. Since 1975, that number has increased many times over.

Sensitivity analysis, chance-constrained programming, the distribution problem of stochastic programming under uncertainty, and stochastic programming with recourse all consider related issues. These techniques most often treat changes in individual parameters or random variation of the right hand side of the system (Vajda 1972, Bawa et. al. 1979, Whittle 1982, Jagannathan 1985). However, this parer considers the effect on LP solution of constraint parameters and error bounds simultaneously obtained from regression equations. This application assumes that there are "true" operative, but unobservable, parameters for the constraints but that estimates of these must be used in the absence of perfect information. Because the computed optimum will differ from the "true" optimum, bounds will be placed on the constraining function. Statistic linear programming under risk as defined by Dempster (1968) provides the general class of problems in which this case is embedded.

The situation herein discussed is introduced by whiteside(1986). Charnes, et. al (1986) introduce a similar situation independently. The primary assumption of the LP model is that the inequality constraint deals with the estimated expected value of a function, h, which has an unobservable unknown terms, β_i, $i=0,1,2$ and ε.

$$h=\beta_0+\beta_1 x_1+\beta_2 X_2+\varepsilon.$$

The true constraint function is

$$E(h)=\beta_0+\beta_1 X_1+\beta_2 X_2 \geq C.$$

A further assumption is that the coefficients of the above constraint are obtained from the sample data in which collinearity among decision variables might exist. This parer considers the simultaneous effect on the left hand side coefficients, and hence LP solutions, of collinearity and sampling error in the data used to estimate the coefficients.

Charnes, et. al (1986) introduced a similar problem in which the natural gas consumption model is expressed as a linear stochastic model. The natural gas model and the model introduced in this paper are the same in terms of estimating the coefficients in the constraint using multiple regression. However, the model of Charnes, et. al. focuses on the decision regimes of the stochastic process rather than optimization of the LP model. Whereas, the model introduced in this paper focuses on the optimization of the linear system under several scenarios.

The study of linear programming and the effects of collinearity is motivated by the Estuarine Linear Program for the Texas Department of Water Resouces. The solutions of this program are recommended fresh water releases into the bays and estuaries of the Gulf of Mexico. A discussion of this particular application is presented in section four of this paper. Section five presents the simulation results with OLS and ridge regression equations that in turn serve as a constraint for a linear program. One thousand replications are observed for a $3\times4\times2\times3$ design. The factors of the design are error variance for the regression, degree of collinearity in the data, regression criteria, and sample size.

Ⅲ. Linear Systems

Suppose the following linear system. Minimize $X_1+X_2+\cdots+X_k$ to :

$$l_1\leq X_1\leq m_1$$
$$l_2\leq X_2\leq m_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$l_k \le X_k \le m_k$$

$$b_0 + b_1X_1 + b_2X_2 + \cdots + b_kX_k \ge h \cdots\cdots\cdots\cdots\cdots (1)$$

$$c_0 + b_1X_1 + c_2X_2 + \cdots + c_kX_k \ge s \cdots\cdots\cdots\cdots\cdots (2)$$

where the left hand side of (1) and (2) are sample least square regression equations use in lieu of the unknown true functional form.

It is well known that collinearity among the variables $X_1, X_2, \cdots, X_k$ yields unstable sample estimates of the regression coefficients. Thus the apparent solution either does not in fact satisfy the true constraints or is not optimal.

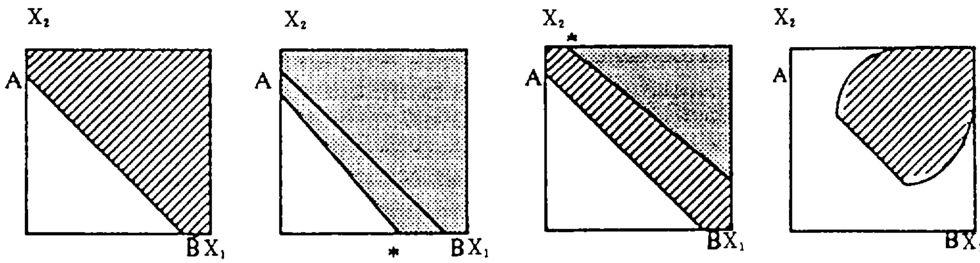

Figure 1-1 Figure 1-2 Figure 1-3 Figure 1-4

: true feasible region : estimated feasible region

A --- B : true constaint function * : estimated optimum

Fig. 1. The effects of multicollinearity on the feasible region of the LP.

A geometric interpretation will clarify these ideas. Consider the case where k=2 and one constraint of type (1) above appears. The first inequalities restrict solutions to the rectangular set indicated in Figure 1-1. The true functional form of constraint (1) cuts across this rectangle as shown forming the true set of feasible solutions shaded in the figure. Thus the optimal solution is where the line segment A B of constraint (1) intersects the rectangle. However, (1) must be estimated from sample data. If there is a collinear relationship between X_1 and X_2 value observed in this sample, then the variance of the estimated regression coefficients will be inflated and the likelihood that the computed optimal soultion is not the true optimal nor truly feasible is increased.

Figures 1-2 and 1-3 illustrate two sample estimates of constraint (1). For Figure 1-2, the computed LP solution (marked by *) is not truly feasible. For Figure 1-3, although feasible, the computed LP solution is not the true optimal. The disparity between the true and estimated constraint results from sampling error and may be exacerbated by collinearity.

Collinearity between X_1 and X_2 introduces another consideration. One effect of this collinearity in a linear programming application is that the solution that apparently optimizes the objective criterion is likely to represent an extrapolation beyond the multivariate range of the X observations. If X_1 and X_2 are functionally related, such that levels of both cannot be controlled completely; then the true feasible solution space is reduced to the intersection of A B and the sample multivariate range indicated by shading in Fig. 1-4. The application discussed in the following section illustrates this later case.

Ⅳ. The Estuarine Linear Program

When dams are built on rivers that empty into the Gulf of Mexico, the fresh water inflows into the the gulf are decreased. This is particularly worrisome in years of low rainfall since the estuaries serve as hatcheries and nurseries for numerous fish and shellfish. Shrimp and other shellfish which are native to the gulf and important to the economy of Texas require levels of low salinity for the juveniles of the species. Hence, a concern for the ecology of the bays and estuaries has motivated legislative provision for fresh water releases from reservoirs built near the gulf to insure appropriate salinity levels in the estuaries and abundance of fish and shellfish populations. However, agricultural and industrial segments of the economy also compete for the fresh water resources. It is therefore desirable to release no more water than necessary to maintain the viability of the wildlife. The Texas Department of Water Resources has used the Estuarine Linear Program to determine recommended monthly releases form each of several lakes and reservoirs into the gulf in order to assure healthful salinity levels and continuing harvests of fish and shellfish near the historic monthly means. For purposes of illustration, one particular alternative wildlife management program for Lake Texana will be discussed. (*Lavaca-Tres Palacios Estuary,* 1980)

The objective of this program is to minimize combined inflow to the estuary while providing freshwater inflows sufficient to generate predicted annual commercial harvest of red drum, seatrout, shrimp, blue crab, and bay oyster at level no less than their mean 1962 through 1976 historical values, satisfying marsh inundation needs, and meeting bounds for salinity. The commercial harvest constraints are represented by five inequalities determined by multiple regression equations on the left hand side and historic mean harvest values on the right hand side. The data for developing the multiple regression equation is given in Tables 1 and 2. For each regression, the response is the commercial harvest and the predictors are appropriately lagged values of seasonal rainfall. For each species, the best equation is determined by significant equations generated by a BMDP stepwise regression procedure. To illustrate the problem of collinearity, consider the following equation used as a constraint to assure adequate shellfish harvest :

$$H=3107.9-11.3Q_1+7.7Q_2-24.2Q_3,$$

Table 1. Commercial fisheries harvests in the Lavaca-Tres Palacios estuary/1 1962~1976, (thousands of pounds)

Year	Shellfish	White Shrimp	Brown & Pink Shrimp	Blue Crab	Bay Oyster	Finfish	Spotted Seatrout	Red Drum
1962	3,843.7	1,405.1	277.3	2,006.8	154.5	232.0	105.6	60.3
1963	2,635.1	1,601.5	169.3	728.4	135.9	174.0	76.2	41.8
1964	3,001.0	2,435.6	199.0	225.9	140.5	116.4	43.5	22.6
1965	2,889.6	1,290.3	1,074.4	401.3	123.6	209.5	80.0	50.7
1966	2,928.9	1,643.0	319.4	477.2	489.3	554.9	274.7	106.8
1967	1,930.0	1,056.0	210.8	360.8	302.4	322.7	138.4	69.0
1968	3,668.5	2,364.5	82.1	933.3	288.6	533.1	267.9	121.2
1969	2,536.2	1,319.1	108.7	891.0	217.4	410.3	168.6	109.0
1970	3,259.0	1,823.0	174.5	782.0	479.5	446.9	173.8	128.7
1971	1,976.1	1,070.0	217.2	394.0	294.6	280.8	140.5	65.5
1972	2,629.3	1,294.3	238.1	882.0	214.9	298.8	123.0	76.9
1973	5,013.3	1,934.2	875.8	1,129.6	73.7	284.4	133.4	70.5
1974	3,044.9	1,418.7	469.8	959.3	197.1	226.9	130.1	52.5
1975	2,978.5	920.5	785.6	897.7	374.7	236.4	94.8	72.1
1976	3,180.5	1,313.5	934.0	651.7	281.2	172.2	65.3	47.9
Mean	3,034.4	1,592.6	409.1	781.4	251.2	300.0	134.4	73.0
	195.1	147.6	86.4	111.4	32.3	34.0	17.2	7.9

Note : 1. Estuary yanks second in shellfish and fifth in finfish commercial harvests of eight Texas estuarine areas

2. Includes blue crab, bay oyster, and white, brown, and pink shrimp
3. Includes croaker, black drum, red drum, flounder, sea catfish, spotted seatrout, and sheepshead
4. Standard error of the mean: two standard errors provide an approximately 95% confiden confidence limits about the mean

Table 2. Seasonal volumes of combined freshwater inflow contributed to Lavaca-Tres Palacios estuary (thousands of acre-feet), 1959~1976

Year	Winter (Jan-March)	Spring (April-June)	Summer (July-August)	Autunm (Sept-Oct.)	Late Fall (Nov.-Dec.)
1959	300.9	378.0	52.0	179.0	119.0
1960	116.1	501.9	202.0	470.0	342.0
1961	474.9	321.0	145.0	519.0/a	160.0
1962	30.9	135.9	22.0	30.0	15.0
1963	59.1	11.1	29.0	5.0/b	21.0
1964	53.3	66.0	16.0	70.0	5.0
1965	188.1	351.9	19.0	30.0	192.0
1966	141.9	360.9	51.0	18.0	3.0
1967	6.9	21.9	33.0	552.0/c	12.0
1968	297.0	848.1	66.0	45.0	53.0
1969	351.0	534.0	14.0	44.0	61.0
1970	185.1	378.0	26.0/d	261.0	8.0
1971	9.9	17.1	89.0	371.0/e	107.0
1972	174.9	584.1	48.0	24.0	14.0
1973	233.1	1,476.9	89.0	479.0/f	57.0
1974	237.9	303.9	41.0	368.0	207.0
1975	62.1	540.0	90.0	37.0	55.0
1976	15.0	237.0	56.0	111.0	423.0
Mean	163.3	392.7	60.4	200.7	103.0
Standard error/g	31.5	83.0	11.5	47.8	28.6

Note : a. Hurricane Carla, Sept. 8~14: near Port Lavaca
b. Hurricane Cindy, Sept. 16~20: near Port Arthur
c. Hurricane Beulah, Sept. 18~23: near Brownsville
d. Hurricane Celia, Aug. 3~5: near Port Aransas
e. Hurricane Fern, Sept. 9~13: near Port Aransas
f. Hurricane Delia, Sept. 4~7: near Galvestone
g. Standard error of the mean: two standard errors provide approximately 95% confidence limits about the mean
Source : Lavaca-Tres Palacios Estuary, 1980, pp. VIII-3 and VIII-5.

where H represent shellfish harvest in thousands of pounds and $Q_1 \cdots Q_3$ represent mean seasonal freshwater inflow in thousands of acre-feet for winter (January-March), spring (April-June), and summer (July-August) respectively.

A simple interpretation of the above equation would suggest that shellfish harvest would be most enhanced if all freshwater inflow at all in winter or summer. If there is no fresh water inflow at all, the model estimates a harvest greater than the historic means (3107.9 vs. 3034.4 thousands pounds). This is clearly ludicrous. It is true that shellfish are most dependent on spring inflow and that extremes of salinity and temperature experienced simultaneously can have a deleterious effect. Thus coefficients for summer and winter would be expected to be relatively smaller than spring, but to actually observe negative coefficients is an indication that the functional form of the relationship is misrepresented.

The presence of negative coefficients in such constraints as the shellfish equation above is an indication of possible collinearity among seasonal inflow levels. The simple correlation between winter and spring inflows is 0.645. Whenever both of these variables appear in a harvest equation, one always has a negative coefficient which is absurd in the context of the problem. It is also reasonable to assume that more complex linear relationships exist among all seasonal inflows since the total annual rainfall is comprised of the sum of the seasonal values.

The effect of this collinearity on the LP solutions is dramatic. When the harvest constraints are nor part of the linear system, the recommended inflows as a percentage of historic mean inflow vary only from 40% in late fall (November-December) to 51% in winter. For the management alternative discussed here, the recommended in flows vary from 40% in late fall to as 105% in the early fall(September-October). For a third alternative where the shellfish harvest equation actually provides the objective criterion for the system, the range is from 40% in late fall to a whopping 127% in spring. The unusually high inflows for spring reflect the collinearity in the harvest equation presented above.

The difficulties described could occur anytime regression equations provide constraints or provide the optimizing criterion for linear programs. Estimated constraint coefficients can introduce misleading results that are even more serious in the presence of collinearity. The simulation study that follows explores the ramifications of this phenomena.

V. Simulation and Results

The linear model for the simulation is as follows. Minimize X_1+X_2 subject to :

$$1\leq X_1\leq 5$$

$$1\leq X_2\leq 5$$

$$h=b_0+b_1X_1+b_2X_2\geq 5 \cdots\cdots\cdots\cdots\cdots (1a)$$

where b_0, b_1, and b_2 are sample regression coefficients. The true functional form of (1a) is

$$E(\hat{h}) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 \geq 5.$$

$$\text{Let} \quad \hat{h} = b_0 + b_1 X_1 + b_2 X_2$$

$$\text{and} \quad E(\hat{h}) = \beta_0 + \beta_1 X_1 + \beta_2 X_2,$$

then the $(1-\alpha)\%$ confidence interval for $E(\hat{h})$ is

$$\hat{h}_g + t_{(1-\alpha/2:n-p)} s(\widehat{E(\hat{h})})$$

If we borrow the probability concept, $P(\hat{h}) \geq 1-\alpha/2$ will be satisfied by the upper bound of the $(1-\alpha)\%$ confidence interval for a given vector of X_g, i.e.,

$$\hat{h}_g t_{(1-\alpha/2:n-p)} S(\hat{h}_g), \cdots\cdots\cdots\cdots\cdots (2a)$$

(For the less than or equal to constraint, the lower bound will satisfy the condition).

A chance constrained programming approach applied to the above system means that the condition, $P(\hat{h} \geq 5) \geq 1-\alpha$ would be satisfied, assuming β_2, β_1, β_2 and variance are known. This paper uses a Modifications of Chance constrained Programming (MCCP) to introduce a confidence interval in order to satisfy the above criterion in probability.

Since the term $S^2(\hat{h}_g)$, which can be defined as follows :

$$S^2(\hat{h}_g) = MSE[X'_g (X'X)^{-1} X_g] = X'_g \, \sigma^2(b) X_g$$

is always strictly convex, the feasible region that is constructed by Eq. 2a would be a convex set. The simulation begins with the construction of a bivariate data set with observed values (X_1, X_2), of size n, where the observed sample correlation between X_1 and X_2 is r. Then a set of n observations of h is constructed with :

$$h = X_1 + X_2 + e$$

where e is a normal, uniform or Double exponential random variable with standard observations are obtained using the Statistical Analysis System (SAS) random number generator. The regression of h on X_1 and X_2 yields b_0, b_1 and b_2 of constraint(1a).

The experimetal design for the simulation is as follows :

Class	Levels			
Regression	OLS	LAV		
Collinearity (r)	0	0.3	0.7	0.9
Error std. dev. (sigma)	0.25	1	2	
Sample size (n)	20	40	100	

The sample sizes are intended to reflect small, moderate and fairly large samples for obtaining regression equations. The values of r allows an orthogonal to a highly collinear structure for X_1 and X_2. The error variance allows for very little to large fluctuation relative to the inequalities $1 \le X_1 \le 5$ and $1 \le X_2 \le 5$. For each factor level combination, 1000 data sets are simulated to obtain solutions for the linear program.

Recall that the true optimal is 5. Figures 2, 3 and 4 depict the mean LP solutions of 1000 replications for n=20, 40 and 100 under three different distributions: normal, uniform and double exponential using two different types of regression. In general, given a fixed sample size and level of collinearity, the mean of the observed optimal solutions approaches 5 as error variance decreases. Similarly, the standard deciation of the observed optimal solution decreases. An analogous results holds for decreases in collinearity, given a fixed sample size and a fixed error variance. As sample size increases, the mean of the observed optimal solutions also approaches 5, holding either variable constant. The variability of solutions is also reduced with larger samples. Error variance in this study is a more important factor than is either collinearity or sample size. These factors cause the observed optimal solution to differ from the true optimal solution. The solutions are much more likely to be in the truly feasible set.

Considering the regression criterion factor, the observed mean LP solution for ordinary least squares (OLS) regression is slightly closer to the true value than is the least absolute value (LAV) mean under normal and uniform error distributions. However, LAV regression provides more accurate LP solutions than OLS under double exponential error distribution, as expected.

It is well known that least absolute value estimators are maximum likelihood and recommended over ordinary least squares if error distribution is double exponential (Rice and White). The simulated LP solution supports the above theoretical property. (For more

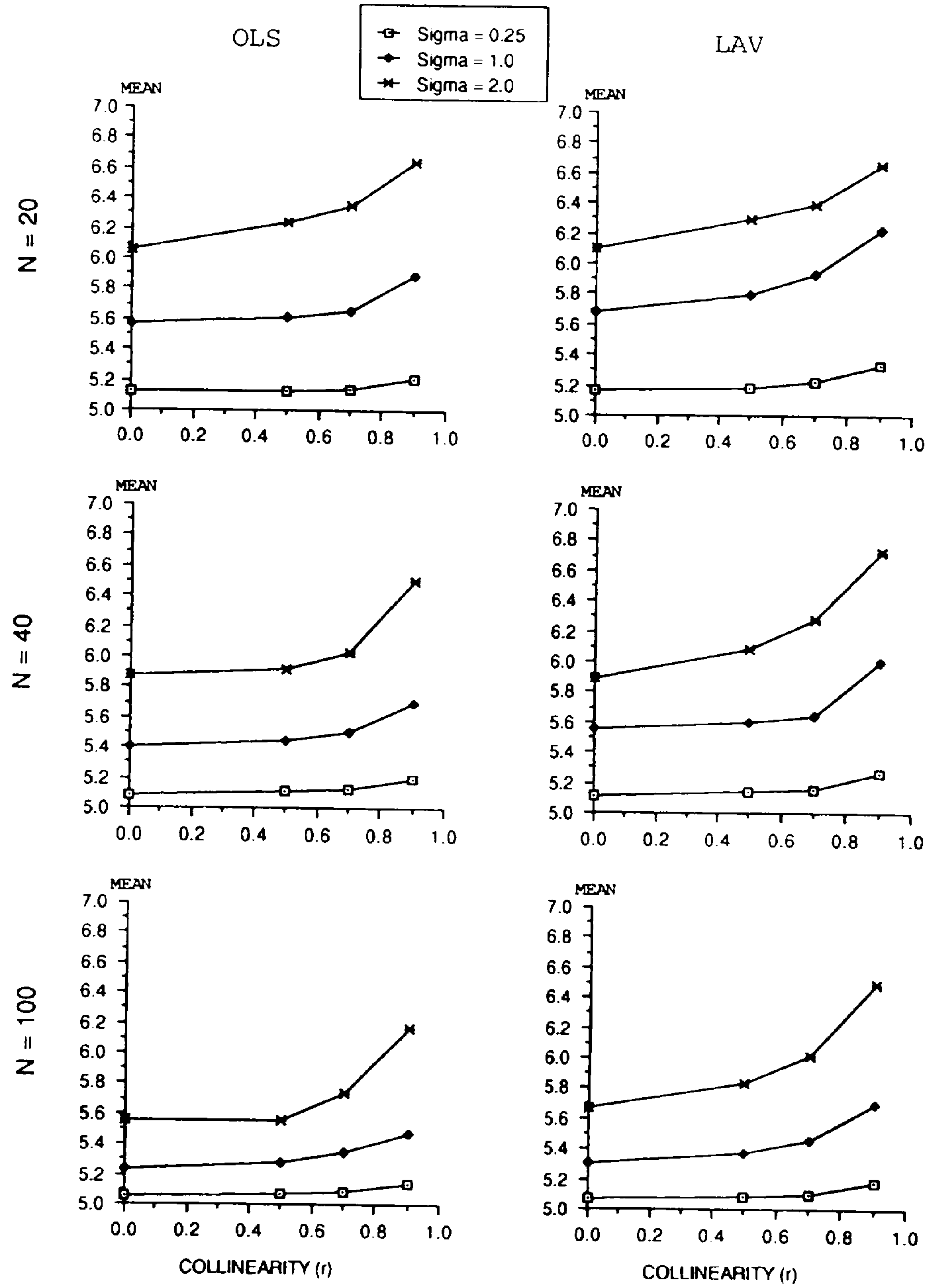

Fig. 2. Mean LP solution for MCCP with normal error

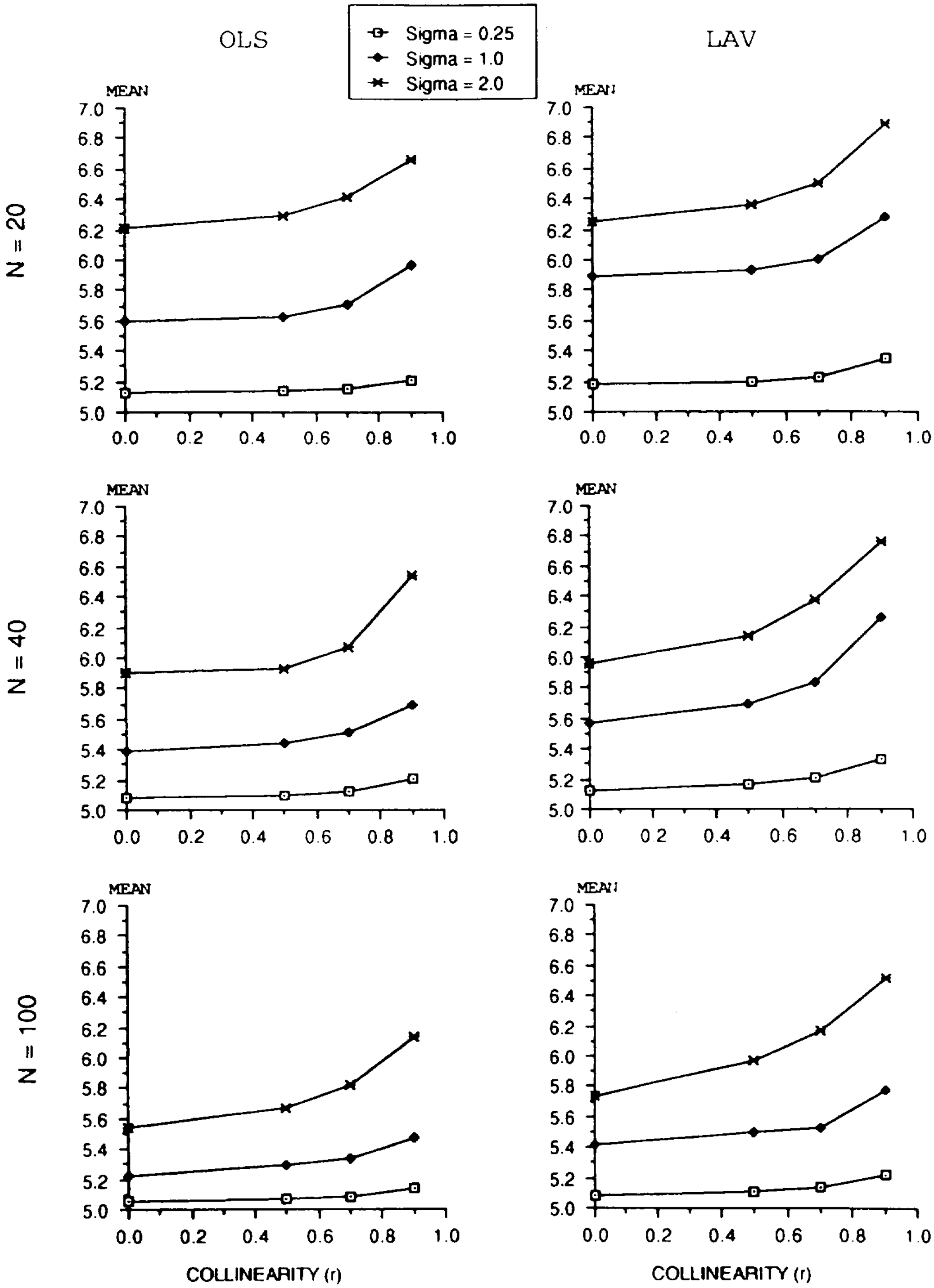

Fig. 3. Mean LP solution for MCCP with uniform error

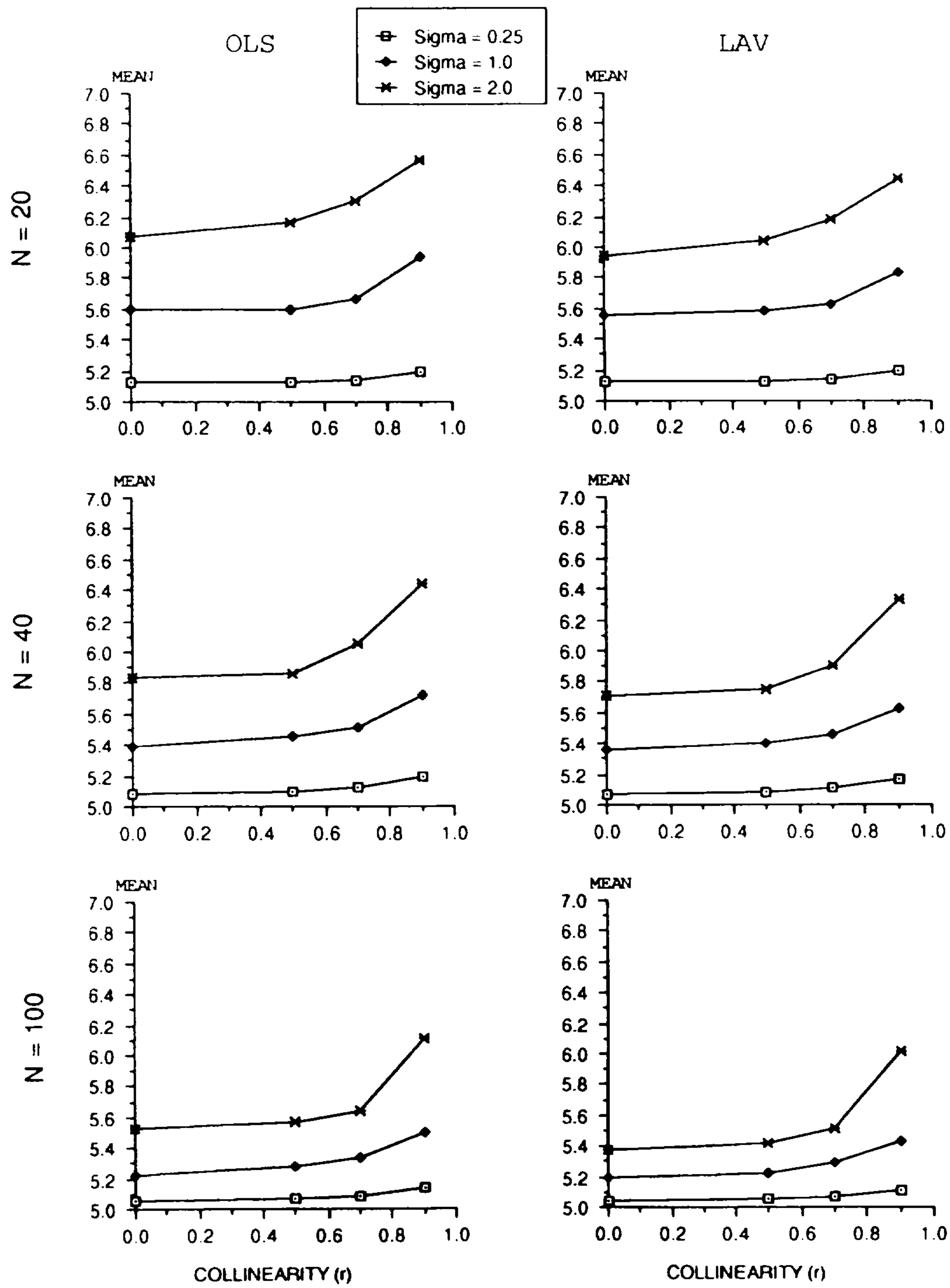

Fig. 4. Mean LP solution for MCCP with double exponential error

detailed analysis, see Choi, 1991).

Ⅵ. Implications

The variability of the sample regression constraint in the LP model introduces a bias for the optimal solution. The case considered here shows a downward bias. The program is structured with a "greater than" constraint and a minimizing objective criterion. This bias is directly related to regression error variance and collinearity among the constraining variables. More importantly, because each of these solutions occur on the boundary of the feasible set, the coordinates of the apparently optimal solution do not reflect the collinear relationship between the variables which may be important for the practical application of the solution.

It is suggested that the data set be examined for outliers and/or kurtosis. In the presence of either of these conditions, least absolute value regression might be a preferable regression criterion. Most importantly, satisfaction of the constraint can be achieved with the desired probability.

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〈국문초록〉

회귀함수의 신뢰구간을 이용한 확률적 선형계획법

최 병 길

선형계획(Linear Programming)은 여러가지 환경적 제약조건(Environmental Constraints)하에서 회소한 자원의 최적배분을 위해 널리 사용되는 기법이다. Linear System의 매개변수(선형계획 모델의 상수 또는 계수)들이 주어져 있거나 알려져 있을때 선형계획의 최적해는 하나만 존재하는 확정적(Deterministic) 모델이 되지만, 모형의 매개변수가 연속적이며 확률변수(Random Variable)의 성격을 가지고 있는 경우 불확실성(Uncertainty)을 전제로 하기 때문에 선형계획의 해를 구하는 방법은 간단하지가 않다. 이에 대한 접근법으로 Chance와 Cooper(1959)가 개발한 확률적 계획법(Chance-Constrained Programming)은 선형계획의 제약조건이 확률로서 표시되어 있는 경우이다. 즉 의사결정자가 인정하는 한도안에서 제약조건의 위반을 인정하면서 최적의 해를 구하는 방법이다.

본 논문의 목적은 선형회귀함수가 Linear System의 일부분을 구성하고 있을때 선형계획의 해의 변화 정도를 알아보고자 하는데 있다. 가정으로는 Linear System의 실제 Parameter는 존재하나, 알려져 있지도 않고 또한 모집단을 통해서 구할수도 없으며, 오직 표본을 통해서만 추정할 수 있는 경우에 적용될 수 있다. 선형회귀함수에 의해서 추정된 제약함수의 계수는 확률적 성격을 가지고 있기 때문에 Linear System은 확률적 모델이 되며, 확률적 선형계획의 최적해를 구하기 위해서는 확정적 모델로 바꾸어 주어야 한다. 이 논문의 특징은 확률적 모델을 확정적 모델로 바꾸는 방법으로 선형회귀의 신뢰구간을 이용하였다.

본 논문의 주요 관심 부문은 선형계획의 최적해의 신뢰성에 있으며, 이에 영향을 줄 수 있는 여러가지 변수들 「회귀함수의 종류(OLS, LAV, Ridge), 분산의 정도, 표본의 크기 그리고 결정함수들간의 관계의 정도(Collinearity)」의 조합에 따른 선형계획의 결과를 Simulation을 이용하여 보여주고 있다.