On Some Properties of B_1-Proximity

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B_1-Proximity의 몇가지 성질에 關하여

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Introduction

The theory of proximity spaces was essentially discovered in 1950 by Efemovič when he axiomatically characterized the proximity relation "A is near B", which is denoted by $A\delta B$, for subsets A and B of a set X. Efremovič's axioms for this nearness relation δ are as follows:

(E1) $A\delta B$ implies $B\delta A$.

(E2) $A\delta(B\cup C)$ if and only if $A\delta B$ or $A\delta C$.
C.

(E3) $A\delta B$ implies $A\neq\phi$

(E4) $A\cap B\neq\phi$ implies $A\delta B$.

(E5) $A\bar{\delta}B$ implies there exists a subset E such that $A\bar{\delta}E$ and $(X-E)\bar{\delta}B$ ($\bar{\delta}$ means the negation of δ).

A binary relation δ satisfying axioms (E1)–(E5) on the power set of X is called the *Efremovič's proximity* on X.

Hayashi introduced the notion of 'paraproximity' by replacing the word 'finite' by 'arbitrary' and thereby strengthening Efremovičs's 'union' axim to read: for an arbitrary index set I, $A\delta(\bigcup_{i\in I} B_i)$ iff $A\delta B_i$ for some $i\in I$. (Hayashi, E., 1964).

A binary realtion δ between X and subsets of X is called the *K-proximity* on X if δ satisfies the following: (Kim, et al, 1973)

(K1) $x\delta A\cup B$ iff $x\delta A$ or $x\delta B$.

(K2) $x\bar{\delta}\phi$ for all $x\in X$.

(K3) $x\in A$ implies $x\delta A$.

(K4) $x\bar{\delta}A$ implies there is a subset E such that $x\bar{\delta}E$ and $y\bar{\delta}A$ for all $y\in X-E$.

In this note we neglect the axiom (K4) and replace (K1) by a stronger axiom, which we call

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a "B_1−proximity" and examine some of its properties.

I. B_0−Proximity and B_1−Proximity

1.1. Definition. Let ξ be a relation between a set X and its power set PX. Consider the following axioms:

(A0) $x \xi (A \cup B)$ if and only if $x \xi A$ or $x \xi B$.

(A1) For any non−void index set I, $A \xi \bigcup_{i \in I} B_i$ if and only if there exists an index $j \in I$ such that $A \xi B_j$.

(A2) $x \bar{\xi} \phi$ for all $x \in X$ ($\bar{\xi}$ means the negation of ξ).

(A3) $x \in A$ implies $x \xi A$.

ξ is called a B_0−*proximity* on X iff ξ satisfies (A0), (A2) and (A3). ξ is called a B_1−*proximity* on X iff ξ satisfies (A1), (A2) and (A3).

In such a case, (X, ξ) is called a (B_0−*proximity*, B_1−*proximity*) space iff ξ is a (B_0−*proximity*, B_1−*proximity*, resp.) on X.

1.2. Remark. Every K−proximity on X is also a B_0−proximity on X.

1.3. Definition. Let (X, ξ_1) and (Y, ξ_2) be two B_0−proximity spaces (or B_1−proximity spaces). A function $f: X \to Y$ is said to be a proximal map iff $x \xi_1 A$ implies $f(x) \xi_2 f(A)$. The category of B_0−proximity spaces and proximal maps is denoted by <u>B_0−*Prox.*</u> Its full subcategory whose objects are the B_1−proximity spaces is denoted by <u>B_1−*Prox.*</u>

1.4. Proposition. Let (X, ξ) be a B_1−proximity space. Define an operator α on the power set PX by $\alpha A = \{x : x \xi A\}$. Then α satisfies following properties:

(1) $\alpha \phi = \phi$.

(2) $A \subset \alpha A$ for each $A \subset X$.

(3) $\alpha (A \cup B) = \alpha A \cup \alpha B$.

(4) $A \subset B$ implies $\alpha A \subset \alpha B$.

Proof. (1) It follows from (A2).

(2) By (A3), if $x \in A$ then $x \xi A$ or $x \in \alpha A$. Therefore $A \subset \alpha A$.

(3) It is clear from (A1).

(4) If $x \in \alpha A$, the $x \xi A$ iff $x \xi B$ by (Al).

1.5. Remark. Since the operator α dosen't satisfy $\alpha \alpha A = \alpha A$ for each $A \subset X$, α is not a Kuratowski's closure operator.

1.6. Proposition. Let (X, ξ) be a B_1−proximity space. Then there exists a topology $\tau(\xi)$ on X such that each closed set in $\tau(\xi)$ is precisely the fixed set under the operator α.

Proof. Consider a family $F = \{A : \alpha A = A\}$ of subsets of X.

i) By (1), (2) in 1.4. we have $\phi \in F$, $X \in F$. resp.

ii) Let $\{A_i : i \in I\}$ be an arbitrary collection of members of F. If $x \xi \bigcap_{i \in I} A_i$ then $x \xi A_i$ for each $i \in I$, and so $x \in \alpha A_i = A_i$ for each $i \in I$. Hence $x \in \bigcap_{i \in I} A_i$.

iii) Let A, B be elements of F. Then from (3) in 1.4, $A \cup B \in F$. Therefore the family $\{X-A : \alpha A = A\}$ forms a topolgy $\tau(\xi)$ on X.

1.7. Propersition. In a B_1−proximity space (X, ξ), the following statements are equivalent:

(1) $x \xi A$.

(2) $x \xi \{y\}$ for some $y \in A$.

(3) $\{x\} \cap \alpha A \neq \phi$

Proof. (1)⇒(2). Since $x \xi A$, i.e. $x \xi \bigcup_{i \in A} \{y\}$, from (A1), there is $y \in A$ with $x \xi \{y\}$.

(2)⇒(3). If $x \xi \{y\}$, then $x \in \alpha \{y\}$ or $x \in \alpha A$, and so $\{x\} \cap \alpha A \neq \phi$.

(3)⇒(1). It is clear.

1.8. Theorem. Let (X, ξ) be a B_1−proximity space. Suppose that ξ satisfies the following condition: $x\xi\{y\}$ implies $y\xi\{x\}$. Then the followings are equivalent:

(1) $x\xi A$.

(2) $x\xi\{y\}$ for some $y\in A$.

(3) $\{x\}\cap\alpha A\neq\phi$.

(4) $\alpha\{x\}\cap A\neq\phi$.

Proof. It is suffice to show that (3) iff (4). Since $\{x\}\cap\alpha A\neq\phi$, $x\in\alpha A$ or $x\xi A$, so $x\xi\{y\}$ for some $y\in A$. But $x\xi\{y\}$ implies $y\xi\{x\}$, hence $y\in\alpha\{x\}\cap A$; $\alpha\{x\}\cap A\neq\phi$. Suppose that $\alpha\{x\}\cap A\neq\phi$. Then there is $y\in X$ such that $y\in\alpha\{x\}\cap A$. That is $y\in\alpha\{x\}$ and $y\in A$. Therefore $x\in\alpha\{y\}$ and $\alpha\{y\}\subset\alpha A$. This implies $\{x\}\cap\alpha A\neq\phi$.

Ⅱ. Main Results

The following theorem is an analogous concept in (Kong, 1980).

2.1 Theorem. $\underline{B_1-Prox}$ is a bicoreflective subcategory of $\underline{B_0-Prox}$.

Proof. Take any object (X, ξ) in $\underline{B_0-Prox}$. Define the relation ξ_1 on the power set of X as follows: $x\xi_1 A$ if and only if there is $y\in A$ such that $x\xi\{y\}$. Then it is clear that ξ_1 satisfies the axiom (A2) and (A3). For any non-void index set I, spppose $x\xi_1\bigcup_{i\in I}A_i$.

Then there is $y\in\bigcup_{i\in I}A_i$ with $x\xi\{y\}$. This implies $x\xi_1 A_j$ for $y\in A_j$. Conversely, if $x\xi_1 A_j$ for some $j\in I$, it is obvious that $x\xi_1\bigcup_{i\in I}A_i$. Thus $(X, \xi_1)\in\underline{B_1-Prox}$.

Let $1_X:(X, \xi_1)\rightarrow(X, \xi)$ be the identity map. Then by the defnition of ξ_1 it is clear that 1_X is a proximal map. Take any object $(Y, \xi')\in\underline{B_1-Prox}$ and take any proximal map $f:(Y, \xi')\rightarrow(X, \xi)$, It remains to show $f:(Y, \xi')\rightarrow(X, \xi_1)$ is a proximal map. Suppose that $x\xi' A$. Then by 1.7, there is $y\in A$ with $x\xi'\{y\}$, so that $f(x)\xi\{f(y)\}$ and $f(y)\in f(A)$. Thus $f(x)\xi_1 f(A)$. This completes the proof.

2.2. Corollary. (Herrlich & Strecker, 1973) $\underline{B_1-Prox}$ is coproductive and cohereditary in $\underline{B_0-Prox}$.

2.3. Definition. (Naimpally & Warrack, 1971) A subset A of a B_1−proximity space (X, ξ) is a *ξ−neighborhod of a point x* in X (in symbols $x\langle A$) iff $x\bar{\xi}(X-A)$.

2.4. Proposition. Given a B_1−proximity (X, ξ) the relation $\langle$ satisfies the following properties:

(1) $x\langle X$ for every x in X.

(2) $x\langle A$ implies $x\in A$.

(3) If $x\langle A$ and $A\subset B$ then $x\langle B$.

(4) If $x\langle A_i$ for $i=1,2,\cdots,n$ iff $x\langle\bigcap_{i=1}^{n}A_i$.

(5) For any index set I, $x\langle\bigcup_{i\in I}A_i$ iff $x\langle A_i$ for every $i\in I$.

(6) If $x\langle A$ then $\{x\}\subset A\subset\alpha A$.

Proof. (1) Since $x\bar{\xi}\phi$, $x\langle X$.

(2) Since $x\langle A$, $x\bar{\xi}(X-A)$, which implies $x\notin(X-A)$, so $x\in A$.

(3) If $A\subset B$, then $X-B\subset X-A$. Thus $x\langle A$ implies $x\bar{\xi}(X-B)$ or $x\langle B$.

(4) For any $i=1,2\cdots,n$, $x\bar{\xi}(X-A_i)$ iff $x\bar{\xi}\bigcup_{i=1}^{n}(X-A_i)$ iff $x\bar{\xi}(X-\bigcap_{i=1}^{n}A_i)$ iff $x\langle\bigcap_{i=1}^{n}A_i$.

(5) For any index set I, $x\langle\bigcup_{i\in I}A_i$

iff $x\bar{\xi}(X-\bigcup_{i\in I}A_i)$

iff $x\bar{\xi}\bigcap_{i\in I}(X-A_i)$

iff $x\bar{\xi}(X-A_i)$ for every $i\in I$

iff $x\langle A_i$ for every $i\in I$

(6) From (2), $x\langle A$ implies $x\xi A$. Therefore $x\in\alpha A$.

2.5. Theorem. If $\langle$ is a binary relation between X and PX satisfylng the properties (1)–(5) in the proposition 2.4 and ξ is defined by $x \bar{\xi} A$ iff $x \langle X-A$, then ξ is a B_1-proximity on X. A is a ξ-neighborhol of x iff $x \langle A$.

Proof. (A1) For any non-void index set I. $x \bar{\xi} A_i$ for each $i \in I$ iff $x \langle X-A_i$ for each $i \in I$ iff $x \langle \bigcap_{i \in I}(X-A_i)$ iff $x \bar{\xi} \bigcup_{i \in I} A_i$.

(A2) If $x \in X$ the $x \langle X$, which implies $x \bar{\xi} \phi$.

(A3) If $x \bar{\xi} A$ the $x \langle X-A$, so $x \in X-A$ or $x \notin A$.

2.6. Lemma. Let (X, ξ) be a B_1-poximity space. Then the followings are equivalent:

(1) $x \langle A$ implies there exists a subset B of X such that $x \langle B$ and $y \langle A$ for every y in B.

(2) If $x \bar{\xi} A$ then there exists a subset E of X such that $x \langle E$ and $y \bar{\xi} A$ for every y in E.

Proof. It is immediate from 2.3.

The condition in 2.6 will ensure that α is a topological closure operator.

2.7. Theorem. If a B_1-proximity space (X, ξ) satisfies the condition in 2.6, the operator α is a topological closure operator.

Proof. By Proposition 1.6, it remains to show that $\alpha\alpha A = \alpha A$ for each $A \subset X$. To do so, we must show that $\alpha\alpha A \subset \alpha A$. Suppose that $x \notin \alpha A$. Then we have $x \bar{\xi} A$, i.e. $x \langle X-A$. Thus there exists a set $E \subset X$ such that $x \langle E$ and $y \bar{\xi} A$ for every $y \in E$. From 2.4(2) and 1.4(2), $X \in E$ and $y \in X - \alpha A$. Therefore $x \in E \subset X - \alpha A \subset X-A$. Consequently $x \bar{\xi} \alpha\alpha A$, that is $x \notin \alpha\alpha A$.

2.8. Remark. The operator α is the closure operator of the topology that it induces: the closed sets are precisely the set of the form αA for each $A \subset X$

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國 文 抄 錄

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